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ISOLATED "HOT SPOTS" ON THE BOUNDARY OF A PLANAR CONVEX DOMAIN

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1. Introduction and Main results

This article is a survey of results related to the hot spots on a domain and it contains an announcement of the paper [10]. Theorems A, B, and C are proved in [10]. The Neumann eigenfunctions of the Laplacian on a bounded planar domain Ω with Lipschitz boundary satisfy $-\Delta u=\mu u$ with the Neumann boundary condition $\partial_\nu u=0$ on $\partial\Omega$. Let $\{\mu_j(\Omega)\}_{j=0}^\infty$ denote the eigenvalues (counting multiplicities). Then

$$0=\mu_0(\Omega)<\mu_1(\Omega)\le\mu_2(\Omega)\le\cdots.$$

We are interested in the characterization of the shape of the second Neumann eigenfunction on a convex domain with the number and locations of the critical points. As mentioned in Conjecture 2.1 below, it is conjectured that if the domain is convex, then none of the second Neumann eigenfunctions have an interior critical point and all critical points are on the boundary. When the domain is a planar polygon, the eigenfunction has a critical point at each corner. The number of the critical points on the boundary can be arbitrary large even if the domain is convex. Hence, we cannot characterize the shape with the number of the critical points on the boundary. Then, in this paper we study the maximum number of the isolated local maximum points on the boundary of a convex domain. The first main result of this article is

Theorem A. *Let $\theta>0$ be small. Let O be the origin of $\mathbb{R}^2$, and let $A_k^{(n)}=(\cos(\frac{n-2k}{2}\theta),\sin(\frac{n-2k}{2}\theta))$. Let $\Omega_{n,\theta}$ denote the convex polygon $OA_0^{(n)}A_1^{(n)}\cdots A_n^{(n)}$. For each integer $n\ge1$, there is a small $\theta>0$ such that $\mu_1(\Omega_{n,\theta})$ is simple, the associated eigenfunction attains its local and global maximum at $A_0^{(n)},\dots,A_n^{(n)}$, and it does not have an interior*

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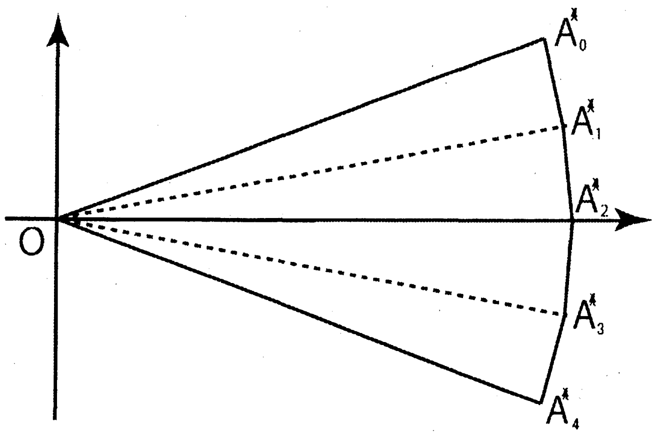

FIGURE 1. The shape of $\Omega_{4,\theta}$ given in Theorem A. The second eigenfunction attains its maximum at five vertices $A_0^{(4)}, \ldots, A_4^{(4)}$.

critical point. In particular, the eigenfunction has exactly $n+1$ isolated local and global maximum points on the boundary. See Figure 1 for the case $n = 4$.

This theorem says that a planar convex domain can have many isolated hot spots on the boundary and that there is no upper bound of the number of the hot spots. Therefore, it is impossible to characterize the shape of the second eigenfunction by the number of the local maximum points.

When the domain is a thin sector (resp. rectangle), each point on the arc (resp. one side) is a maximum point, hence there are infinitely many maximum points on the boundary. However, they are not isolated.

Remark 1.1. *It is known that the first Dirichlet eigenfunction on a planar domain has exactly one local and global interior maximum point if the domain is strictly convex. See* [12]. *For a nonlinear version of the Dirichlet problem, see* [3].

