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ON THE RANGE OF THE FIRST TWO DIRICHLET EIGENVALUES OF THE LAPLACIAN WITH VOLUME AND PERIMETER CONSTRAINTS

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ABSTRACT. In this paper we study the set of points, in the plane, defined by $\mathcal{E}^A := \{(x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ |\Omega| = 1\}$ on the one hande and $\mathcal{E}^P := \{(x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ P(\Omega) = 2\sqrt{\pi}\}$, on the other hand where $(\lambda_1(\Omega), \lambda_2(\Omega))$ are the two first eigenvalues of the Dirichlet-Laplacian. We give some qualitative properties of these sets and show some pictures obtained through numerical computations.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $|\Omega|$ its area and $P(\Omega)$ its perimeter. Let us consider the Dirichlet eigenvalue problem,

$$\left\{\begin{array}{ll} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

defined on the Sobolev space $H_0^1(\Omega)$. We will denote the eigenvalues by $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ...$ (counted with their multiplicities) and the corresponding orthonormal real eigenfunctions by $u_i,\ i = 1, 2, ...$

A natural question in spectral geometry is the following:
let $0 < a \leq b$ be two given real numbers, does there exist a domain Ω of area 1 which has a and b as their two first eigenvalues? In acoustic, the question corresponds to: is there exist a drum of given area (say 1) whose two first fundamental frequencies are a and b?. In [WK] and [BBF], it was studied the region

$$\mathcal{E}^A = \left\{(x,y) \in \mathbb{R}^2 : (x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ \Omega \subset \mathbb{R}^2, |\Omega| = 1\right\},$$

which is the range of the first two Dirichlet eigenvalues of planar sets with unit area. We also refer to [LY] for a similar study for the three first eigenvalues. Obviously, the complete knowledge of the set $\mathcal{E}^A$ allows to answer the previous questions. The same question can be raised by replacing the area by the perimeter. It leads to study the set

$$\mathcal{E}^P = \left\{(x,y) \in \mathbb{R}^2 : (x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ \Omega \subset \mathbb{R}^2, P(\Omega) = 2\sqrt{\pi}\right\},$$

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These two questions will be discussed in this paper.

The plan of this paper is the following. In the next section we study the set $\mathcal{E}^A$ and we mainly recall results already contained in [WK], [BBF] and [BNP] with some proofs. Then in section 3, we study the set $\mathcal{E}^P$ and give some of its properties.

2. The range of $\{\lambda_1, \lambda_2\}$ with an area constraint

We recall that we want to study the set

$$\mathcal{E}^A = \left\{(x,y) \in \mathbb{R}^2 : (x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ \Omega \subset \mathbb{R}^2, |\Omega| = 1\right\},$$

Let us begin with some elementary facts. Obviously $\mathcal{E}^A$ lies in the first quadrant and within the sector $0 < x \leq y$, because we defined the eigenvalues to be ordered. The behavior of eigenvalues with respect to homothety ($\lambda_k(t\Omega) = \lambda_k(\Omega)/t^2$) has two consequences. First we can also write

$$\mathcal{E}^A = \left\{(x,y) \in \mathbb{R}^2 : (x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ \Omega \subset \mathbb{R}^2, |\Omega| \leq 1\right\} \tag{2}$$

$$\left\{(x,y) \in \mathbb{R}^2 : (x,y) = (|\Omega|\lambda_1(\Omega), |\Omega|\lambda_2(\Omega)),\ \Omega \subset \mathbb{R}^2\right\}. \tag{3}$$

Moreover the region $\mathcal{E}^A$ is conical with respect to the origin in the sense,

$$(x,y) \in \mathcal{E}^A \Rightarrow (\alpha x, \alpha y) \in \mathcal{E}^A,\ \forall \alpha \geq 1.$$

Indeed, we can consider a homothety of ratio $1/\sqrt{\alpha}$ of the original domain and complete with a collection of small balls to reach volume 1 without changing the two first eigenvalues. This proves also the first equality in (2).

Now, we can get more precise information about $\mathcal{E}^A$ thanks to some important results on the low eigenvalues of the Laplacian. This region can be reduced using the famous Faber-Krahn inequality proved in [F1] and [K], (see [H, Theorem 3.2.1]) which states that the ball minimizes λ_1 among all planar domains with the same area. We can write this result as

$$|\Omega|\lambda_1(\Omega) \geq \lambda_1(\mathcal{B}) = \pi j_{0,1}^2 \approx 18.16842,$$

where $j_{n,k}$ denotes the k-th positive zero of the Bessel function J_n and $\mathcal{B}$ denotes the ball of unit area. Equality holds if and only if Ω is a ball (up to a set of zero capacity). For the second eigenvalue, we know that the minimum is attained by two balls of equal area. This result is due to Krahn and has been rediscovered by Szegö, and some other authors, see [H, Theorem 4.1.1] for more details. It can be written as

$$|\Omega|\lambda_2(\Omega) \geq 2\lambda_1(\mathcal{B}) = 2\pi j_{0,1}^2 \approx 36.33684.$$

The quotient λ_2/λ_1 is maximized at the ball (cf. [AB1] or [H, Theorem 6.2.1]) or equivalently,

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\mathcal{B})}{\lambda_1(\mathcal{B})} = \frac{j_{1,1}^2}{j_{0,1}^2} := \gamma \approx 2.539.$$

Now we recall two convexity results due to D. Bucur, G. Buttazzo and I. Figueiredo in [BBF].

