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Stability of hypersurfaces with constant mean curvature and applications to isoperimetric problems

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1 Introduction

A surface with constant mean curvature (CMC surface) is an equilibrium surface of the area functional among surfaces which enclose the same volume with given boundary condition. A CMC surface is said to be stable if the second variation of the area is nonnegative for all volume-preserving variations which satisfy the boundary condition. In general, it is not easy to judge whether a given CMC surface is stable or not. In this article, we first give some known criteria for the stability (§2). They are given by the properties of the eigenvalues and the eigenfunctions of the eigenvalue problems associated with the second variation of the area. Moreover, a new criterion for the stability is given by using bifurcations (§5). By choosing the volume V or the mean curvature H as parameter, we can construct existence theorems of bifurcation of CMC surfaces satisfying the given boundary condition (§3), and give some results to estimate the eigenvalues (§4).

In this article, for simplicity, we discuss CMC surfaces with fixed boundary. Our methods can be applied to more general variational problems, for example, to CMC hypersurfaces in $(n+1)$-dimensional Riemannian manifolds, and to hypersurfaces with constant anisotropic mean curvature in $\mathbf{R}^{n+1}$. Also, they can be applied to problems with other boundary conditions, for example, to CMC surfaces with free boundary on the union of a finite number of smooth surfaces (§6).

This article includes an announcement of a recent joint work with Bennett Palmer and Paolo Piccione. The proofs of the main results which will be mentioned in §3, §4, and §5 will be given in a forthcoming paper [6].

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2 Stability and two associate eigenvalue problems

Let $X : \tilde{\Sigma} \to \mathbf{R}^3$ be a $C^{3+\alpha}$ $(0 < \alpha < 1)$ immersion from an orientable two-dimensional smooth manifold $\tilde{\Sigma}$ to the three dimensional euclidean space $\mathbf{R}^3$. Denote by $\nu = (\nu_1, \nu_2, \nu_3) : \tilde{\Sigma} \to S^2 \subset \mathbf{R}^3$ the Gauss map of X (that is, a unit normal vector field along X). Let Σ be a compact two-dimensional submanifold of $\tilde{\Sigma}$ with or without boundary. We will denote by X the restriction $X|_\Sigma$. The area $A(X)$ and the volume $V(X)$ of X are defined by

$$A(X) = \int_\Sigma d\Sigma, \qquad V(X) = \frac{1}{3}\int_\Sigma \langle X, \nu\rangle \, d\Sigma,$$

where $d\Sigma$ is the area element of X (in other words, the volume form of Σ induced by X), and $\langle\ ,\ \rangle$ is the standard inner product in $\mathbf{R}^3$. $V(X)$ is the algebraic volume of the cone-like domain constructed by $X(\Sigma)$ and the origine of $\mathbf{R}^3$. If X is an embedding, $X(\Sigma)$ is the boundary of a bounded domain Ω in $\mathbf{R}^3$, and ν is the outward-pointing unit normal, then $V(X)$ is the usual volume of Ω.

If a one-parameter family $X_t : \Sigma \to \mathbf{R}^3$ $(\exists t_0 > 0, -t_0 < t < t_0)$ of immersions is sufficiently differentiable with respect to t, and $X_0 = X$, then we say that X_t is a variation of X. A variation X_t of X is said to be volume-preserving if $V(X_t) = V(X)$ $(\forall t)$, and we say that it fixes the boundary if $X_t|_{\partial\Sigma} = X|_{\partial\Sigma}$ $(\forall t)$ holds. Let X_t be a volume-preserving variation of X which fixes the boundary. Then the first variation $\partial A := \left.\dfrac{\partial A(X_t)}{\partial t}\right|_{t=0}$ of the the area vanishes since X is CMC. The second variation of the the area is

$$\partial^2 A := \left.\frac{\partial^2 A(X_t)}{\partial t^2}\right|_{t=0} = -\int_\Sigma \varphi L[\varphi] \, d\Sigma =: I(\varphi), \quad \varphi := \left\langle \left.\frac{\partial X_t}{\partial t}\right|_{t=0}, \nu \right\rangle,$$

where L is the second order linear elliptic self-adjoint operator which is defined as follows.

$$L[\varphi] = \Delta\varphi + \|d\nu\|^2\varphi, \tag{1}$$

here Δ is the Laplacian on Σ with the metric induced by X. We should note our choice about the sign of Δ. For example, if X is the identity mapping on the plane, say $X(u, v) \equiv (u, v, 0)$, then $\Delta\varphi = \varphi_{uu} + \varphi_{vv}$.

