<table>
<thead>
<tr>
<th>Title</th>
<th>Construction of a single-peak solution of the Liouville-Gel'fand equation on a two-dimensional domain with a hole (Geometry of solutions of partial differential equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kan, Toru</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2013-09-1850: 48-57</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195127">http://hdl.handle.net/2433/195127</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
</tr>
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Construction of a single-peak solution of the Liouville-Gel'fand equation on a two-dimensional domain with a hole

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1 Introduction

We are concerned with the Liouville-Gel'fand equation

$$\begin{cases}
\Delta u + \lambda e^u = 0 & \text{in } \Omega_{\epsilon}, \\
u = 0 & \text{on } \partial \Omega_{\epsilon}.
\end{cases} \quad (LG)$$

Here $\lambda > 0$ is a parameter and $\Omega_{\epsilon} \subset \mathbb{R}^2$ is a planar domain with a hole whose size is $\epsilon > 0$. The precise definition of $\Omega_{\epsilon}$ will be introduced later. What we discuss in this article is construction of a solution of (LG) caused by a hole in $\Omega_{\epsilon}$.

The equation (LG) has an interesting solution structure when a domain is non-simply connected. The case where $\Omega_{\epsilon}$ is an annulus was investigated by S.-S. Lin [7] and Nagasaki and Suzuki [8]. They independently showed that radially symmetric solutions make a branch and it emanates from $(\lambda, u) = (0, 0)$, bends back once and blows up at each point in $\Omega_{\epsilon}$ as $\lambda \downarrow 0$. Moreover, S.-S. Lin found that the branch has infinitely many secondary bifurcation points from which non-radially symmetric solutions emanate. Nagasaki and Suzuki also obtained non-radially symmetric solutions which have rotational symmetry of order $k$ ($k \in \mathbb{N}$) and is large in some sense. Additionally, Dancer [2] showed that the set of bifurcating non-radially symmetric solutions are unbounded in the bifurcation diagram. These results indicate that bifurcating non-radially symmetric solutions connect to the large solutions obtained by Nagasaki and Suzuki. In [5, 6], suggestive evidence of this expectation was given provided that the inside radius of $\Omega_{\epsilon}$ is small.

For a general non-simply connected domain, Chen and C.-C. Lin [1] revealed the existence of a solution whose mass is not equal to $8\pi k$ ($k \in \mathbb{N}$). Furthermore, del Pino, Kowalczyk and Musso [3] proved that for each $k \in \mathbb{N}$, (LG) has a solution blowing up at $k$ different points as $\lambda \to 0$.

Our motivation is to obtain more detailed information on the solution structure for general non-simply connected domains by extending the results in [5, 6]. What we consider in particular is a solution with one maximum point. In this article, only by a formal argument, we explain how such a solution can be constructed.
2 Construction of a formal solution

We begin with the definition of the domain $\Omega_{\epsilon}$. Let $\Omega$ and $D \subset \mathbb{R}^2$ be bounded domains including the origin. Then, for small $\epsilon > 0$, we define $\Omega_{\epsilon}$ by

$$\Omega_{\epsilon} := \Omega \setminus (\epsilon D) = \{x \in \Omega ; \epsilon^{-1}x \notin D\}.$$ 

The following figure is an example of $\Omega_{\epsilon}$.

![Figure: Domain $\Omega_{\epsilon}$](image)

As will be seen below, an important factor to construct a formal solution is the regular part of a Green's function for Dirichlet Laplacian in $\Omega$. We denote it by $H^{\Omega} = H^{\Omega}(x, y)$. Then, through this section, we assume that

$$\nabla_x H^{\Omega}(0,0) \neq 0. \tag{2.1}$$

This assumption leads to success of argument.

