PARTIAL STATIONARY REFLECTION PRINCIPLES

TOSHIMICHI USUBA

Toshimichi Usuba (薄葉 季路)
Institute for Advanced Research,
Nagoya University

1. INTRODUCTION

Throughout this paper, $\kappa$ denotes a regular uncountable cardinal and $\lambda$ a cardinal $\geq \kappa^+$, unless otherwise specified.

Partial stationary reflection on $\mathcal{P}_{\omega_1}\omega_2$ was introduced by H. Sakai [2]. First we extend his notion to arbitrary $\kappa$ and $\lambda$.

Definition 1.1. Let $S^*$ be a stationary subset of $\mathcal{P}_\kappa\lambda$. For a stationary set $T \subseteq \mathcal{P}_{\kappa+}\lambda$, we say that $\text{RP}(S^*, T)$ holds if for every stationary subset $S \subseteq S^*$ there exists $X \in T$ such that $\kappa \subseteq X$ and $S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$. $\text{RP}(S^*)$ means $\text{RP}(S^*, \mathcal{P}_\kappa+\lambda)$.

It is known that total stationary reflection $\text{RP}(\mathcal{P}_\kappa\lambda)$ is a large cardinal property (e.g., see Velickovic [3]), but Sakai [2] showed that partial stationary reflection on $\mathcal{P}_{\omega_1}\omega_2$ is not:

Fact 1.2 ([2]). Suppose CH. If $\Box_{\omega_1}$ holds, then there are a stationary set $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ and a $\sigma$-Baire, $\omega_2$-c.c. poset $\mathbb{P}$ such that $\mathbb{P}$ forces $\text{RP}(S^*)$.

In this paper, we generalize his result as follows:

Theorem 1.3. Suppose $\kappa^{<\kappa} = \kappa$. Let $T \subseteq \mathcal{P}_{\kappa+}\lambda$ be a stationary set such that $\forall X \in T (\kappa \subseteq X)$. Then there exists a $\kappa$-closed, $\kappa^+$-c.c. poset which forces the following statements:

(1) $T$ is stationary.
(2) There exists a stationary set $S^* \subseteq \mathcal{P}_\kappa\lambda$ such that
   (a) $\forall X \in T (S^* \cap \mathcal{P}_\kappa X$ contains a club in $\mathcal{P}_\kappa X$),
   (b) $\text{RP}(S^*, T)$ holds.
This theorem shows that, even $\kappa > \omega_1$ and $\lambda > \omega_2$, our partial stationary reflection is not a large cardinal property.

Next we consider a natural strengthening of partial stationary reflection, simultaneous partial stationary reflection.

**Definition 1.4.** For stationary sets $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda$ and $T \subseteq \mathcal{P}_{\kappa^+} \lambda$, we say that $RP^2(S_0^*, S_1^*, T)$ holds if for every stationary subsets $S_0 \subseteq S_0^*$ and $S_1 \subseteq S_1^*$ in $\mathcal{P}_\kappa \lambda$, there exists $X \in T$ such that $\kappa \subseteq X$ and both $S_0 \cap \mathcal{P}_\kappa X$ and $S_1 \cap \mathcal{P}_\kappa X$ are stationary in $\mathcal{P}_\kappa X$. $RP^2(S_0^*, S_1^*)$ means $RP^2(S_0^*, S_1^*, \mathcal{P}_{\kappa^+} \lambda)$.

We prove that our simultaneous partial stationary reflection is a large cardinal property by showing the following:

**Definition 1.5.** For a regular uncountable cardinal $\mu$, $\square(\mu)$ holds if there exists a sequence $\langle C_\xi : \xi < \mu \rangle$ satisfying the following:

(1) for all $\xi < \mu$, $C_\xi$ is club in $\xi$ and for all $\eta \in \text{lim}(C_\xi)$, $C_\eta = C_\xi \cap \eta$,

(2) for all club $C$ in $\mu$, there exists $\xi \in \text{lim}(C)$ such that $C \cap \xi \neq C_\xi$.

Such an sequence $\langle C_\xi : \xi < \mu \rangle$ is called a $\square(\mu)$-sequence.

**Theorem 1.6.** Suppose $RP^2(S_0^*, S_1^*)$ holds for some stationary $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda$. Then for every regular $\mu$ with $\kappa^+ \leq \mu \leq \lambda$, $\square(\mu)$ fails.