X is said to be stable if $\partial^2 A \geq 0$ for all volume-preserving variations of X which fix the boundary. It can be proved that X is stable if and only if $I(\varphi) \geq 0$ for all $\varphi \in C_0^{3+\alpha}(\Sigma)$ which satisfy $\int_\Sigma \varphi \, d\Sigma = 0$. The identity $\int_\Sigma \varphi \, d\Sigma = 0$ corresponds to the condition that the considered variation is volume-preserving (cf. [4]).

Now set

$$F_0 := \left\{ \varphi \in H_0^1(\Sigma) \,\middle|\, \varphi|_{\partial\Sigma} = 0, \ \int_\Sigma \varphi \, d\Sigma = 0 \right\}.$$

Consider the eigenvalue problem:

$$\tilde{L}[\varphi] := L[\varphi] + c = -\tilde{\lambda}\varphi, \quad \varphi \in F_0 - \{0\}, \tag{2}$$

where c is any real constant. Note that $\tilde{L}$ is self-adjoint in the space F_0. Note that the equation in (2) is equivalent to

$$\int_\Sigma (L[\varphi] + \tilde{\lambda}\varphi)u \, d\Sigma = 0, \quad \forall u \in F_0. \tag{3}$$

Denote by $\tilde{\lambda}_n$ the n'th eigenvalue of (2): $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$.

Lemma 2.1 (Patnaik [7]) *X is stable if and only if all eigenvalues of (2) is non-negative.*

Actually, the number of negative eigenvalues is the dimension of the volume-preserving variation vector fields which diminish the area. However, it is difficult to know the sign of $\tilde{\lambda}_1$ in general. So, we consider the following eigenvalue problem:

$$L[\varphi] = -\lambda\varphi, \qquad \varphi|_{\partial\Sigma} = 0, \qquad \varphi \in H_0^1(\Sigma) - \{0\}. \tag{4}$$

Denote by λ_n the n'th eigenvalue of (4): $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$. Then, one can show the following:

Lemma 2.2 *$\lambda_1 < \tilde{\lambda}_1 \leq \lambda_2$ holds.*

Lemmas 2.1, 2.2 imply that if $\lambda_1 \geq 0$, X is stable, and that if $\lambda_2 < 0$, then X is unstable. More precisely, we have the following criterion for the stability:

Theorem 2.1 ([5]) *Let $X : \Sigma \to \mathbf{R}^3$ be a CMC immersion.*
(I) If $\lambda_1 \geq 0$, then X is stable.
(II) Assume that $\lambda_1 < 0 \leq \lambda_2$ holds.
(II-i) Assume further that there exists a function $\varphi \in C_0^\infty(\Sigma)$ which satisfies $L[\varphi] = 1$. Then, X is stable if and only if $\int_\Sigma \varphi \, d\Sigma \geq 0$ holds.
(II-ii) If there is no such φ as above, then X is unstable.
(III) If $\lambda_2 < 0$, then X is unstable.

We give another criterion for the stability which is essentially equivalent to Theorem 2.1. For a one-parameter family $\{X_t\}$ $(X_0 = X)$ of immersions, we set

$H(t) :=$ the mean curvature of X_t, $\quad V(t) :=$ the volume of X_t.

Theorem 2.2 ([5]) *Let $X : \Sigma \to \mathbf{R}^3$ be a CMC immersion.*
(I) If $\lambda_1 \geq 0$, then X is stable.
(II) Assume that $\lambda_1 < 0 \leq \lambda_2$ holds.
(II-i) Assume further that there exists a variation X_t of X s.t. $H'(0) = \text{constant} \neq 0$, then X is stable if and only if $H'(0)V'(0) \geq 0$ holds.
(II-ii) If there is no such X_t as above, then X is unstable.
(III) If $\lambda_2 < 0$, then X is unstable.