Theorem 2.1 (Bucur-Buttazzo-Figueiredo). **(i):** *The set $\mathcal{E}^A$ is convex in the x-direction, namely:*

$$\forall (x,y) \in \mathcal{E}^A,\ \forall t \in [0,1],\ ((1-t)x + ty, y) \in \mathcal{E}^A.$$

(ii): *The set $\mathcal{E}^A$ is convex in the y-direction, namely:*

$$\forall (x,y) \in \mathcal{E}^A,\ \forall t \in [0,1],\ (x, (1-t)y + t\gamma x) \in \mathcal{E}^A.$$

Proof. We just give here the main ideas of the proof, for the details we refer to [BBF] and [BB]. Let $(x,y)=(\lambda_1(\Omega),\lambda_2(\Omega))$. For the horizontal convexity, one can construct a decreasing continuous sequence (or homotopy) $\Omega_1\subset\Omega_t\subset\Omega$, $t\in[0,1]$ such that

- $\Omega_0=\Omega$
- $\lambda_2(\Omega_t)=\lambda_2(\Omega)$
- $\lambda_1(\Omega_1)=\lambda_2(\Omega_1)$.

Roughly speaking, Ω_t is obtained from Ω by removing an increasing portion of the nodal line of u_2 and $\Omega_1=\{x\in\Omega,\ u_2(x)\neq 0\}$ is the open set Ω without the whole nodal line for which we already know that $\lambda_1(\Omega_1)=\lambda_2(\Omega_1)=\lambda_2(\Omega)$

The vertical convexity relies on properties of Steiner symmetrization and continuous Steiner symmetrization. We consider a point $(x,y)=(\lambda_1(\Omega),\lambda_2(\Omega))$ in $\mathcal{E}^A$. We denote by $\mathcal{B}$ the ball of area 1. We want to prove that the segment $(x,(1-s)y+s\gamma x)$, $s\in[0,1]$ is included in $\mathcal{E}^A$. If $y\geq\lambda_2(B)$ the result is obvious using horizontal convexity, so we can assume $y<\lambda_2(\mathcal{B})$. Let us fix $\alpha\in(\lambda_2(\Omega),\lambda_2(\mathcal{B}))$. We can construct a sequence of Steiner symmetrizations of Ω, say $\Omega_n, n\in\mathbb{N}$ such that Ω_n converges to $\mathcal{B}$. Moreover, we assume that we go from Ω_n to Ω_{n+1} thanks to a continuous Steiner symmetrization. We denote by Ω_t, $t\in\mathbb{R}_+$ this family of sets. According to classical properties of the continuous Steiner symmetrization, see [BR], $\lambda_1(\Omega_t)$ decreases with t. Now, the sequence $\lambda_2(\Omega_t)$ has possibly discontinuities, but converges to $\lambda_2(\mathcal{B})$, therefore there exists n_0 such that $\lambda_2(\Omega_{n_0})\geq\alpha$. We introduce

$$t^*=\sup\{t\in[0,n_0]:\ \lambda_2(\Omega_t)\leq\alpha\}.$$

By lower semi-continuity on the left and upper semi-continuity on the right of the eigenvalues with respect to continuous Steiner symmetrization, see [BH], we have $\lambda_2(\Omega_{t^*})=\alpha$. We conclude by using one more time the horizontal convexity between the points $(\lambda_1(\Omega_{t^*}),\alpha)$ and (α,α) (which belong to $\mathcal{E}^V$), the point $(\lambda_1(\Omega),\alpha)$ is on this segment, so belongs to $\mathcal{E}^A$. □

As a consequence, they also proved:

Theorem 2.2 (Bucur, Buttazzo, Figueiredo). *The set $\mathcal{E}^A$ is closed in $\mathbb{R}^2$.*

Proof. The idea of the proof is the following. Let us consider $(x,y)\in\overline{\mathcal{E}^A}$ and a sequence Ω_n such that $\lambda_1(\Omega_n)\to x$ and $\lambda_2(\Omega_n)\to y$. Then, we can find a subsequence, still denoted by Ω_n and a set Ω such that

$$\lambda_1(\Omega)\leq\liminf\lambda_1(\Omega_n)=x\quad\text{and}\quad\lambda_2(\Omega)\leq\liminf\lambda_2(\Omega_n)=y.$$

This is a consequence of the so-called compactness for the weak γ-convergence, see [BBF] and [BB] for more details.

Let us assume first that $y\geq\lambda_2(\mathcal{B})$ where $\mathcal{B}$ is the ball of volume 1. Then, there is an homothetic ball B' of volume smaller than 1 such that $y=\lambda_2(B')$. The horizontal convexity of $\mathcal{E}^A$ proved in Theorem 2.2 shows that the segment joining the points $(\lambda_1(B'),\lambda_2(B'))$ and $(\lambda_2(B'),\lambda_2(B'))$ is contained in $\mathcal{E}^A$. Therefore, (x,y) which belongs to this segment lies in $\mathcal{E}^A$.

Now, if $y<\lambda_2(\mathcal{B})$, from the vertical convexity, the segment joining the points $(\lambda_1(\Omega),\lambda_2(\Omega))$ and $(\lambda_1(\Omega),\gamma\lambda_1(\Omega))$ is contained in $\mathcal{E}^A$ and the point $(\lambda_1(\Omega),y)$ belongs to this segment. We conclude, as above, by using the horizontal convexity between $(\lambda_1(\Omega),y)$ and (y,y). □

The above results show that the only unknown part of the set $\mathcal{E}^A$ is the lower part, the curve γ joining the point A corresponding to one ball and the point B corresponding to two balls. It turns out that the tangents of γ at these extremal points are known, see [WK] for the vertical tangent at point A and the recent [BNP] for the horizontal tangent at point B.