In what follows, we find a formal expansion of a solution $(\lambda, u) = (\lambda_{\epsilon}, u_{\epsilon})$ by using the method of matched asymptotic expansions. To do this we separate $\Omega_{\epsilon}$ into three parts. Two of them are regions near the boundary ($|x| \sim 1$ and $|x| \sim \epsilon$) and the other is a region between them. The latter region is supposed to be $|x| \sim \delta_{\epsilon}$, where $\delta_{\epsilon}$ has a property $\epsilon \ll \delta_{\epsilon} \ll 1 \ (\epsilon \rightarrow 0)$ and is determined later. To obtain the expansion in this region, it is convenient to perform the change of variables $x = \delta_{\epsilon} y$ and $v_{\epsilon}(y) = u_{\epsilon}(x) + \log(\delta_{\epsilon}^{2}\lambda_{\epsilon})$. Then we see that $v_{\epsilon}$ satisfies

$$\Delta v_{\epsilon} + e^{v_{\epsilon}} = 0 \quad \text{in} \quad (\delta_{\epsilon}^{-1}\Omega) \setminus (\epsilon\delta_{\epsilon}^{-1}D).$$

Assuming that $v_{\epsilon}$ can be expanded as $v_{\epsilon}(y) = v_{0}(y) + \delta_{\epsilon} v_{1}(y) + \cdots$, we have

$$\Delta v_{0} + e^{v_{0}} = 0, \quad \Delta v_{1} + e^{v_{0}} v_{1} = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}.$$ 

Since a solution which we find has one peak, it is appropriate to choose $v_{0}$ as

$$v_{0}(y) = \log \frac{8(1 - \rho^2)}{(1 - \rho^2 + |y - \rho\omega|^2)^2},$$
or, in polar coordinates \( y = (r \cos \theta, r \sin \theta) \),
\[
    v_0(y) = \log \frac{8(1-\rho^2)}{r^2 (r + r^{-1} - 2\rho \cos(\theta - \gamma))^2}.
\]  
(2.2)

Here \( \rho \in (0, 1) \) and \( \gamma \in \mathbb{R}/2\pi\mathbb{Z} \) are parameters and \( \omega = (\cos \gamma, \sin \gamma) \). Substituting this into the equation for \( v_1 \), we have
\[
    \Delta v_1 + \frac{8(1-\rho^2)}{r^2 (r + r^{-1} - 2\rho \cos(\theta - \gamma))^2} v_1 = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}.
\]  
(2.3)

To determine \( v_1 \), boundary conditions at the origin and infinity is needed. They are obtained as matching conditions, and therefore we consider the expansion near the boundary. First we treat the region \(|x| \sim 1\). We formally expand \( u_\epsilon(x) = u_0(x) + \delta_\epsilon u_1(x) + \cdots \) as \( \epsilon \to 0 \). Then, for \( j = 0, 1 \), we have
\[
    \begin{cases}
        \Delta u_j = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \\
        u_j = 0 \quad \text{on} \quad \partial \Omega.
    \end{cases}
\]  
(2.4)

Since the maximum principle implies that \( u_\epsilon \) is positive, \( u_0 \) must be nonnegative. Hence \( u_0 \) is given by
\[
    u_0(x) = c_0 G_0^\Omega(x).
\]  
(2.5)

Here \( c_0 \) is a nonnegative constant and \( G_0^\Omega \) is a Green’s function for the Dirichlet Laplacian in \( \Omega \) with a singularity at the origin.

We substitute \( x = \delta_\epsilon^{\frac{1}{2}} \tilde{x} \) in (2.5) and \( y = \delta_\epsilon^{-\frac{1}{2}} \tilde{x} \) in (2.2), and compare the asymptotic behavior as \( \epsilon \to 0 \). As \( \epsilon \to 0 \),
\[
    u_0(\delta_\epsilon^{\frac{1}{2}} \tilde{x}) = c_0 \left( \frac{1}{2\pi} \log \frac{1}{|\delta_\epsilon^{\frac{1}{2}} \tilde{x}|} - H_0^\Omega(\delta_\epsilon^{\frac{1}{2}} \tilde{x}) \right)
\]
\[
    \sim c_0 \left( \frac{1}{4\pi} \log \frac{1}{\delta_\epsilon} + \frac{1}{2\pi} \log \frac{1}{|\tilde{x}|} - H_0^\Omega(0) - \delta_\epsilon^{\frac{1}{2}} \nabla H_0^\Omega(0) \cdot \tilde{x} \right)
\]
\[
    = c_0 \left( \frac{1}{4\pi} \log \frac{1}{\delta_\epsilon} + \frac{1}{2\pi} \log \frac{1}{\tilde{r}} - H_0^\Omega(0) - \delta_\epsilon^{\frac{1}{2}} \mu \cos(\tilde{\theta} - \tau) \right),
\]
\[
    v_0(\delta_\epsilon^{-\frac{1}{2}} \tilde{x}) = \log \frac{8(1-\rho^2)}{(\delta_\epsilon^{-\frac{1}{2}} \tilde{r})^2 ((\delta_\epsilon^{-\frac{1}{2}} \tilde{r})^2 + (\delta_\epsilon^{-\frac{1}{2}} \tilde{r})^{-1} - 2\rho \cos(\tilde{\theta} - \gamma))^2}
\]
\[
    \sim \log \{8(1-\rho^2)\delta_\epsilon^2\} + 4 \log \frac{1}{\tilde{r}} + \delta_\epsilon^{\frac{1}{2}} \frac{4\rho \cos(\tilde{\theta} - \gamma)}{\tilde{r}}
\]
\[
    = \log \{8(1-\rho^2)\delta_\epsilon^2\} + 4 \log \frac{1}{\tilde{r}} + \delta_\epsilon^{\frac{1}{2}} \frac{4\rho \tilde{x} \cdot \tilde{\omega}}{|\tilde{x}|^2},
\]