Sometimes we can find zero eigenvalues and eigenfunctions belonging to zero by using geometric properties of X and the considered functionals. In fact, we have the following:

Proposition 2.1 *Let $X : \Sigma \to \mathbf{R}^3$ be an immersion with constant mean curvature H, and let $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \to S^2$ be its Gauss map. Set $E_1 := (1,0,0)$, $E_2 := (0,1,0)$, $E_3 := (0,0,1)$. Then,*

$$L[\nu_j] = 0, \quad j = 1,2,3, \tag{5}$$

$$L[\langle E_j \times X, \nu\rangle] = 0, \quad j = 1,2,3, \tag{6}$$

$$L[\langle X, \nu\rangle] = -2H. \tag{7}$$

Before giving a proof of Proposition 2.1, we give an important result:

Proposition 2.2 *Let $X : \Sigma \to \mathbf{R}^3$ be an immersion. Let $X_t = X + (\xi^i X_i + f\nu)t + \mathcal{O}(t^2)$ be a variation of X. Then the first variation of the mean curvature H is given by the following:*

$$\delta H = L[f]/2 + \xi^i H_i. \tag{8}$$

If H is constant, then $\delta H = L[f]/2$ holds.

Proof of Proposition 2.1 For a constant vector v in $\mathbf{R}^3$, a parallel translation $X_t = X + tv$ does not change the mean curvature H. Hence, by Proposition 2.2, we obtain (5). Similarly, since a rotation $X_t = X + tE_j \times X + \mathcal{O}(t^2)$ does not change H, by Proposition 2.2, we obtain (6). On the other hand, since, by a homothety $X_t = (1+t)X$, H becomes $H/(1+t)$, we obtain (7). □

In the case where $\lambda_1 < 0 \le \lambda_2$, the criteria for the stability for X (Theorems 2.1, 2.2) we gave are not very simple. The results on the bifurcation which we will give in the next section will be sometimes useful to judge the stability in combination with the criteria for the stability in terms of the eigenvalue problems above.

3 Existence of bifurcation

In this section, we will give two sufficient conditions for existence of bifurcations of CMC surfaces with fixed boundary condition.

First we give a result about non-existence of bifurcation.

Theorem 3.1 (Existence and uniqueness of CMC deformation. [5]) *Let $X : \Sigma \to \mathbf{R}^3$ be a CMC immersion. Set $E := \{e \in C_0^{2+\alpha}(\Sigma) \mid L[e] = 0\}$. Assume either the following (i) or (ii) holds.*

(i) $E = \{0\}$. (ii) $\dim E = 1$ and $\int_\Sigma e\, d\Sigma \neq 0$ for all $e \in E - \{0\}$.

Then, in a small neighborhood of X, there exists a unique (up to diffeomorphisms of Σ) one-parameter family $\{X_t\}$ ($X_t : \Sigma \to \mathbf{R}^3$, $X_0 = X$) of CMC immersions with the same boundary values as X.

Therefore, there is no bifurcation in this case. It is well-known that the multiplicity of λ_1 is one and that any eigenfunction belonging to λ_1 does not change sign. Hence, if $\lambda_1 = 0$, then (ii) in Theorem 3.1 is satisfied. Therefore, bifurcation may occur only in the case where $\lambda_k = 0$ for some $k \geq 2$.

On the other hand, we have a criterion for the stability of CMC surfaces. For a one-parameter family $\{X_t : \Sigma \to \mathbf{R}^3\}$ of immersions, denote by $H(t)$ and $V(t)$, the mean curvature and the volume of X_t, respectively. Also denote by Δ_t, ν_t the Laplacian, the Gauss map of X_t, respectively, and let L_t, $\tilde{L}_t$ be the self-adjoint operators associated with the second variation of the area for X_t (see (1), (2)).

Theorem 3.2 (Existence of bifurcation 1. [6]) *Assume we have a one-parameter family $X_t = X + \varphi(t)\nu : \Sigma \to \mathbf{R}^3$, $(t \in I = (-\epsilon, \epsilon) \subset \mathbf{R})$, of CMC $C^{3+\alpha}$ immersions with $X = X_0$ and $X|_{\partial\Sigma} = X_t|_{\partial\Sigma}$, which satisfy the following (i)-(iii).*

(i) X_t is differentiable with respect to t.

(ii) $H'(0) \neq 0$.

(iii) $E = \{ae\,;\, a \in \mathbf{R}\}$, $\quad \exists e \in (C_0^{2+\alpha}(\Sigma) - \{0\})$.