Theorem 2.3 (Wolff-Keller, Brasco-Nitsch-Pratelli). *Let γ denotes the curve, lower part of the set $\mathcal{E}^A$, then*

- *the tangent at the point A corresponding to one ball is vertical,*
- *the tangent at the point B corresponding to two identical balls is horizontal*

(sketch of the) proof: Because of Faber-Krahn inequality, to prove the first item, it suffices to find a (continuous) sequence of open sets Ω_ε of area 1, converging to the ball $\mathcal{B}$ and such that

$$\frac{\lambda_2(\Omega_\varepsilon)-\lambda_2(\mathcal{B})}{\lambda_1(\Omega_\varepsilon)-\lambda_1(\mathcal{B})}\to-\infty. \tag{4}$$

For that purpose, S. Wolf and J. Keller use the following expansion of the two first eigenvalues of a nearly circular domain. If a domain Ω_ε is given in polar coordinates as

$$r:=\frac{1}{\sqrt{\pi}}+\varepsilon\sum_{n=-\infty}^{+\infty}a_n e^{in\theta}+\varepsilon^2\sum_{n=-\infty}^{+\infty}b_n e^{in\theta}+O(\varepsilon^3), a_n=\overline{a_{-n}}, b_n=\overline{b_{-n}} \tag{5}$$

then its area is preserved (at order two) if

$$a_0=0,\ b_0=-\frac{1}{2}\sum_{n=1}^{+\infty}|a_n|^2,$$

while the two first eigenvalues satisfy

$$\lambda_1(\Omega_\varepsilon)=\pi j_{01}^2\left\{1+4\varepsilon^2\sum_{n=1}^{+\infty}\left[1+j_{01}\frac{J_n'(j_{01})}{J_n(j_{01})}\right]|a_n|^2\right\}+O(\varepsilon^3), \tag{6}$$

(this expansion is actually due to Lord Rayleigh who proved, in particular, that the coefficient in ε^2 is positive) and

$$\lambda_2(\Omega_\varepsilon)=\pi j_{11}^2\{1-2|\varepsilon||a_2|\}+O(\varepsilon^2) \tag{7}$$

in these expression j_{01} and j_{11} denote respectively the first (positive) zeroes of the Bessel functions J_0 and J_1. In particular $\lambda_1(\mathcal{B})=\pi j_{01}^2$ and $\lambda_2(\mathcal{B})=\pi j_{11}^2$ Choosing now $a_2\neq 0$, we get

$$\frac{\lambda_2(\Omega_\varepsilon)-\lambda_2(\mathcal{B})}{\lambda_1(\Omega_\varepsilon)-\lambda_1(\mathcal{B})}=\frac{-2\pi j_{11}^2|\varepsilon||a_2|+O(\varepsilon^2)}{4\pi j_{01}^2\varepsilon^2\sum_{n=1}^{+\infty}\left[1+j_{01}\frac{J_n'(j_{01})}{J_n(j_{01})}\right]+O(\varepsilon^3)} \tag{8}$$

and the result follows when ε goes to 0, the tangent at point A is vertical.

Let us denote by Θ the union of two identical balls of total area 2π. In [BNP] (see also [vdB] for similar results), the authors introduce the set (see Figure 1)

$$\Omega_\varepsilon:=\{(x,y):(x-1+\varepsilon)^2+y^2<1 \text{ or } (x+1-\varepsilon)^2+y^2<1\}$$

and they prove the following estimates (using appropriate test functions in the Rayleigh quotient defining λ_1 and λ_2)

$$\forall\varepsilon \text{ small enough}\qquad \lambda_1(\Omega_\varepsilon)\leq\lambda_1(\Theta)-\gamma_1\varepsilon \tag{9}$$

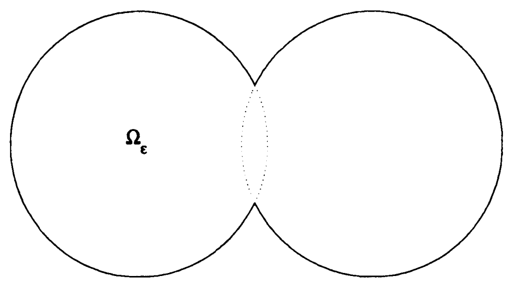

FIGURE 1. The set Ω_ε.

$$\forall\varepsilon \text{ small enough} \qquad \lambda_2(\Omega_\varepsilon) \le \lambda_2(\Theta) + \gamma_2\varepsilon^{3/2} \tag{10}$$

where γ_1, γ_2 are two positive constants. Thus introducing $\widetilde{\Omega}_\varepsilon$ and $\widetilde{\Theta}$ which are rescaled version of Ω_ε and Θ of area 1, we can estimate the following ratio using (9) and (10)

$$\frac{\lambda_2(\widetilde{\Omega}_\varepsilon) - \lambda_2(\widetilde{\Theta})}{\lambda_1(\widetilde{\Theta}) - \lambda_1(\widetilde{\Omega}_\varepsilon)} \le \frac{\gamma_2'\varepsilon^{3/2}}{\gamma_1'\varepsilon} \to 0 \text{ when } \varepsilon \to 0$$

which shows that the tangent at point B is horizontal. □

In Figure 2, we have determined numerically this curve with the same procedure as in [WK], solving a minimization problem with a convex combination of λ_1 and λ_2. Our results were obtained with the gradient method to solve the minimization problems, as in [AA2]. The solver that we used was the Method of Fundamental Solutions (MFS), as studied in [AA1] or in some cases an enriched version of the MFS, as in [AV]. We recall the conjecture already stated in [BBF]:

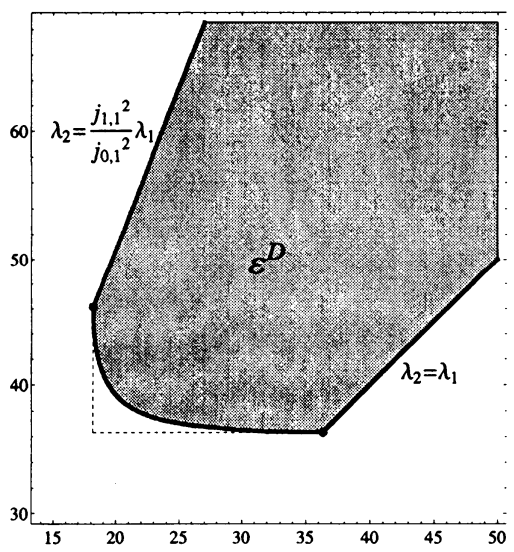

FIGURE 2. The region $\mathcal{E}^A$.

Conjecture 1. The set $\mathcal{E}^A$ is convex.

3. The range of $\{\lambda_1, \lambda_2\}$ with a perimeter constraint

We want now to study the set

$$\mathcal{E}^P = \{(x,y) \in \mathbb{R}^2 : (x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ \Omega \subset \mathbb{R}^2, P(\Omega) = 2\sqrt{\pi}\}.$$

The choice of the value $2\sqrt{\pi}$ is done to ensure that the set $\mathcal{E}^P$ contains the same ball (of area 1) than the previous set $\mathcal{E}^A$ and then the two sets can be more easily compared.

Let us observe, in that context that $\mathcal{E}^P \subset \mathcal{E}^A$. The first remark is that we can also take as a definition for $\mathcal{E}^P = \{(x,y) : (x,y) = (\lambda_1(\Omega), \lambda_2(\Omega)),\ P(\Omega) \le 2\sqrt{\pi}\}$ and the set $\mathcal{E}^P$ is conical with respect to the origin. The proof is the same as in the area constraint: take an homothetic version of a set Ω and complete with a collection of small discs without changing the two first eigenvalues. Then, if the point (x,y) belongs to $\mathcal{E}^P$ corresponding to some Ω of perimeter $2\sqrt{\pi}$, by the classical isoperimetric inequality $|\Omega| \le 1$ and therefore Ω defines an admissible set for the class $\mathcal{E}^A$, so $(x,y) \in \mathcal{E}^A$.

Now, as in the area constraint case, we have

- $\mathcal{E}^P \subset \{(x,y) : 0 < x \le y\}$
- $\mathcal{E}^P \subset \{(x,y) : x \ge \lambda_1(\mathcal{B}) = \pi j_{0,1}^2\}$ because Faber-Krahn's inequality holds true by the classical isoperimetric inequality (as we already mentioned, it follows from the inclusion $\mathcal{E}^P \subset \mathcal{E}^A$)
- $\mathcal{E}^P \subset \{(x,y) : y/x \le \frac{\lambda_2(\mathcal{B})}{\lambda_1(\mathcal{B})} = \frac{j_{1,1}^2}{j_{0,1}^2}\}$ because Ashbaugh-Benguria Theorem still holds true.

The first big difference comes from the fact that the lowest point of $\mathcal{E}^P$, i.e. the point corresponding to the domain minimizing λ_2 with a perimeter constraint is no longer the union of two balls. It has been proved in [BBH] that this domain is a regular convex domain (see Figure 3). It will also be a consequence of the more general Theorem 3.1 below.

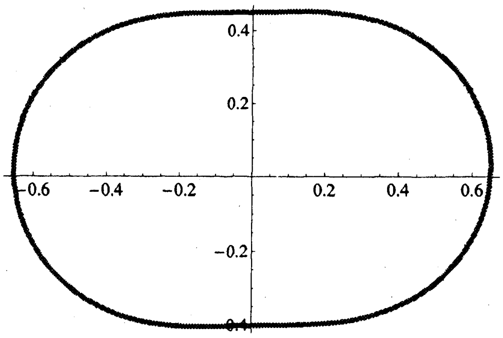

FIGURE 3. The set which minimizes λ_2 with a perimeter constraint.

Theorem 3.1. *For any $\beta \in [0, \pi/2]$, there exists a minimizer Ω for the problem*

$$\min\{\cos\beta\lambda_1(\Omega) + \sin\beta\lambda_2(\Omega), P(\Omega) = 2\sqrt{\pi}\}. \tag{11}$$

Moreover, the domain Ω is convex, C^∞ and it satisfies the overdetermined condition

$$\cos\beta|\nabla u_1|^2 + \sin\beta|\nabla u_2|^2 = \frac{\cos\beta\lambda_1(\Omega) + \sin\beta\lambda_2(\Omega)}{\sqrt{\pi}} \mathcal{C} \quad \text{on } \partial\Omega \tag{12}$$

where u_1 and u_2 are the two first normalized eigenfunctions (which are both simple) and C is the curvature of the boundary. In particular, the boundary of Ω does not contain any segment.