We also prove the following:

**Theorem 1.7.** For every stationary $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda$ and regular $\mu$ with $\kappa^+ \leq \mu \leq \lambda$, $RP^2(S_0^*, S_1^*, \{X \in \mathcal{P}_{\kappa^+} \lambda : \text{cf}(X \cap \mu) < \kappa\})$ fails, where $\text{cf}(X) = \text{cf}(\text{ot}(X))$.

Todorcevic showed that $RP(\mathcal{P}_{\omega_1} \omega_2)$ implies that $2^\omega \leq \omega_2$. However we prove the following, which shows that our partial stationary reflection does not affect the size of the continuum:

**Theorem 1.8.** (1) Suppose $RP(S^*)$ for some stationary $S^* \subseteq \mathcal{P}_\kappa \lambda$. Then every $\kappa$-c.c. forcing preserves $RP(S^*)$.

(2) Suppose PFA$^{++}$. Let $\lambda \geq \omega_2$. Then every c.c.c. forcing notion forces $RP^2(\mathcal{P}_{\omega_1}^\kappa \lambda, \mathcal{P}_{\omega_1}^\kappa \lambda)$.

2. Preliminaries

For a set $X$ of ordinals, let $\text{cf}(X) = \text{cf}(\text{ot}(X))$.

For regular cardinals $\nu < \mu$, let $E^\nu_\mu = \{\alpha < \mu : \text{cf}(\alpha) = \nu\}$ and $E^\nu_{<\nu} = \{\alpha < \mu : \text{cf}(\alpha) < \nu\}$.
The proofs of the following lemmata are easy:

**Lemma 2.1.** For a stationary $S \subseteq \mathcal{P}_{\kappa}\lambda$ and a $\kappa$-c.c. poset $\mathbb{P}$, $\mathbb{P}$ preserves the stationarity of $S$.

**Lemma 2.2.** For $S \subseteq \mathcal{P}_{\kappa}\lambda$, if $\{X \in \mathcal{P}_{\kappa^{+}}\lambda : X \cap S \subseteq \mathcal{P}_{\kappa}X\}$ is stationary in $\mathcal{P}_{\kappa^{+}}\lambda$, then $S$ is stationary in $\mathcal{P}_{\kappa}\lambda$.

**Lemma 2.3.** For stationary sets $S^{*} \subseteq \mathcal{P}_{\kappa}\lambda$ and $T \subseteq \mathcal{P}_{\kappa^{+}}\lambda$, suppose $RP(S^{*}, T)$ holds. Then for every stationary $S \subseteq S^{*}$, $\{X \in T : X \cap S \subseteq \mathcal{P}_{\kappa}X\}$ is stationary in $\mathcal{P}_{\kappa^{+}}\lambda$.

We define club shootings into $\mathcal{P}_{\kappa}\lambda$, which was observed in [2].

**Definition 2.4.** For $S \subseteq \mathcal{P}_{\kappa}\lambda$, let $\mathbb{C}(S)$ be the poset which consists of all functions $p$ such that:

1. $|p| < \kappa$,
2. $p : d(p) \times d(p) \rightarrow \kappa$ for some $d(p) \in \mathcal{P}_{\kappa}\lambda$, and
3. $\forall x \subseteq d(p) (x \in S \Rightarrow x$ is not closed under $p)$.

For $p, q \in \mathbb{C}(S)$, $p \leq q \iff q \subseteq p$.

Let $\mathbb{C} = \mathbb{C}(\emptyset)$.

**Lemma 2.5.**

1. $\mathbb{C}(S)$ satisfies the $(2^{<\kappa})^{+}$-c.c.
2. For every $x \in \mathcal{P}_{\kappa}\lambda$, $\{p \in \mathbb{C}(S) : x \subseteq d(p)\}$ is a dense open set in $\mathbb{C}(S)$.
3. Whenever $G$ is $(V, \mathbb{C}(S))$-generic, $\bigcup G$ is a function from $\lambda \times \lambda$ to $\kappa$, and every $x \in S$ is not closed under the function.

Proof. For (1), take $A \subseteq \mathbb{C}(S)$ with size $(2^{<\kappa})^{+}$. By $\Delta$-system lemma, we can find $B \subseteq A$ and $a \in \mathcal{P}_{\kappa}\lambda$ such that $|B| = (2^{<\kappa})^{+}$ and $d(p) \cap d(q) = a$ for every distinct $p, q \in B$. Moreover we may assume that $p|a \times a = q|a \times a$ for every $p, q \in B$. We check that $B$ is a pairwise compatible set.