Then, $\int_\Sigma e\, d\Sigma = 0$. And there exists a one-parameter family $\lambda(t)$ $(t \in (-\epsilon_0, \epsilon_0) \subset I)$ of real values such that $\lambda(t)$ is differentiable with respect to t, $\lambda(0) = 0$, $\lambda(t)$ is a simple eigenvalue of L_t, and there is no other eigenvalue of L_t near 0.

Assume further that

(iv) $\lambda'(0) \neq 0$.

Let $E^\perp$ be any complement of E in $C_0^{3+\alpha}$. Then there exist an open interval $\hat{I}$ $(0 \in \hat{I} \subset \mathbf{R})$ and C^1 functions $\zeta : \hat{I} \to E^\perp$ and $t : \hat{I} \to \mathbf{R}$, such that $t(0) = 0$, $\zeta(0) = 0$, and $Y(s) := X + (\varphi(t(s)) + se + s\zeta(s))\nu$ is a CMC immersion with mean curvature $\hat{H}(s) := H(t(s))$. Moreover, in a small neighborhood of X, CMC immersions with the same boundary values as X consists of $\{X_t\,;\, t \in I\}$ and $\{Y(s)\,;\, s \in \hat{I}\}$. Furthermore, surfaces $\{X_t\,;\, t \in I\}$ and $\{Y(s)\,;\, s \in \hat{I}\}$ are all different except for $X_0 = Y(0)$.

Theorem 3.2 is proved by applying a general result on bifurcation by Crandall-Rabinowitz [2]. On the other hand, Patnaik [7] was trying to obtain a similar result to Theorem 3.2, where he used the volume instead of the mean curvature. We can prove the following result by using an idea which was introduced by [7].

Theorem 3.3 (Existence of bifurcation 2. [7], [6]) *Assume we have a one- parameter family $X_t = X + \varphi(t)\nu : \Sigma \to \mathbf{R}^3$, $(t \in I = (-\epsilon, \epsilon) \subset \mathbf{R})$, of CMC $C^{3+\alpha}$ immersions with $X = X_0$ and $X|_{\partial\Sigma} = X_t|_{\partial\Sigma}$, which satisfy the following (i)-(iii).*

(i) X_t is differentiable with respect to t.

(ii) $V'(0) \neq 0$, $H'(0) \neq 0$.

(iii) $E = \{ae\,;\, a \in \mathbf{R}\}$, $\quad \exists e \in (C_0^{2+\alpha}(\Sigma) - \{0\})$.

Then, $\int_\Sigma e\, d\Sigma = 0$, and $\lambda_j = \tilde{\lambda}_k = 0$ for $\exists j \geq 2$ and $\exists k \geq 1$. There exists a one-parameter family $\tilde{\lambda}(t)$ $(t \in (-\epsilon_0, \epsilon_0) \subset I)$ of real values such that $\tilde{\lambda}(t)$ is differentiable with respect to t, $\tilde{\lambda}(0) = 0$, $\tilde{\lambda}(t)$ is a simple eigenvalue of $\tilde{L}_t$, and there is no other eigenvalue of $\tilde{L}_t$ near 0. Assume further that

(iv) $\tilde{\lambda}'(0) \neq 0$.
Let $E^{\perp}$ *be the complement of* E *in* $C_0^{3+\alpha}(\Sigma)$. *Then there exist an open interval* $\hat{I}$ *(*$0 \in \hat{I} \subset \mathbf{R}$*) and* C^1 *mappings* $\eta : \hat{I} \to C_0^{3+\alpha}(\Sigma)$ *and* $\tau : \hat{I} \to \mathbf{R}$*, such that* $\tau(0) = 0$*,* $\eta(0) = 0$*, and* $Y(s) := X + (\varphi(\tau(s)) + se + s\eta(s))\nu$ *is a CMC immersion with volume* $\hat{V}(s) := V(\tau(s))$. $\eta(s)$ *can be written as* $\eta(s) = c(s)\varphi'(0) + \xi(s)$*, where* $c : \hat{I} \to \mathbf{R}$ *and* $\xi : \hat{I} \to \{u \in C_0^{3+\alpha}(\Sigma) \mid \int_\Sigma u\, d\Sigma = 0\} \cap E^{\perp}$ *are* C^1 *mappings such that* $c(0) = 0$*,* $\xi(0) = 0$*. Moreover, in a small neighborhood of* X*, CMC immersions with the same boundary values as* X *consists of* $\{X_t\,;\, t \in I\}$ *and* $\{Y(s)\,;\, s \in \hat{I}\}$*. Furthermore, surfaces* $\{X_t\,;\, t \in I\}$ *and* $\{Y(s)\,;\, s \in \hat{I}\}$ *are all different except for* $X_0 = Y(0)$*.*