We study the eigenfunction on $\Omega_{n,\theta}$, using that on a thin isosceles triangle. The main part of this paper is to study the shape of the eigenfunction on an isosceles triangle. In order to state the next main result we need some notation. Let $a > 0$. Throughout the present paper we define $O = (0,0)$, $P = (0,a)$, $Q = (0,-a)$, $R = (\sqrt{3},0)$, $S = (-\sqrt{3},0)$ in the xy-plane and denote the open triangle PQR by T. Note that if $a = 1$, then T is an equilateral triangle. Let $T_+ = T \cap \{y > 0\}$ and $T_- = T \cap \{y < 0\}$. Following [6], we call T *a superequilateral* *triangle* if $a > 1$ and *a subequilateral triangle* if $0 < a < 1$. Laugesen-Siudeja [6] studied the shape of the second Neumann eigenfunction on T and obtained the following:

Proposition 1.2 ([6, Theorems 3.1 and 3.2]). *(i) Every second Neumann eigenfunction on a subequilateral triangle T is even with respect to the x-axis.*
(ii) Every second Neumann eigenfunction on a superequilateral triangle T is odd with respect to the x-axis.

In this paper we study the shape of the second Neumann eigenfunction on an isosceles triangle, using this proposition. The second main result of this article is the following two theorems:

Theorem B. *Let u be a second Neumann eigenfunction on T. Suppose that T is a subequilateral triangle. Then $\mu_1(T)$ is simple, u is even with respect to the x-axis, and $u(O) \neq 0$. Moreover, suppose without loss of generality that $u(O) > 0$. Then the following holds:*
(i) $u_x < 0$ in $\overline{T}\backslash(\{x = 0\} \cup \{R\})$, $u_y > 0$ in $\overline{T_+}\backslash(\{y = 0\} \cup \{P\})$, and $u_y < 0$ in $\overline{T_-}\backslash(\{y = 0\} \cup \{Q\})$. Here $\overline{T}$ and $\overline{T_+}$ denote the closures of T and T_+, respectively.
(ii) u has exactly four critical points O, P, Q, and R in $\overline{T}$.
(iii) P and Q are the local and global maximum points of u and $u(P) = u(Q) > 0$.
(iv) R is the local and global minimum point of u and $u(R) < 0$.
(v) O is the saddle point of u.
See the left figure of Figure 2.

Theorem C. *Let u be a second Neumann eigenfunction on T. Suppose that T is a superequilateral triangle. Then $\mu_1(T)$ is simple, u is odd with respect to the x-axis, and $u(P) \neq 0$. Moreover, suppose without loss of generality that $u(O) > 0$. Then the following holds:*
(i) $u_y > 0$ in $\overline{T}\backslash\{P, Q, R\}$, $u_x < 0$ in $\overline{T_+}\backslash(\{y = 0\} \cup \{x = 0\})$, and $u_x > 0$ in $\overline{T_-}\backslash(\{y = 0\} \cup \{x = 0\})$.
(ii) u has exactly three critical points P, Q, and R in $\overline{T}$.
(iii) P and Q are the maximum and minimum points of u, respectively.
(iv) R is neither local maximum nor local minimum point.
See the right figure of Figure 2.

Bañuelos-Burdzy [2] showed that if one of the angles of a triangle is greater than $\pi/2$, then the maximum and minimum points are located at most distinct vertices. Theorem C (iii) is partially included in [2]. However, they did not study the case where every angle is smaller than or equal to $\pi/2$.

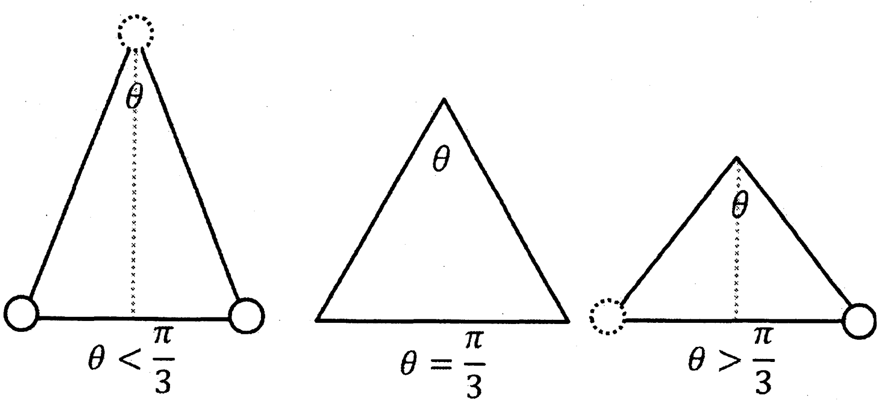

FIGURE 2. Circles stand for the hot spots and dashed circles stand for the cold spots. The second eigenvalue is double in the case of the equilateral triage.