Take $p, q \in B$. Pick $\alpha < \kappa$ with $\alpha > \sup(d(p) \cap \kappa) + 1, \sup(d(q) \cap \kappa) + 1$. Then define $r$ as $\text{dom}(r) = (d(p) \cup d(q)) \times (d(p) \cup d(q))$ and

$$r(\xi, \eta) = \begin{cases} p(\xi, \eta) & \text{if } \xi, \eta \in d(p). \\ q(\xi, \eta) & \text{if } \xi, \eta \in d(q). \\ \alpha & \text{otherwise}. \end{cases}$$

We have $r \leq p, q$. (2) follows from a similar argument, and (3) is straightforward. \(\square\)
3. The proof of Theorem 1.3

Suppose $\kappa^{<\kappa} = \kappa$. Fix a stationary set $T \subseteq \mathcal{P}_{\kappa^+}\lambda$ such that $\forall X \in T (\kappa \subseteq X)$.

We consider the following poset $\mathbb{P}_T$, which adds a new stationary subset $S^*$ of $\mathcal{P}_\kappa\lambda$.

**Definition 3.1.** $\mathbb{P}_T$ is the set of all functions $p$ satisfying the following:

1. $|p| < \kappa$ and $\text{dom}(p) \subseteq T$,
2. For every $X \in \text{dom}(p)$, $p(X)$ is a $\subseteq$-increasing continuous set $\{x_i : i \leq \gamma\}$ in $\mathcal{P}_\kappa X$ such that $\gamma < \kappa$ and $x_i \cap \kappa \in \kappa$ for all $i \leq \gamma$.

For $p \in \mathbb{P}_T$ and $X \in \text{dom}(p)$, $\text{max}(p(X))$ denotes the maximum element of $p(X)$.

Let $u(p) = \bigcup\{p(X) : X \in \text{dom}(P)\}$. Note that $u(p) \subseteq \mathcal{P}_\kappa\lambda$ and $|u(p)| < \kappa$. For $p, q \in \mathbb{P}_T$, define $p \leq q$ if

(a) $\text{dom}(p) \supseteq \text{dom}(q)$,
(b) $\forall X \in \text{dom}(q) (q(X) = \{x \in p(X) : x \leq \max(q(X))\})$ (hence $u(p) \supseteq u(q)$),
(c) $\forall x \in u(p) (x \subseteq \bigcup u(q) \Rightarrow x \in u(q))$,
(d) $\forall X \in \text{dom}(p) \setminus \text{dom}(q) (\max(p(X)) \not\subseteq \bigcup u(q))$,
(e) $\forall X \in \text{dom}(q) \forall x \in p(X) \setminus q(X) (x \not\subseteq \bigcup u(q))$.

**Lemma 3.2.**

1. $\mathbb{P}_T$ is $\kappa$-closed,
2. $\mathbb{P}_T$ satisfies the $\kappa^+$-c.c. (if $\kappa^{<\kappa} = \kappa$),
3. For all $X \in T$ and $x \in \mathcal{P}_\kappa X$, $\{p \in \mathbb{P}_T : X \in \text{dom}(p) \text{ and } x \subseteq \max(p(X))\}$ is dense in $\mathbb{P}_T$.

**Proof.** (1). Let $\gamma < \kappa$ be a limit ordinal and $(p_i : i < \gamma)$ be a decreasing sequence in $\mathbb{P}_T$. Then define the function $p^*$ as the following manner:

(i) $\text{dom}(p^*) = \bigcup_{i < \gamma} \text{dom}(p_i)$,
(ii) for $X \in \text{dom}(p^*)$, $p^*(X) = \bigcup\{p_i(X) : i < \gamma, X \in \text{dom}(p_i)\} \cup \{\max(p_i(X)) : i < \gamma, X \in \text{dom}(p_i)\}$.

Since the $p_i$'s are decreasing, it is easy to show that $p^* \in \mathbb{P}_T$. For $i < \gamma$, we show $p \leq p_i$. It is easily verified that the conditions (a) and (b) in the definition of the order are satisfied.