Proof. First, let us observe that the monotonicity of the eigenvalues of the Dirichlet-Laplacian with respect to the inclusion has two easy consequences:

(1) If Ω^* denotes the convex hull of Ω, since *in two dimensions* and for a connected set, $P(\Omega^*) \leq P(\Omega)$, it is clear that we can restrict ourselves to look for minimizers in the class of *convex sets* with perimeter less or equal than $c := 2\sqrt{\pi}$.

(2) Obviously, it is equivalent to consider the constraint $P(\Omega) \leq c$ or $P(\Omega) = c$.

We will need the simplicity of $\lambda_2(\Omega)$ (the simplicity of $\lambda_1(\Omega)$ is a consequence of the connectedness of Ω_β)

Lemma 3.2. *If Ω is a minimizer of problem* (11), *then $\lambda_2(\Omega)$ is simple.*

Proof. The idea of the proof is to show that a double eigenvalue would split under boundary perturbation of the domain, with one of the eigenvalues going down. A very similar result is proved in [H, Theorem 2.5.10]. The new difficulties here are the perimeter constraint (instead of the volume) and the fact that the domain Ω is convex, but not necessarily regular. Nevertheless, we know that any eigenfunction of a convex domain is in the Sobolev space $H^2(\Omega)$, see [Gri]. Let us assume, for a contradiction, that $\lambda_2(\Omega)$ is not simple, then it is double because Ω is a convex domain in the plane, see [Lin]. Let us recall the result of derivability of eigenvalues in the multiple case (see [Cox] or [R]). Assume that the domain Ω is modified by a regular vector field $x \mapsto x + tV(x)$. We will denote by Ω_t the image of Ω by this transformation. Of course, Ω_t may be not convex but we have actually no convexity constraint (since convexity come for free) and this has no consequence on the differentiability of $t \mapsto \lambda_2(\Omega_t)$. Let us denote by u_2, u_3 two orthonormal eigenfunctions associated to λ_2, λ_3. Then, the first variation of $\lambda_2(\Omega_t), \lambda_3(\Omega_t)$ are the repeated eigenvalues of the 2×2 matrix

$$\mathcal{M} = \begin{pmatrix} -\int_{\partial\Omega} \left(\frac{\partial u_2}{\partial n}\right)^2 V.n\, d\sigma & -\int_{\partial\Omega} \left(\frac{\partial u_2}{\partial n}\frac{\partial u_3}{\partial n}\right) V.n\, d\sigma \\ -\int_{\partial\Omega} \left(\frac{\partial u_2}{\partial n}\frac{\partial u_3}{\partial n}\right) V.n\, d\sigma & -\int_{\partial\Omega} \left(\frac{\partial u_3}{\partial n}\right)^2 V.n\, d\sigma \end{pmatrix}. \tag{13}$$

Now, let us introduce the Lagrangian $L(\Omega) = \cos\beta\lambda_1(\Omega) + \sin\beta\lambda_2(\Omega)\mu P(\Omega)$. As we will see below, the perimeter is differentiable and the derivative is a linear form in $V.n$ supported on $\partial\Omega$ (see e.g. [HP, Corollary 5.4.16]). We will denote by $\langle dP_{\partial\Omega}, V.n\rangle$ this derivative. Moreover the first eigenvalue is also differentiable (see [HP]) since it is simple, we will denote by $d\lambda_1(\Omega; V)$ its derivative. So the Lagrangian $L(\Omega_t)$ has a derivative which is the smallest eigenvalue of the matrix $\sin\beta\mathcal{M} + (\cos\beta\langle d\lambda_1(\Omega; V), V.n\rangle + \mu\langle dP_{\partial\Omega}, V.n\rangle)\, I$ where I is the identity matrix. Therefore, to reach a contradiction (with the optimality of Ω), it suffices to prove that one can always find a deformation field V such that the smallest eigenvalue of this matrix is negative. Let us consider two points A and B on $\partial\Omega$ and two small neighborhoods γ_A and γ_B of these two points of same length, say 2δ. Let us choose any regular function $\varphi(s)$ defined on $(-\delta, +\delta)$ (vanishing at the extremities of the interval) and a deformation field V such that

$$V.n = +\varphi \text{ on } \gamma_A, \quad V.n = -\varphi \text{ on } \gamma_B, \quad V.n = 0 \text{ elsewhere}.$$

Then, the matrix $\sin\beta\mathcal{M} + (\cos\beta\langle d\lambda_1(\Omega; V), V.n\rangle + \mu\langle dP_{\partial\Omega}, V.n\rangle)\, I$ splits into two matrices $\mathcal{M}_A - \mathcal{M}_B$ which are obtained from the previous formula. In particular, it is clear that the exchange of A and B replaces the matrix $\mathcal{M}_A - \mathcal{M}_B$ by its opposite. Therefore, the only case where one would be unable to choose two points A, B and a deformation φ such that the matrix has a negative eigenvalue is if $\mathcal{M}_A - \mathcal{M}_B$ is identically zero for any φ. But this implies, in particular

$$\int_{\gamma_A} \frac{\partial u_2}{\partial n}\frac{\partial u_3}{\partial n}\varphi\, d\sigma = \int_{\gamma_B} \frac{\partial u_2}{\partial n}\frac{\partial u_3}{\partial n}\varphi\, d\sigma \tag{14}$$