where \( H_0^\Omega(x) = H^\Omega(x, 0), \nabla H_0^\Omega(0) = (\mu \cos \tau, \mu \sin \tau), \tilde{x} = (\tilde{r} \cos \tilde{\theta}, \tilde{r} \sin \tilde{\theta}) \) and \( \tilde{\omega} = (\cos \gamma, \sin \gamma) \). By matching two expansions log 1/(\delta_\epsilon^2\lambda_\epsilon) + v_0(\tilde{x}) + \delta_\epsilon v_1(\tilde{x}) + \cdots and u_0(x) +
\[ \delta_{\epsilon}u_{1}(x) + \cdots \text{ in the region } |x| \sim \delta_{\epsilon}^{\frac{1}{2}}, \text{ we have } c_{0} = 8\pi \text{ and} \]

\[ u_{1}(x) = 4\rho \frac{x \cdot \tilde{\omega}}{|x|^{2}} + o \left( \frac{1}{|x|} \right) \text{ as } x \to 0. \]  

(2.6)

(2.4) and (2.6) give

\[ u_{1}(x) = 4\rho \left( \frac{x \cdot \tilde{\omega}}{|x|^{2}} - 2\pi \nabla_{y}H(x, 0) \cdot \tilde{\omega} \right) + c_{1}G_{\Omega}^{0}(x), \]

where \( c_{1} \in \mathbb{R} \) is an undetermined constant. From this,

\[ u_{1}(\delta_{\epsilon}^{\frac{1}{2}}\tilde{x}) \sim 4\rho \left( \delta_{\epsilon}^{-\frac{1}{2}} \frac{\tilde{x} \cdot \omega}{|\tilde{x}|^{2}} - 2\pi \mu \omega \cdot N \right) + c_{1}(\frac{1}{2\pi} \log \frac{1}{|\delta_{\epsilon}^{\frac{1}{2}}\tilde{x}|} - H_{0}^{\Omega}(0)). \]

Thus it is appropriate to impose the condition

\[ v_{1}(y) = -c_{0} \mu \rho \cos(\theta - \tau) + \frac{c_{1}}{2\pi} \log \frac{1}{r} + a_{1} + o(1) \text{ as } r \to \infty. \]  

(2.7)

Here \( a_{1} \) is a constant determined later. Moreover,

\[ \frac{c_{0}}{4\pi} \log \frac{1}{\delta_{\epsilon}} - c_{0}H_{0}^{\Omega}(0) + \frac{c_{1}}{2\pi} \delta_{\epsilon} \log \frac{1}{\delta_{\epsilon}} - \delta_{\epsilon} \left( 8\pi \rho \mu \cos(\gamma - \tau) + c_{1}H_{0}^{\Omega}(0) \right) \]

\[ = \log \frac{1}{\delta_{\epsilon}^{2} \lambda_{\epsilon}} + \log \{8(1 - \rho^{2})\delta_{\epsilon}^{2} \} - \delta_{\epsilon} a_{1}, \]

which gives

\[ \lambda_{\epsilon} = 8(1 - \rho^{2})\delta_{\epsilon}^{2} \exp \left[ c_{0}H_{0}^{\Omega}(0) + \frac{c_{1}}{2\pi} \delta_{\epsilon} \log \frac{1}{\delta_{\epsilon}} \right. \]

\[ \left. + \delta_{\epsilon} \left\{ a_{1} - 8\pi \rho \mu \cos(\gamma - \tau) - c_{1}H_{0}^{\Omega}(0) \right\} \right]. \]  