(c). Take $x \in u(p^*)$ such that $x \subseteq \bigcup u(p_i)$. Take $X \in \text{dom}(p^*)$ such that $x \in p^*(X)$. If $x \neq \max(p^*(X))$, then $x \in p_j(X)$ for some $j > i$ with $X \in \text{dom}(p_j)$. Since $p_j \leq p_i$, we have $x \in p_i(X)$. Next suppose $x = \max(p^*(X))$. Take $k < \gamma$ such that $i < k$ and $X \in \text{dom}(p_k)$. Then $\max(p_k(X)) \subseteq \max(p^*(X)) = x \subseteq \bigcup u(p_i)$ holds. Hence $X \in \text{dom}(p_i)$ by (d). For each $j \geq i$, $\max(p_j(X)) \subseteq \max(p^*(X)) = x \subseteq$
$\bigcup u(p_i)$ holds. Thus we have $\max(p_j(X)) \in p_i(X)$ by (e). Therefore $\{\max(p_j(X)): i \leq j < \gamma\} \subseteq p_i(X)$, and we have $\max(p^*(X)) = \bigcup \{\max(p_j(X)): i \leq j < \gamma\} \in p_i(X)$.

(d). Take $X \in \text{dom}(p^*) \setminus \text{dom}(p_i)$. Then there exists $j > i$ such that $X \in \text{dom}(p_j)$. We know $\max(p_j(X)) \not\in \bigcup u(p_i)$. Because $\max(p_j(X)) \subseteq \max(p^*(X))$, we know $\max(p^*(X)) \not\in \bigcup u(p_i)$.

(e). Take $X \in \text{dom}(p_i)$ and $x \in p^*(X) \setminus p_i(X)$. Then there exist $j \geq i$ and $y \in \text{dom}(p_j)$ such that $y \subseteq x$ and $y \notin p_i(X)$. Hence $y \not\in \bigcup u(p_i)$ and $x \not\in \bigcup u(p_i)$.

(2). Take an arbitrary $A \subseteq \mathbb{P}_T$ with $|A| \geq \kappa^+$. We prove that $A$ is not an antichain. By $\Delta$-system lemma, we can find $r \in \mathbb{P}_T$, $s \in \mathbb{P}_{\kappa \lambda}$, and $B \subseteq A$ with $|B| \geq \kappa^+$ such that $\forall p, q \in B$ $(\text{dom}(p) \cap \text{dom}(q) = r \text{ and } \bigcup u(p) \cap \bigcup u(q) = s)$. By our cardinal arithmetic assumption, there exists $C \subseteq B$ with $|C| \geq \kappa^+$ such that $\forall p, q \in B \forall X \in r (p(X) = q(X))$ and $\mathcal{P}_\kappa s \cap u(p) = \mathcal{P}_\kappa s \cap u(q))$. We check that any two elements of $C$ are pairwise compatible. Take $p, q \in C$. For each $X \in \text{dom}(p) \cup \text{dom}(q)$, fix $a_X \in \mathbb{P}_\kappa X$ such that $(\bigcup u(p) \cup \bigcup u(q)) \cap X \not\subseteq a_X$. Define the function $r$ as the following:

(i) $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$,

(ii) $r(X) = \max(a_X)$ if $X \in \text{dom}(p)$, and $r(X) = \max(a_X)$ if $X \in \text{dom}(q)$.

This is well-defined because $p(X) = q(X)$ for all $X \in \text{dom}(p) \cap \text{dom}(q)$. We see that $r$ is a lower bound of $p$ and $q$. $r \in \mathbb{P}_T$ is easily verified. For $r \leq p$, the conditions (a) and (b) are clear.

(c). Take $x \in u(r)$ such that $x \not\in \bigcup u(p)$. Then $x \not\in a_X$ for all $X \in \text{dom}(p) \cup \text{dom}(q)$. Hence $x \in u(p) \cup u(q)$. If $x \in u(p)$ then we have done. Assume $x \in u(q)$. Then $x \subseteq \bigcup u(q)$. Since $x \subseteq \bigcup u(p)$, we have $x \subseteq \bigcup u(p) \cap \bigcup u(q) = s$ and $x \in \mathcal{P}_\kappa s$. Because $\mathcal{P}_\kappa s \cap u(p) = \mathcal{P}_\kappa s \cap u(q)$, we have $x \in \mathcal{P}_\kappa s \cap u(p)$ and $x \in u(p)$.

(d). Take $X \in \text{dom}(r) \setminus \text{dom}(p)$. Then $\max(r(X)) = a_X \not\subseteq \bigcup u(p) \cap X$, thus $\max(r(X)) \not\subseteq \bigcup u(p)$.