and

$$\int_{\gamma_A} \left[\left(\frac{\partial u_2}{\partial n}\right)^2 - \left(\frac{\partial u_3}{\partial n}\right)^2\right]\varphi\, d\sigma = \int_{\gamma_B} \left[\left(\frac{\partial u_2}{\partial n}\right)^2 - \left(\frac{\partial u_3}{\partial n}\right)^2\right]\varphi\, d\sigma \tag{15}$$

for any regular φ and any points A and B on $\partial\Omega$. This implies that the product $\left(\frac{\partial u_2}{\partial n}\frac{\partial u_3}{\partial n}\right)^2$ and the difference $\left(\frac{\partial u_2}{\partial n}\right)^2 - \left(\frac{\partial u_3}{\partial n}\right)^2$ should be constant a.e. on $\partial\Omega$. As a consequence $\left(\frac{\partial u_2}{\partial n}\right)^2$ has to be constant. Since the nodal line of the second eigenfunction touches the boundary in two points see [Mel], $\frac{\partial u_2}{\partial n}$ has to change sign. So we get a function belonging to $H^{1/2}(\partial\Omega)$ taking values c and $-c$ on sets of positive measure, which is absurd, unless $c = 0$. This last issue is impossible by the Holmgren uniqueness theorem. □

We are now in a position to prove the existence and regularity of optimal domains for problem (11). To show the existence of a solution we use the direct method of calculus of variations. Let Ω_n be a minimizing sequence that, according to point 1 above, we can assume made by convex sets. Moreover, Ω_n is a bounded sequence because of the perimeter constraint. Therefore, there exists a convex domain Ω and a subsequence still denoted by Ω_n such that:

- Ω_n converges to Ω for the Hausdorff metric and for the L^1 convergence of characteristic functions (see e.g. [HP, Theorem 2.4.10]); since Ω_n and Ω are convex this implies that $\Omega_n \to \Omega$ in the γ-convergence;
- $P(\Omega) \le c$ (because of the lower semicontinuity of the perimeter for the L^1 convergence of characteristic functions, see [HP, Proposition 2.3.6]);
- $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ and $\lambda_2(\Omega_n) \to \lambda_2(\Omega)$ (continuity of the eigenvalues for the γ-convergence, see [BB, Proposition 2.4.6] or [H, Theorem 2.3.17].

Therefore, Ω is a solution of problem (11).

We go on with the proof of regularity, which is classical, see e.g. [CL]. Let us consider (locally) the boundary of $\partial\Omega$ as the graph of a (concave) function $h(x)$, with $x \in (-a, a)$. We make a perturbation of $\partial\Omega$ using a regular function ψ compactly supported in $(-a, a)$, i.e. we look at Ω_t whose boundary is $h(x) + t\psi(x)$. The function $t \mapsto P(\Omega_t)$ is differentiable at $t = 0$ (see [Gi] or [HP]) and its derivative $dP(\Omega, \psi)$ at $t = 0$ is given by:

$$dP(\Omega, \psi) := \int_{-a}^{+a} \frac{h'(x)\psi'(x)\, dx}{\sqrt{1 + h'(x)^2}}\,. \tag{16}$$

In the same way, thanks to Lemma 3.2, the functions $t \mapsto \lambda_1(\Omega_t)$ and $t \mapsto \lambda_2(\Omega_t)$ are differentiable (see [HP, Theorem 5.7.1]) and since the (normalized) eigenfunctions u_1, u_2 belong to the Sobolev space $H^2(\Omega)$ (due to the convexity of Ω, see [Gri, Theorem 3.2.1.2]), the derivative of $J(\Omega) := \cos\beta\lambda_1(\Omega) + \sin\beta\lambda_2(\Omega)$ at $t = 0$ is given by

$$dJ(\Omega, \psi) := -\int_{-a}^{+a} [\cos\beta|\nabla u_1(x, h(x))|^2 + \sin\beta|\nabla u_2(x, h(x))|^2]\psi(x)\, dx. \tag{17}$$

The optimality of Ω implies that there exists a Lagrange multiplier μ such that, for any $\psi \in C_0^\infty(-a, a)$

$$\mu dJ(\Omega, \psi) + dP(\Omega, \psi) = 0$$

which implies, thanks to (16) and (17), that h is a solution (in the sense of distributions) of the o.d.e.:

$$-\left(\frac{h'(x)}{\sqrt{1 + h'(x)^2}}\right)' = \mu[\cos\beta|\nabla u_1(x, h(x))|^2 + \sin\beta|\nabla u_2(x, h(x))|^2]\,. \tag{18}$$

Since $u_1, u_2 \in H^2(\Omega)$, their first derivatives $\frac{\partial u_j}{\partial x}$ and $\frac{\partial u_j}{\partial y}, j = 1, 2$ have a trace on $\partial\Omega$ which belong to $H^{1/2}(\partial\Omega)$. Now, the Sobolev embedding in one dimension $H^{1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$ for any $p > 1$ shows that $x \mapsto |\nabla u_j(x, h(x))|^2, j = 1, 2$ is in $L^p(-a, a)$ for any $p > 1$. Therefore, according to (18), the function $h'/\sqrt{1 + h'^2}$ is in $W^{1,p}(-a, a)$ for any $p > 1$ (recall that h' is bounded because Ω is convex), so it belongs to some Hölder space $C^{0,\alpha}([-a, a])$ (for any $\alpha < 1$, according to Morrey-Sobolev embedding). Since h' is bounded, it follows immediately that h belongs to $C^{1,\alpha}([-a, a])$. Now, we come back to the partial differential equation and use an intermediate Schauder regularity result (see [GH] or the remark after Lemma 6.18 in [GT]) to claim that if $\partial\Omega$ is of class $C^{1,\alpha}$, then the eigenfunctions u_j are $C^{1,\alpha}(\overline{\Omega})$ and $|\nabla u_j|^2$ is $C^{0,\alpha}$ for $j = 1, 2$. Then, looking again to the o.d.e. (18) and using the same kind of Schauder's regularity result yields that $h \in C^{2,\alpha}$. We iterate the process, thanks to a classical bootstrap argument, to conclude that h is C^∞.