Remark 3.1 Let us denote by "$\cdot$" the derivative with respect to t. In Theorem 3.2, the variation vector field of X_t at $t = 0$ is $(\dot{\varphi}(0))\nu$, that of $Y(s)$ is $(t'(0)\dot{\varphi}(0) + e)\nu$, and $\int_\Sigma e\, d\Sigma = 0$. It seems that this implies that $Y(s)$ does not have the same symmetry as X_t.

$$\hat{H}'(0) = (1/2)L[t'(0)\dot{\varphi}(0) + e] = (1/2)t'(0)L[\dot{\varphi}(0)] = t'(0)\dot{H}(0),$$
$$\hat{V}'(0) = \int_\Sigma (t'(0)\dot{\varphi}(0) + e)\, d\Sigma = t'(0)\int_\Sigma \dot{\varphi}(0)\, d\Sigma = t'(0)\dot{V}(0).$$

In Theorem 3.3, the same formulas hold by exchanging t for τ.

4 Eigenvalue estimate

By applying a general result on bifurcation by Crandall-Rabinowitz [3], we can show the following lemmas.

Lemma 4.1 ([6]) *We assume (i) - (iv) in Theorem 3.2. We use the same notations as those in Theorem 3.2. Then, there exist an open interval* $J \subset \hat{I}$ *with* $0 \in J$ *and continuously differentiable functions* $\mu : J \to \mathbf{R}$ *and* $w : J \to C_0^{2+\alpha}(\Sigma)$ *such that*

$$L_{Y(s)}[w(s)] = -\mu(s)w(s), \quad \forall s \in J. \tag{9}$$

Moreover,

$$|st'(s)\lambda'(0) + \mu(s)| \leq o(1)(|st'(s)| + |\mu(s)|) \text{ as } s \to 0, \tag{10}$$

$$\lim_{s \to 0, \mu(s) \neq 0} \frac{-st'(s)\lambda'(0)}{\mu(s)} = 1, \tag{11}$$

and

$$\|y'(s) - w(s)\| \leq C \min\{|st'(s)|, |\mu(s)|\}, \quad \forall s \in J, \tag{12}$$

where $y(s) := se + s\zeta(s)$*, and* C *is a certain constant. Especially,* $\mu(s)$ *and* $-st'(s)\lambda'(0) = -s\hat{H}'(s)(H'(t(s)))^{-1}\lambda'(0)$ *have the same zeroes and, where* $\mu(s) \neq 0$*, the same sign.*

Lemma 4.2 ([6]) *We assume (i) - (iv) in Theorem 3.3. We use the same notations as those in Theorem 3.3. Then, there exist an open interval $J \subset \hat{I}$ with $0 \in J$ and continuously differentiable functions $\mu : J \to \mathbf{R}$ and $w : J \to \{u \in C_0^{2+\alpha}(\Sigma) \mid \int_\Sigma u \, d\Sigma = 0\}$ such that*

$$L_{Y(s)}[w(s)] = -\mu(s)w(s) + c(s), \quad \forall s \in J, \tag{13}$$

where $c(s)$ is a constant. Moreover,

$$|s\tau'(s)\tilde{\lambda}'(0) + \mu(s)| \leq o(1)(|s\tau'(s)| + |\mu(s)|) \text{ as } s \to 0, \tag{14}$$

$$\lim_{s \to 0, \mu(s) \neq 0} \frac{-s\tau'(s)\tilde{\lambda}'(0)}{\mu(s)} = 1, \tag{15}$$

and

$$\|y'(s) - w(s)\| \leq C \min\{|s\tau'(s)|, |\mu(s)|\}, \quad \forall s \in J, \tag{16}$$

where $y(s) := se + s\eta(s)$, and C is a certain constant. Especially, $\mu(s)$ and $-s\tau'(s)\tilde{\lambda}'(0) = -s\hat{V}'(s)(V'(\tau(s)))^{-1}\tilde{\lambda}'(0)$ have the same zeroes and, where $\mu(s) \neq 0$, the same sign.

5 Stability of surfaces in bifurcation branch

In view of Theorems 3.1, 2.2, in order to study the stability of CMC surfaces in a bifurcation branch, we need to study only the case where $\lambda_2 = 0$ holds.