(2.8)

Next we consider the expansion in \(|x| \sim \epsilon\). Performing the change of variables \( x = \epsilon z \) and putting \( w_{\epsilon}(z) = u_{\epsilon}(x) \), we have

\[ \begin{cases} \Delta w_{\epsilon} + \epsilon^{2} \lambda_{\epsilon} e^{w_{\epsilon}} = 0 & \text{in } \epsilon^{-1}\Omega_{\epsilon} = (\epsilon^{-1}\Omega) \setminus \overline{D}, \\ w_{\epsilon} = 0 & \text{on } \partial(\epsilon^{-1}\Omega_{\epsilon}). \end{cases} \]

Hence the formal expansion \( w_{\epsilon}(z) = w_{0}(z) + \delta_{\epsilon}w_{1}(z) + \cdots \) gives

\[ \begin{cases} \Delta w_{j} = 0 & \text{in } \mathbb{R}^{2} \setminus D, \\ w_{j} = 0 & \text{on } \partial D. \end{cases} \]
for $j = 0,1$. To find a solution of the above equation, we perform the Kelvin transform

$$w_j^*(z^*) = w_j(z), \quad z^* = \frac{z}{|z|^2}.$$ 

Then $w_j^*$ satisfies

$$\begin{cases} \Delta w_j^* = 0 \quad \text{in} \quad D^* \setminus \{0\}, \\ w_j^* = 0 \quad \text{on} \quad \partial D^*, \end{cases}$$

where $D^* := \{z^* = z/|z|^2; z \in D\}$. Since $w_0^*$ is nonnegative, we see that $w_0^*(z^*) = d_0 G_0^{D^*}(z^*)$ for some constant $d_0 \geq 0$. Thus

$$w_0(z) = d_0 G_0^{D^*}(z^*) = d_0 G_0^{D^*}(z/|z|^2).$$

If $d_0 > 0$, this function has logarithmic growth at $z = \infty$, while $v_0$ has no such a singularity at $y = 0$. This implies that $d_0 = 0$, that is, $w_0 \equiv 0$. Since $w_1$ satisfies the same equation as $w_0$ and must be nonnegative, we have

$$w_1(z) = d_1 G_0^{D^*}(z^*) = d_1 G_0^{D^*}(z/|z|^2), \quad (2.9)$$

where $d_1 \geq 0$ is some undetermined constant.

We compare the expansions in the region $|x| \sim \epsilon^{1/2} \delta^{1/2} \hat{x}$. By putting $z = \epsilon^{-1/2} \delta^{1/2} \hat{x}$ in (2.9) and $y = \epsilon^{1/2} \delta^{-1/2} \hat{x}$ in (2.2), we have

$$w_1(\epsilon^{-1/2} \delta^{1/2} \hat{x}) = d_1 \left( \frac{1}{2\pi} \log |\epsilon^{-1/2} \delta^{1/2} \hat{x}| - H_0^{D^*} \left( \epsilon^{1/2} \delta^{1/2} \frac{\hat{x}}{|\hat{x}|^2} \right) \right)$$

$$\sim d_1 \left( \frac{1}{2\pi} \log(\epsilon^{-1} \delta_e) + \frac{1}{2\pi} \log |\epsilon^{1/2} \delta_e^{-1/2} \hat{x}| - H_0^{D^*}(0) \right),$$

$$v_0(\epsilon^{1/2} \delta_e^{-1/2} \hat{x}) = \log \frac{8(1 - \rho^2)}{(\epsilon^{1/2} \delta_e^{-1/2} \hat{r})^2 \left( (\epsilon^{1/2} \delta_e^{-1/2} \hat{r}) + (\epsilon^{1/2} \delta_e^{-1/2} \hat{r})^{-1} - 2\rho \cos(\hat{\theta} - \gamma) \right)^2}$$

$$\sim \log \{8(1 - \rho^2)\}.$$

Thus, assuming that two expansions $\log 1/(\delta^2 \lambda_e) + v_0(y) + \delta_e v_1(y) + \cdots$ and $w_0(z) + \delta_e w_1(z) + \cdots$ give the same expansion in $|x| \sim \epsilon^{1/2} \delta^{1/2}$, we deduce

$$v_1(y) = \frac{d_1}{2\pi} \log r + a_2 + o(1) \quad \text{as} \quad r \to 0 \quad (2.10)$$

and

$$\log \frac{8(1 - \rho^2)}{\delta^2 \lambda_e} - \delta_e a_2 = d_1 \delta_e \left( \frac{1}{2\pi} \log(\epsilon^{-1} \delta_e) - H_0^{D^*}(0) \right). \quad (2.11)$$