Since we know that the minimizers are of class C^∞, we can now write rigorously the optimality condition. Under variations of the boundary (replace Ω by $\Omega_t = (I + tV)(\Omega)$), the shape derivative of the perimeter is given by (see [HP, Corollary 5.4.16])

$$dP(\Omega; V) = \int_{\partial\Omega} C\, V.n\, d\sigma$$

where C is the curvature of the boundary and n the exterior normal vector. Using the expression of the derivative of the eigenvalues given in (17) (see also [HP, Theorem 5.7.1]), the proportionality of these two derivatives through some Lagrange multiplier yields the existence of a constant μ such that

$$\cos\beta|\nabla u_1|^2 + \sin\beta|\nabla u_2|^2 = \mu C \tag{19}$$

Setting $X = (x_1, x_2)$, multiplying the equality in (19) by $X.n$ and integrating on $\partial\Omega$ yields, thanks to Gauss formulae $\int_{\partial\Omega} C\, X.n\, d\sigma = P(\Omega)$, and a classical application of the Rellich formulae $\int_{\partial\Omega} |\nabla u_j|^2\, X.n\, d\sigma = 2\lambda_j(\Omega), j = 1, 2$, the value of the Lagrange multiplier. So, we have proved that any minimizer Ω satisfies

$$\cos\beta|\nabla u_1|^2 + \sin\beta|\nabla u_2|^2 = \frac{\cos\beta\lambda_1(\Omega_\beta) + \sin\beta\lambda_2(\Omega_\beta)}{\sqrt{\pi}}\, C(x)\,, \quad x \in \partial\Omega \tag{20}$$

where $C(x)$ is the curvature at point x.

As a consequence, we easily see that the boundary of the optimal domain does not contain any segment. Indeed, an easy consequence of Hopf's lemma (applied to each nodal domain) is that the normal derivative of u_2 only vanishes on $\partial\Omega$ at points where the nodal line hits the boundary while the normal derivative of u_1 never vanishes. This proves, together with (20) that the curvature cannot be zero (for $\beta < \pi/2$) or can be zero only at two points (for $\beta = \pi/2$). $\square$

As we did in the previous section, we use Theorem 3.1 to determine the lower part of the set $\mathcal{E}^P$ since looking for solutions of the minimization problem (11) provides the lower point of $\mathcal{E}^P$ in the direction orthogonal to $(\cos\beta, \sin\beta)$. The result we get numerically is shown in Figure 4. In this Figure, the black point on the left, say A,

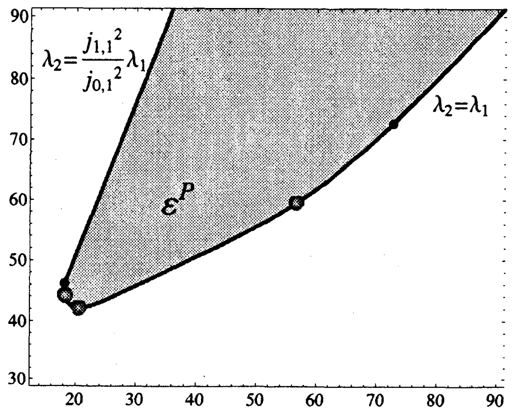

FIGURE 4. The region $\mathcal{E}^P$.

corresponds to one ball, the black point on the right, say B, to two identical balls, while the three red points correspond to solutions of the minimization problem (11) for $\beta = 0.2$, $\beta = 1.6$ and $\beta = 2.18$ respectively, see Figure 5

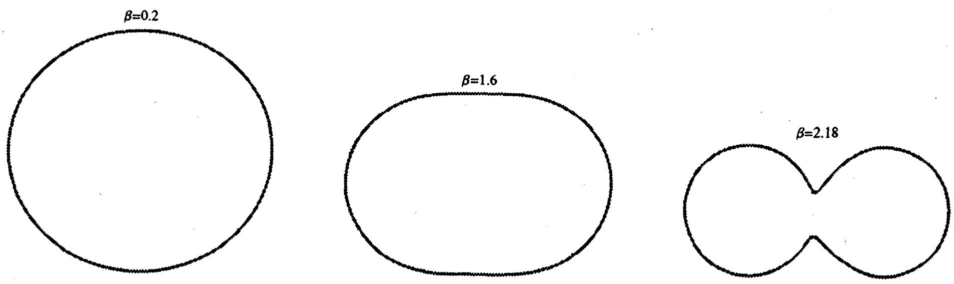

FIGURE 5. Three optimal domains for $\beta = 0.2$, $\beta = 1.6$ and $\beta = 2.18$.