From Theorem 2.2, we obtain the following lemma.

Lemma 5.1 *We assume (i) - (iii) in Theorem 3.3. We use the same notations as those in Theorem 3.3. Also, we assume that $\lambda_2 = 0$ holds. Then, the following (A) and (B) hold.*

(A) If $H'(0)V'(0) \geq 0$, then X is stable and $\tilde{\lambda}_1 = 0$ holds.

(B) If $H'(0)V'(0) < 0$, then X is unstable and $\tilde{\lambda}_2 = 0$ holds.

Theorem 5.1 (Stability of bifurcation branch [6]) *We assume (i) - (iv) in Theorem 3.3. We use the same notations as those in Theorem 3.3. Also, we assume that $\lambda_2 = 0$ holds. Let $\mu(s)$ be the eigenvalue for $Y(s)$ which is obtained by Lemma 4.2. Then, the following (A1), (A2) and (B) hold.*

(A1) Assume $H'(0) > 0$ and $V'(0) > 0$ holds. Then, X is stable. Assume further that $\mu(s)$ is the smallest eigenvalue of the eigenvalue problem (2) for $Y(s)$. If $\tilde{\lambda}'(0) < 0$ (resp. $\tilde{\lambda}'(0) > 0$), then, for the CMC-bifurcation $Y(s)$ with volume $\hat{V}(s)$ obtained in Theorem 3.3, the following result about stability holds. If $\hat{V}'(s) = 0$, then $Y(s)$ is stable. Assume that $\hat{V}'(s) \neq 0$ holds. Then, for $s > 0$, $Y(s)$ is stable if and only if $\hat{V}'(s) > 0$ (resp. $\hat{V}'(s) < 0$) holds. And for $s < 0$, $Y(s)$ is stable if and only if $\hat{V}'(s) < 0$ (resp. $\hat{V}'(s) > 0$) holds.

(A2) Assume $H'(0) < 0$ and $V'(0) < 0$ holds. Then, X is stable. Assume further that $\mu(s)$ is the smallest eigenvalue of the eigenvalue problem (2) for $Y(s)$. If $\tilde{\lambda}'(0) < 0$ (resp. $\tilde{\lambda}'(0) > 0$), then, for the CMC-bifurcation $Y(s)$ with volume $\hat{V}(s)$ obtained in

Theorem 3.3, the following result about stability holds. If $\hat{V}'(s) = 0$, then $Y(s)$ is stable. Assume that $\hat{V}'(s) \neq 0$ holds. Then, for $s > 0$, $Y(s)$ is stable if and only if $\hat{V}'(s) < 0$ (resp. $\hat{V}'(s) > 0$) holds. And for $s < 0$, $Y(s)$ is stable if and only if $\hat{V}'(s) > 0$ (resp. $\hat{V}'(s) < 0$) holds.

(B) If $H'(0)V'(0) < 0$, Then, X is unstable, and $Y(s)$ is unstable for small $|s|$.

Theorem 5.1 can be proved by using Theorem 3.3 and Lemmas 4.2, 5.1.

Remark 5.1 Theorem 5.1 implies that if $H'(0)V'(0) > 0$ (that it, the original surface X is stable), then, only the following three types of bifurcations can occur: a super-critical pitchfork bifurcation, a subcritical pitchfork bifurcation, and a transcritical bifurcation. There are many interesting examples which have symmetry. If, for the surfaces $Y(s)$ in Theorem 5.1, $Y(-s) = \Phi \circ Y(s) \circ \Psi$ holds for an isometry Φ of $\mathbf{R}^3$ and a diffeomorphism Ψ of Σ, then if $H'(0)V'(0) > 0$, only pitchfork bifurcations can occur.

6 Applications and generalizations

The method developed in the previous sections is applied to various examples.

(I) A free boundary problem for CMC hypersurfaces between two parallel hyper-planes in $\mathbf{R}^{n+1}$ (cf. [8]). We have a bifurcation from a one-parameter family of cylinders to produce a half period of an unduloid-type solutions. Symmetry with respect to a hyperplane breaks. The stability of the unduloid depends on the dimension. In order to judge the stability of each unduloid, the results obtained in the previous sections are very useful. We can also apply a similar method to study the isoperimetric problem in $S^1 \times \mathbf{R}^n$.