When the domain is a disk, sector, rectangle, or special triangle, the second Neumann eigenfunctions can be written in terms of the Bessel, sine, and cosine functions. Hence, the detailed analysis of the shape can be done. In particular, when the domain is an equilateral triangle, the second eigenvalue is double and there is an eigenfunction, which is u_2 in (1.1) below, having two maximum points on the boundary. Theorem B tells us that a subequilateral triangle also has the second eigenfunction with two local maximum points on the boundary. It seems that a subequilateral triangle is the first example having a second Neumann eigenfunction with two maximum points on the boundary except for an equilateral triangle.

When the domain is an equilateral triangle, the second eigenvalue is double and Lamé derived an exact expression of the eigenfunctions. Let us consider the equilateral triangle with vertices at $(0,0)$, $(1,0)$, and $(1/2, \sqrt{3}/2)$. Then the two second eigenfunctions are

(1.1)

$$u_1(x,y) = 2\left\{\cos\left(\frac{\pi}{3}(2x-1)\right) + \cos\left(\frac{2\pi y}{\sqrt{3}}\right)\right\}\sin\left(\frac{\pi}{3}(2x-1)\right),$$

$$u_2(x,y) = \cos\left(\frac{2\pi}{3}(2x-1)\right) - 2\cos\left(\frac{\pi}{3}(2x-1)\right)\cos\left(\frac{2\pi y}{\sqrt{3}}\right).$$

We see by direct calculation that for $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}\backslash\{(0,0)\}$, $\alpha u_1 + \beta u_2$ does not have an interior critical point. Combining this fact and Theorems B and C, we have

Corollary 1.3. *None of the second Neumann eigenfunctions on an arbitrary isosceles triangle have an interior critical point.*

2. Background

Let us explain background of studying the number of the maximum points on the boundary. One of the motivations is the following "hot spots" conjecture of Rauch:

Conjecture 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then every second Neumann eigenfunction on Ω attains its maximum only on the boundary.*

This conjecture does not hold in this level of generality. There are counter-examples which are planar domains with hole(s). Kawohl added the convexity assumption of the domain, and the "hot spots" conjecture now means Conjecture 2.1 with the convex domains. Bañuelos-Burdzy [2] and Jerison-Nadirashivili [5] proved the conjecture for planar convex domains with two axes of symmetry. (An additional technical assumption is imposed in [2].) Pascu [11] proved the conjecture if the domain is a planar convex one with one axis of symmetry and if the second Neumann eigenfunction is odd with respect to the axis. For positive answers for certain classes of planar domains without symmetry, see [1, 9]. Conjecture 2.1 is believed to be true for a general convex domain. However, it remains open even for a general triangle. Corollary 1.3 is the positive answer for the isosceles triangles.

Yanagida posed the following nonlinear "hot spots" conjecture and pointed out that Conjecture 2.1 is a special case of Conjecture 2.2 below:

Conjecture 2.2 ([13]). *Let Ω be a bounded convex domain, and let f be a smooth function. If a non-constant solution u of the Neumann problem*

$$\Delta u + f(u) = 0 \quad in \ \Omega, \qquad \partial_\nu u = 0 \quad on \ \partial\Omega \tag{2.1}$$

has an interior critical point, then the second eigenvalue of the eigenvalue problem

$$\Delta\phi + f'(u)\phi = -\mu\phi \quad in \ \Omega, \qquad \partial_\nu \phi = 0 \quad on \ \partial\Omega$$

is negative.

When $f(u) = \mu_1(\Omega)u$, Conjecture 2.2 indicates that the second Neumann eigenfunction does not have an interior critical point, hence Conjecture 2.1 immediately follows. Conjecture 2.2 holds for a disk [7] and a rectangle [8]. A slightly weak statement of Conjecture 2.2 was proved in [4] for the domain $I \times D \subset \mathbb{R} \times \mathbb{R}^N$, where I is an interval and D is an arbitrary domain. Conjecture 2.2 remains also open for a general convex domain.

Even if Conjecture 2.1 were proved, the information of the shape of the eigenfunction on the boundary cannot be obtained, hence our problem is different from Conjecture 2.1.

3. Nonlinear hot spots conjecture for the interval

In general Conjecture 2.2 is difficult to prove. However, In the case of the interval it is not difficult to prove the conjecture. In this section we prove Conjecture 2.2 for the interval.

Theorem 3.1. *Let $I(\subset \mathbb{R})$ be a (connected) interval, and let f be a smooth function. If the non-constant solution u of the Neumann problem*

$$u_{xx} + f(u) = 0 \ \ in \ \ I, \quad u_x = 0 \ \ at \ \ \partial I \tag{3.1}$$

has an interior critical point, then the second eigenfunction of the eigenvalue problem

$$\phi_{xx} + f'(u)\phi = -\mu\phi \ \ in \ \ I, \quad u_x = 0 \ \ at \ \ \partial I \tag{3.2}$$

is negative.