We conclude by giving the tangents of the curve bounding $\mathcal{E}^P$ at point A and B:

Theorem 3.3. *Let γ_2 denotes the curve, lower part of the set $\mathcal{E}^P$, then*

- *the tangent of γ_2 at the point A corresponding to one ball is vertical,*
- *the tangent of γ_2 at the point B corresponding to two identical balls is the first bissectrix.*

Proof. We proceed in the same way as in the proof of Theorem 2.3. Because of Faber-Krahn inequality, to prove the first item, it suffices to find a (continuous) sequence of open sets Ω_ε of perimeter $2\sqrt{\pi}$, converging to the ball $\mathcal{B}$ and such that

$$\frac{\lambda_2(\Omega_\varepsilon)-\lambda_2(\mathcal{B})}{\lambda_1(\Omega_\varepsilon)-\lambda_1(\mathcal{B})} \to -\infty. \tag{21}$$

We use a family of domains Ω_ε given in polar coordinates as

$$r := \frac{1}{\sqrt{\pi}} + 2\varepsilon a \cos 2\theta \tag{22}$$

with a a positive real number. then its perimeter is given by

$$P(\Omega_\varepsilon) = \int_0^{2\pi} \sqrt{r^2+r'^2}\, d\theta = 2\sqrt{\pi} + 4\pi^{3/2}a^2\varepsilon^2 + O(\varepsilon^3)$$

while the two first eigenvalues satisfy

$$\lambda_1(\Omega_\varepsilon) = \pi j_{01}^2 \left\{1 + 4\varepsilon^2 \left[1 + j_{01}\frac{J_2'(j_{01})}{J_2(j_{01})}\right] a^2\right\} + O(\varepsilon^3), \tag{23}$$

and

$$\lambda_2(\Omega_\varepsilon) = \pi j_{11}^2 \{1 - 2|\varepsilon| a\} + O(\varepsilon^2) \tag{24}$$

By homogeneity, we can consider $P^2(\Omega_\varepsilon)\lambda_j(\Omega_\varepsilon)$ instead of fixing the perimeter and considering $\lambda_j(\Omega_\varepsilon)$. Therefore, we get

$$\frac{P^2(\Omega_\varepsilon)\lambda_2(\Omega_\varepsilon) - P^2(\mathcal{B})\lambda_2(\mathcal{B})}{P^2(\Omega_\varepsilon)\lambda_1(\Omega_\varepsilon) - P^2(\mathcal{B})\lambda_1(\mathcal{B})} = \frac{-8\pi^2 j_{11}^2|\varepsilon| a + O(\varepsilon^2)}{O(\varepsilon^2)} \tag{25}$$

and the result follows when ε goes to 0, the tangent at point A is vertical.

Now we want to determine the tangent at the point corresponding to $\widetilde{\Omega}_\varepsilon$ the union of two identical balls of total area 1. We use the same set as previously namely (see Figure 1)

$$\Omega_\varepsilon := \{(x,y) : (x-1+\varepsilon)^2 + y^2 < 1 \text{ or } (x+1-\varepsilon)^2 + y^2 < 1\}$$

First of all, it is easy to check by a straightforward computation that

$$|\Omega_\varepsilon| = 2\pi - O(\varepsilon^{3/2}),\ P(\Omega_\varepsilon) = 4\pi - 4\sqrt{2\varepsilon} + O(\varepsilon^{3/2}), \text{ and } \frac{P(\Omega_\varepsilon)^2}{|\Omega_\varepsilon|} = 8\pi - 16\sqrt{2\varepsilon} + O(\varepsilon).$$

We want to find the limit of the ratio

$$Q(\varepsilon) := \frac{P^2(\Omega_\varepsilon)\lambda_2(\Omega_\varepsilon) - P^2(\Theta)\lambda_2(\Theta)}{P^2(\Omega_\varepsilon)\lambda_1(\Omega_\varepsilon) - P^2(\Theta)\lambda_1(\Theta)}$$

when $\varepsilon \to 0$. We write it

$$Q(\varepsilon) := \frac{\frac{P^2(\Omega_\varepsilon)}{|\Omega_\varepsilon|}|\Omega_\varepsilon|\lambda_2(\Omega_\varepsilon) - \frac{P^2(\Theta)}{|\Theta|}|\Theta|\lambda_2(\Theta)}{\frac{P^2(\Omega_\varepsilon)}{|\Omega_\varepsilon|}|\Omega_\varepsilon|\lambda_1(\Omega_\varepsilon) - \frac{P^2(\Theta)}{|\Theta|}|\Theta|\lambda_1(\Theta)}.$$

If we introduce $x(\varepsilon) = |\Omega_\varepsilon|\lambda_1(\Omega_\varepsilon)$ and $y(\varepsilon) = |\Omega_\varepsilon|\lambda_2(\Omega_\varepsilon)$, we already know, according to Theorem 2.3 that $y(\varepsilon) - y(0) = g(\varepsilon)(x(\varepsilon) - x(0))$ with $g(\varepsilon) \to 0$ when $\varepsilon \to 0$. Now we write

$$Q(\varepsilon) = \frac{(8\pi - 16\sqrt{2\varepsilon})y(\varepsilon) - 8\pi y(0)}{(8\pi - 16\sqrt{2\varepsilon})x(\varepsilon) - 8\pi x(0)} = \frac{8\pi g(\varepsilon)(x(\varepsilon)-x(0)) - 16\sqrt{2\varepsilon}y(\varepsilon)}{8\pi(x(\varepsilon)-x(0)) - 16\sqrt{2\varepsilon}x(\varepsilon)}.$$

Moreover, according to (9), $x(\varepsilon) - x(0) = O(\varepsilon)$ and $y(\varepsilon)/x(\varepsilon) \to 1$ therefore we have $Q(\varepsilon) \to 1$ which proves the desired result. □

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