Here $a_2 \in \mathbb{R}$ is a constant determined later.
Now we solve (2.3) under the conditions (2.7) and (2.10). First we observe that the functions
\[
\Phi_{\rho,\gamma,1}(z) = \frac{r - r^{-1}}{r + r^{-1} - 2\rho \cos(\theta - \gamma)},
\]
\[
\Phi_{\rho,\gamma,2}(z) = \frac{2 \cos(\theta - \gamma) - \rho (r + r^{-1})}{r + r^{-1} - 2\rho \cos(\theta - \gamma)},
\]
\[
\Phi_{\rho,\gamma,3}(z) = \frac{\sin(\theta - \gamma)}{r + r^{-1} - 2\rho \cos(\theta - \gamma)}
\]
are bounded solutions of (2.3). Furthermore, every bounded solution of (2.3) is linear combination of these solutions (see [4], [5]). We also observe what is necessary to solve the equation. Suppose that (2.3), (2.10) and (2.7) has a solution. By a simple computation, we have
\[
\Phi_{\rho,\gamma,j}(z) = \begin{cases} 
1 + 2\rho r^{-1} \cos(\theta - \gamma) + O(r^{-2}) & (j = 1) \\
- \rho \{1 - 2(\rho^{-1} - \rho)r^{-1} \cos(\theta - \gamma)\} + O(r^{-2}) & (j = 2) \\
- \rho \cos(\theta - \gamma) + O(r^{-2}) & (j = 3)
\end{cases} \quad \text{as } r \to \infty,
\]
\[
\Phi_{\rho,\gamma,j}(z) = \begin{cases} 
- 1 + 2\rho r \cos(\theta - \gamma) + O(r^2) & (j = 1) \\
- \rho \{1 - 2(\rho^{-1} - \rho)r \cos(\theta - \gamma)\} + O(r^2) & (j = 2) \\
r \cos(\theta - \gamma) + O(r^2) & (j = 3)
\end{cases} \quad \text{as } r \to 0.
\]
Hence, as \( r \to \infty, \)
\[
r \left( \frac{\partial w_1}{\partial r} \Phi_{\rho,\gamma,j} - w_1 \frac{\partial \Phi_{\rho,\gamma,j}}{\partial r} \right) = \begin{cases} 
- c_0 \mu r \cos(\theta - \tau) - \frac{c_1}{2\pi} - 4c_0 \rho \mu \cos(\theta - \gamma) \cos(\theta - \gamma) + o(1) & (j = 1) \\
- \rho \{- c_0 \mu r \cos(\theta - \tau) - \frac{c_1}{2\pi} + 4c_0 (\rho^{-1} - \rho) \mu \cos(\theta - \tau) \cos(\theta - \gamma)\} + o(1) & (j = 2) \\
- 2c_0 \mu \cos(\theta - \tau) \sin(\theta - \gamma) + o(1) & (j = 3)
\end{cases}
\]
and as \( r \to 0, \)
\[
r \left( \frac{\partial w_1}{\partial r} \Phi_{\rho,\gamma,j} - w_1 \frac{\partial \Phi_{\rho,\gamma,j}}{\partial r} \right) = \begin{cases} 
- d_1/(2\pi) + o(1) & (j = 1) \\
- (\rho d_1)/(2\pi) + o(1) & (j = 2) \\
o(1) & (j = 3)
\end{cases}.
\]
Thus multiplying both sides of (2.3) by \( \Phi_{\rho,\gamma,j} \) and integrating give
\[
0 = \left[ \int_0^{2\pi} r \left( \frac{\partial w_1}{\partial r} \Phi_{\rho,\gamma,j} - w_1 \frac{\partial \Phi_{\rho,\gamma,j}}{\partial r} \right) d\theta \right]_{r=0}^\infty.
\]
\[\begin{align*}
-c_1 - 4\pi c_0 \rho \mu \cos(\gamma - \tau) + d_1 & \quad (j = 1) \\
-\rho \{-c_1 + 4\pi c_0 (\rho^{-1} - \rho) \mu \cos(\gamma - \tau) - d_1\} & \quad (j = 2) \\
2\pi c_0 \mu \sin(\gamma - \tau) & \quad (j = 3)
\end{align*}\]