Proof. We assume that u has a critical point. Since u is not a constant solution, u_x has at least one simple zero inside I. Moreover, $\{u_x > 0\} \neq \emptyset$ and $\{u_x < 0\} \neq \emptyset$. We define $v_1(x)$ and $v_2(x)$ by

$$v_1(x) := \begin{cases} u_x(x) & \text{on } \{u_x(x) > 0\}, \\ 0 & \text{on } \{u_x(x) \leq 0\}, \end{cases}$$

and

$$v_2(x) := \begin{cases} -u_x(x) & \text{on } \{u_x(x) < 0\}, \\ 0 & \text{on } \{u_x(x) \geq 0\}, \end{cases}$$

respectively. We define $z(x)$ by

$$z(x) := v_1(x) - cv_2(x).$$

Let $\phi_1(x)(> 0)$ be the first eigenfunction of (3.2). Then there is $c > 0$ such that

$$\int_I \phi_1(x)z(x)dx = 0,$$

since $\int_I \phi_1(x)v_2(x)dx \neq 0$. We define

$$\mathcal{H}[\psi] := \int_I \left(\psi_x^2 - f'(u)\psi^2\right) dx.$$

We have

$$\begin{aligned}
\mathcal{H}[z] &= \int_I \left(z^2 - f'(u)z^2\right) dx \\
&= \int_I \left((v_1)_x^2 - f'(u)v_1^2\right) dx + c^2 \int_I \left((v_2)_x^2 - f'(u)v_2^2\right) dx \\
&= [(v_1)_z v_1]_{\partial I} - \int_I \left((v_1)_{xx} + f'(u)v_1\right) v_1 dx \\
&\quad + [(v_2)_z v_2]_{\partial I} - \int_I \left((v_2)_{xx} + f'(u)v_2\right) v_2 dx \\
&= 0,
\end{aligned}$$

where we use $(v_1)_x = (v_2)_x = 0$ at ∂I and $(v_j)_{xx} + f'(u)v_j = 0$ for $j = 1, 2$. By a variational characterization of the second eigenvalue μ_1 we have

$$\begin{aligned}
\mu_1 &= \inf_{\psi \in H^1(I)\setminus\{0\}, \int_I \phi_1 \psi dx = 0} \frac{\mathcal{H}[\psi]}{\|\psi\|_2^2} \\
&\leq \frac{\mathcal{H}[z]}{\|z\|_2^2} = 0,
\end{aligned}$$

where $\|\cdot\|_2$ denotes the L^2-norm. We prove that $\mu_1 \neq 0$. Suppose the contrary, i.e., $\mu_1 = 0$. Then, z is the second eigenfunction. Therefore, z satisfies the Neumann boundary conditions, i.e., $z_x = 0$ at ∂I. Hence, $z_x = 0$ at ∂I. On the other hand, $z = 0$ at ∂I. Since z satisfies the ODE $z_{xx} + f'(u)z = 0$, we see by the uniqueness of the solution of the ODE that $z \equiv 0$ in I. We obtain a contradiction, because z should be a (non-zero) eigenfunction. Hence $\mu_1 < 0$. □

The proof of Conjecture 2.2 for the domain $I \times D$ is similar to that of Theorem 3.1. Let $\Omega = I \times D$. Let $u(x, y_1, y_2, \cdots, y_N)$ be a non-constant solution of (2.1). Then $v := u_x$ satisfies

$$\partial_\nu v = 0 \quad \text{or} \quad v = 0$$

at each point on $\partial\Omega$. Therefore,

$$\begin{aligned}
\mathcal{H}[v] &= \int_\Omega \left(|\nabla v|^2 - f'(u)v^2\right) dx dy_1 \cdots dy_N \\
&= \int_{\partial\Omega} v \partial_\nu v d\sigma - \int_\Omega \left(\Delta v + f'(u)v\right) v dx dy_1 \cdots dy_N \\
&= 0.
\end{aligned}$$

Using this equality, we can similarly prove Conjecture 2.2 for this case.

In the case of a general domain $v\partial_\nu v$ is not necessarily zero. Hence, we cannot determine the sign of the term $\int_{\partial\Omega} v\partial_\nu v d\sigma$. This is a point.

The following corollary immediately follows from Theorem 3.1:

Corollary 3.2. *Let u be a non-constant solution of (3.1). If the second eigenvalue is negative, then the maximum and minimum points are on the boundary ∂I and u has no interior critical point.*

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