(II) We can apply our method to more general variational problems: Free or fixed boundary problem for surfaces with constant anisotropic mean curvature, which are critical points of an anisotropic surface energy with volume constraint (cf. [1]).

7 Appendix

We will quote some results from Crandall and Rabinowitz [2] and [3], and we will generalize some of them, which are used in the proofs of our results mentioned above.

Let Y, Z be real Banach spaces, V be an open interval of 0 in Y, $I = (a, b)$ be an open interval, and $F : I \times V \to Z$ be a twice continuously Fréchet differential mapping. For a linear mapping T, denote by $N(T)$ the kernel of T, and by $R(T)$ the image of T.

Theorem 7.1 ([2], Theorem 1.7) *Assume that $t_0 \in I$ and that the following (i) - (iii) hold.*

(i) $F(t, 0) = 0$ for all $t \in I$,

(ii) $\dim N(D_yF(t_0,0)) = \operatorname{codim} R(D_yF(t_0,0)) = 1$,

(iii) $D_{ty}F(t_0,0)y_0 \notin R(D_yF(t_0,0))$, *where* $y_0 \in Y$ *spans* $N(D_yF(t_0,0))$.

Let W *be any complement of span*$\{y_0\}$ *in* Y. *Then there exists an open interval* $\hat{I}$ *containing* 0 *and continuously differentiable functions* $t : \hat{I} \to \mathbf{R}$ *and* $\zeta : \hat{I} \to W$ *such that* $t(0) = t_0$, $\zeta(0) = 0$, *and if* $y(s) = sy_0 + s\zeta(s)$, *then* $F(t(s), y(s)) = 0$. *Moreover,* $F^{-1}(\{0\})$ *near* $(t_0, 0)$ *consists precisely of the curves* $(t, 0)$, $t \in I$, *and* $(t(s), y(s))$, $s \in \hat{I}$.

Denote by $B(Y,Z)$ the set of bounded linear maps of Y into Z.

Definition 7.1 ([3], Definition 1.2) *Let* T, $K \in B(Y,Z)$. *Then* $\mu \in \mathbf{R}$ *is a* K*-simple eigenvalue of* T *if*

$$\dim N(T - \mu K) = \operatorname{codim} R(T - \mu K) = 1$$

and, if $N(T - \mu K) = \operatorname{span}\{e\}$,

$$Ke \notin R(T - \mu K).$$

Lemma 7.1 ([3], Lemma 1.3) *Let* T_0, $K \in B(Y,Z)$ *and assume that* r_0 *is a* K*-simple eigenvalue of* T_0. *Then there exists a value* $\delta > 0$ *such that whenever* $T \in B(Y,Z)$ *and* $\|T - T_0\| < \delta$, *there is a unique* $r(T) \in \mathbf{R}$ *satisfying* $|r(T) - r_0| < \delta$ *for which* $T - r(T)K : Y \to Z$ *is not a homeomorphism. The map* $T \to r(T)$ *is analytic and* $r(T)$ *is a* K*-simple eigenvalue of* T. *Finally, if* $N(T_0 - r_0K) = \operatorname{span}\{y_0\}$ *and* W *is a complement of* $\operatorname{span}\{y_0\}$ *in* Y, *there is a unique null vector* $x(T)$ *of* $T - r(T)K$ *satisfying* $x(T) - y_0 \in W$. *The map* $T \to x(T)$ *is also analytic.*

Corollary 7.1 ([3], Corollary 1.13) *We assume the same assumptions as those in Theorem 7.1. We use the same notations as in Theorem 7.1. Let* $K \in B(Y,Z)$ *and assume that* 0 *is a* K*-simple eigenvalue of* $D_yF(t_0,0)$. *Then, there exist open intervals* J_1, J_2 *with* $t_0 \in J_1$, $0 \in J_2$ *and continuously differentiable functions* $\lambda : J_1 \to \mathbf{R}$, $\mu : J_2 \to \mathbf{R}$, $u : J_1 \to Y$, $w : J_2 \to Y$ *such that*

(i) $D_yF(t,0)u(t) = \lambda(t)Ku(t)$, $\forall t \in J_1$,

(ii) $D_yF(t(s), y(s))w(s) = \mu(s)Kw(s)$, $\forall s \in J_2$.