Note that $\mu > 0$ from (2.1) and $d_1 \geq 0$. Therefore the above relations yield

\[\begin{align*}
\gamma &= \tau, \\
c_1 &= 2\pi c_0 \mu \left(\frac{1}{\rho} - 2\rho\right) = 16\pi^2 \mu \left(\frac{1}{\rho} - 2\rho\right), \\
d_1 &= \frac{2\pi c_0 \mu}{\rho} = \frac{16\pi^2 \mu}{\rho}.
\end{align*}\]

Conversely, it can be checked that the function

\[V(y) = -c_0 \mu \left\{\left(\frac{1}{\rho} - \rho\right) \Phi_{\rho, \tau, 1}(y) \log r + \Phi_{\rho, \tau, 1}(y) \log r - \frac{1}{\rho} + r \cos(\theta - \tau)\right\}\]

is a solution of (2.3), (2.10), (2.7) provided that $\gamma, c_1$ and $d_1$ satisfy the above relations. Thus, by setting $a_2 = a_3 = (c_0 \mu)/\rho$, we see that $v_1$ is given by

\[v_1(y) = V(y) + \alpha \Phi_{\rho, \tau, 3}(y),\]

where $\alpha \in \mathbb{R}$ is an arbitrary constant.

From (2.8) and (2.11), it can be shown that

\[\delta_\varepsilon = \frac{\rho}{2\pi \mu} \frac{\log \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}} (1 + o(1))\]

as $\varepsilon \to 0$. Hence setting $\eta_\varepsilon = 2\pi \mu \delta_\varepsilon / \rho$, we have

\[\eta_\varepsilon = \frac{\log \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}} (1 + o(1)),\]

\[\lambda_\varepsilon = \frac{4\rho^2 (1 - \rho^2) e^{8\pi H_0^\Omega(0)}}{\mu \pi} \eta_\varepsilon^2 (1 + o(1)).\]

This indicates that $u_\varepsilon$ appears through a saddle-node bifurcation when $\rho \sim 1/\sqrt{2}$.

Finally we discuss how the constant $\alpha$ is determined. From the formal expansion obtained above, the solution $u_\varepsilon$ is expected to expand as

\[u_\varepsilon(x) = \log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} + v_0(y) + \delta_\varepsilon V(y) + \alpha \delta_\varepsilon \Phi_{\rho, \tau, 3}(y) + (h.o.t.)\]

provided that $|y| \sim 1$. This expansion is valid only in the region $|y| \sim 1$, and therefore we add a correction term to obtain an approximation in the whole region of $\Omega_\varepsilon$. We define a
correction function $v_c$ as a solution of
\[
\begin{dcases}
\Delta v_c = 0 & \text{in } \delta^{-1}_e \Omega_e, \\
v_c = -\log \frac{1}{\delta^2_e \lambda_e} - v_0 - \delta_e V & \text{on } \partial(\delta^{-1}_e \Omega_e).
\end{dcases}
\]

Then one can show that
\[|v_c(y)| \leq C (\varepsilon^{-1} + \delta^2_e r^2)\]
for all $y \in \delta^{-1}_e \Omega_e$, and
\[v_c(y) = \delta^2_e \xi(y) + o(\delta^2_e)\]
locally uniformly for $y \in \mathbb{R} \setminus \{0\}$ as $\varepsilon \to 0$. Here $C > 0$ is a constant independent of $\varepsilon$ and $\xi$ is a function determined by the regular part of a Green’s function in $\Omega$ (we omit the detail of $\xi$). Consequently, we obtain the expansion
\[u_\varepsilon(x) = \log \frac{1}{\delta^2_e \lambda_e} + U_\varepsilon(y) + \alpha \delta_e \Phi_{\rho,\tau,3}(y) + r_\varepsilon(y),\]
where $U_\varepsilon = v_0 + \delta_e V + v_c$ and $r_\varepsilon$ is a remainder term. $r_\varepsilon$ is expected to be small on whole domain $\Omega_\varepsilon$ in some appropriate topology.

We set $\eta_\varepsilon(y) = \alpha \delta_e \Phi_{\rho,\tau,3}(y) + r_\varepsilon(y)$ and substitute the above expansion into (LG). Then the equation is rewritten as
\[\mathcal{L}(\eta_\varepsilon) + F(\eta_\varepsilon) + R_\varepsilon = 0,\]
where
\[\mathcal{L}(\eta_\varepsilon) = \Delta \eta_\varepsilon + e^{U_\varepsilon} \eta_\varepsilon,\]
\[F(\eta_\varepsilon) = e^{U_\varepsilon}(e^{\eta_\varepsilon} - 1 - \eta_\varepsilon),\]
\[R_\varepsilon = \Delta U_\varepsilon + e^{U_\varepsilon}.\]

To determine the constant $\alpha$, we multiply the above equation by $\Phi_{\rho,\tau,3}$ and integrate over $\delta^{-1}_e \Omega_e$. Then we have
\[
\int_{\delta^{-1}_e \Omega_e} \mathcal{L}(\eta_\varepsilon) \Phi_{\rho,\tau,3} dx \sim \int_{\delta^{-1}_e \Omega_e} \eta_\varepsilon \mathcal{L}(\Phi_{\rho,\tau,3}) dx
\]
\[
\sim \alpha \delta_e \int_{\delta^{-1}_e \Omega_e} (e^{U_\varepsilon} - e^{v_0}) \Phi_{\rho,\tau,3}^2 dx
\]
\[
\sim \alpha \delta^2_e \int_{\mathbb{R}^2} e^{v_0} V \Phi_{\rho,\tau,3}^2 dx
\]
\[
= \frac{4\pi^2 \mu}{\rho} \alpha \delta^2_e,
\]
\[
\int_{\delta^{-1}_e \Omega_e} F(\eta_\varepsilon) \Phi_{\rho,\tau,3} dx \sim \int_{\delta^{-1}_e \Omega_e} e^{U_\varepsilon} \eta^2_\varepsilon \Phi_{\rho,\tau,3} dx
\]
\[ \sim \alpha^2 \delta^2 \int_{\mathbb{R}^2} e^{v_0} \Phi_{\rho,\tau,3}^3 dx = 0. \]

From the definition of \( U_\epsilon \), we see that
\[ R_\epsilon = e^{v_0} (e^{\delta_\epsilon V + v_c} - 1 - \delta_\epsilon V). \]
Hence
\[
\int_{\delta_\epsilon^{-1} \Omega_\epsilon} R_\epsilon \Phi_{\rho,\tau,3} dx \sim \int_{\delta_\epsilon^{-1} \Omega_\epsilon} e^{v_0} \{v_c + (\delta_\epsilon V + v_c)^2\} \Phi_{\rho,\tau,3} dx
\sim \delta_\epsilon^2 \int_{\mathbb{R}^2} e^{v_0} (\xi + V^2) \Phi_{\rho,\tau,3} dx
= \delta_\epsilon^2 \int_{\mathbb{R}^2} e^{v_0} \xi \Phi_{\rho,\tau,3} dx.
\]
Thus \( \alpha \) is given by
\[ \alpha = \frac{\rho}{4\pi^2 \mu} \int_{\mathbb{R}^2} e^{v_0} \xi \Phi_{\rho,\tau,3} dx. \]

At the end, we summarize what we obtained.

**Main Result 1.** Assume (2.1). Then, for small \( \epsilon \) and \( \rho \in (0, 1) \), we can construct a "formal" solution \((\lambda_\epsilon, u_\epsilon)\) of (LG) with the following expansion:

\[ \lambda_\epsilon \sim \frac{4\rho^2 (1 - \rho^2) e^{8\pi H_{0}(0)}}{\mu \pi} \left( \log \frac{1}{\log \frac{1}{\epsilon}} \right)^2, \quad \text{as} \quad \epsilon \to 0. \]

\[ u_\epsilon(x) \sim \log \frac{1}{\delta_\epsilon^2 \lambda_\epsilon} + v_0(\delta_\epsilon^{-1} x) + \delta_\epsilon v_1(\delta_\epsilon^{-1} x) + v_c(\delta_\epsilon^{-1} x). \]

*Here constants and functions are chosen suitably as discussed above.*

**References**