Moreover,

$$\lambda(t_0) = \mu(0) = 0, \quad u(t_0) = y_0 = w(0), \quad u(t) - y_0 \in W, \quad w(s) - y_0 \in W.$$

By a similar way to the proof of Lemma 7.1, we can prove the following.

Lemma 7.2 *Let* T_0, $K_0 \in B(Y,Z)$ *and assume that* r_0 *is a* K_0*-simple eigenvalue of* T_0. *Then there exists a value* $\delta > 0$ *such that whenever* $T, K \in B(Y,Z)$ *and* $\|K - K_0\|, \|T - T_0\| < \delta$, *there is a unique* $r(T,K) \in \mathbf{R}$ *satisfying* $|r(T,K) - r_0| < \delta$ *for which* $T - r(T,K)K : Y \to Z$ *is not a homeomorphism. The map* $(T,K) \to r(T,K)$ *is analytic and* $r(T,K)$ *is a* K*-simple eigenvalue of* T. *Finally, if* $N(T_0 - r_0K_0) = \operatorname{span}\{y_0\}$ *and* W *is a complement of* $\operatorname{span}\{y_0\}$ *in* Y, *there is a unique null vector* $x(T,K)$ *of* $T - r(T,K)K$ *satisfying* $x(T,K) - y_0 \in W$. *The map* $(T,K) \to x(T,K)$ *is also analytic.*

By a similar way to the proof of Corollary 7.1, we can prove the following.

Corollary 7.2 *We assume the same assumptions as those in Theorem 7.1. We use the same notations as in Theorem 7.1. Let $K(t,y) \in B(Y,Z)$ be differentiable with respect to $(t,y) \in I \times V$, and assume that 0 is a $K(t_0,0)$-simple eigenvalue of $D_yF(t_0,0)$. Then, there exist open intervals J_1, J_2 with $t_0 \in J_1$, $0 \in J_2$ and continuously differentiable functions $\lambda : J_1 \to \mathbf{R}$, $\mu : J_2 \to \mathbf{R}$, $u : J_1 \to Y$, $w : J_2 \to Y$ such that*

(i) $D_yF(t,0)u(t) = \lambda(t)K(t,0)u(t)$, $\forall t \in J_1$,

(ii) $D_yF(t(s),y(s))w(s) = \mu(s)K(t(s),y(s))w(s)$, $\forall s \in J_2$.

Moreover,

$$\lambda(t_0) = \mu(0) = 0, \quad u(t_0) = y_0 = w(0), \quad u(t) - y_0 \in W, \quad w(s) - y_0 \in W.$$

The following result can be proved by the same way as the proof of Theorem 1.16 in [3], which assumes the same assumptions as those of Corollary 7.1 and obtains the same result as Theorem 7.2

Theorem 7.2 *Let the assumptions of Corollary 7.2 hold and let λ, μ be the functions provided by the corollary. Then, $\lambda'(t_0) \neq 0$, and near $s = 0$ the functions $\mu(s)$ and $-st'(s)\lambda'(t_0)$ have the same zeroes, and, whenever $\mu(s) \neq 0$, the same sign. More precisely,*

$$|st'(s)\lambda'(t_0) + \mu(s)| \leq o(1)(|st'(s)| + |\mu(s)|) \text{ as } s \to 0,$$

$$\lim_{s \to 0, \mu(s) \neq 0} \frac{-st'(s)\lambda'(t_0)}{\mu(s)} = 1.$$

Moreover, there is a constant C such that

$$\|y'(s) - w(s)\| \leq C \min\{|st'(s)|, |\mu(s)|\}$$

near $s = 0$.

References

[1] J. Arroyo, M. Koiso, and B. Palmer, *Stability of non liquid bridges*, preprint (2009).

[2] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Func. Analysis **54** (1971), 321–340.

[3] M. G. Crandall and P. H. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues, and linearized stability*, Arch. Rat. Mech. Anal. **54** (1971), 321–340.

[4] J. L. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), 339–353.

[5] M. Koiso, *Deformation and stability of surfaces with constant mean curvature*, Tohoku Math. J. (2) **54** (2002), 145–159.

[6] M. Koiso, B. Palmer, and P. Piccione, in preparation.

[7] U. Patnaik, *Volume constrained Douglas problem and the stability of liquid bridges between two coaxial tubes*, Dissertation, University of Toledo, USA, 1994.

[8] R. Pedrosa and M. Ritoré, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J. **48** (1999), 1357–1394.

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