(e). Take $X \in \text{dom}(p)$ and $x \in r(X) \setminus p(X)$. By the definition of $r(X)$, we have $r(X) = p(X) \cup \{a_X\}$. Hence $x = a_X \not\subseteq \bigcup u(p)$.

$r \leq q$ can be proved by the same argument.

(3). Take $X \in \mathbb{T}$, $x \in \mathbb{P}_\kappa X$ and $q \in \mathbb{P}$. Take $x^* \in \mathbb{P}_\kappa X$ such that $\bigcup u(q) \cap X \not\subseteq x^*$. Define $p$ as $\text{dom}(p) = \text{dom}(q) \cup \{X\}$, $\text{dom}(q) = q$ and $\text{dom}(X) = \{x^*\}$ if $X \notin \text{dom}(q)$, and $q(X) \cup \{x^*\}$ if $X \in \text{dom}(q)$. Then $p \leq q$ can be verified. \qed
Note that the following: For $\gamma < \kappa$ and a decreasing sequence $\langle p_i : i < \gamma \rangle$ in $\mathbb{P}_T$, let $p^*$ be a lower bound of the $p_i$'s as constructed in the proof of (1) above. Then $p^*$ is the largest lower bound of the $p_i$'s and $\bigcup u(p^*) = \bigcup_{i<\gamma}(\bigcup u(p_i))$.

**Definition 3.3.** For a canonical name of $(V, \mathbb{P}_T)$-generic filter $\dot{G}$, let $\dot{S}^*$ be a $\mathbb{P}_T$-name such that

$$\Vdash_{\mathbb{P}_T} \langle \{ u(p) : p \in \dot{G} \} \rangle$$

The following are easily verified by the definition of $\mathbb{P}_T$.

**Lemma 3.4.** (1) $\Vdash_{\mathbb{P}_T} \forall X \in T (\dot{S}^* \cap \mathcal{P}_\kappa X$ contains a club in $\mathcal{P}_\kappa X)$”,

(2) for all $p \in \mathbb{P}_T$, $p \Vdash_{\mathbb{P}_T} \{ y \in \dot{S}^* : y \subseteq \bigcup u(p) \} = u(p)$.

Now fix a name $\dot{S}$ such that

$$\Vdash_{\mathbb{P}_T} \dot{S} \subseteq \dot{S}^* \text{ and } \forall X \in T (\mathcal{P}_\kappa X \cap \dot{S} \text{ is non-stationary in } \mathcal{P}_\kappa X).$$

We see that $\mathbb{P}_T \ast \mathbb{C}(\dot{S})$ has good properties.

For each $X \in T$, fix a name $\dot{g}_X$ such that

$$\Vdash_{\mathbb{P}_T} \dot{g} : [X]^\omega \rightarrow X \text{ and } \forall x \in \mathcal{P}_\kappa X (x \text{ is closed under } \dot{g}_X \Rightarrow x \notin \dot{S}).$$

Let $\dot{Q}$ be a name such that $\Vdash \langle \dot{Q} = \mathbb{C}(\dot{S}) \rangle$. We prove that $\mathbb{P}_T \ast \dot{Q}$ has a $\kappa$-closed dense subset.

**Lemma 3.5.** Let $D = \{ p \in \mathbb{P}_T : \forall X \in \text{dom}(p) (p \Vdash_{\mathbb{P}_T} \text{max}(p_X) \text{ is closed under } \dot{g}_X) \}$. Then $D$ is dense in $\mathbb{P}_T$.

**Proof.** Take $p \in \mathbb{P}_T$. We want to find $q \in D$ such that $q \leq p$. We take a decreasing sequence $p_i (i < \omega)$ in $\mathbb{P}_T$ by induction on $i < \omega$. Let $p_0 = p$. Suppose $p_i$ is defined.

By the $\kappa$-closedness of $\mathbb{P}_T$, we can choose $p' \leq p_i$ and $a \in \mathcal{P}_\kappa \lambda$ such that $p' \Vdash \langle \dot{g}_X \text{max}(p_i(X))^{\omega} \subseteq a \cap X \rangle$ for all $X \in \text{dom}(p_i)$.

Then choose $p_{i+1} \leq p'$ such that $a \cap X \subseteq \text{max}(p_{i+1}(X))$ for all $X \in \text{dom}(p_{i+1})$.

Finally let $q$ be the greatest lower bound of the $p_i$’s. By our construction, it is easy to see that $q \in D$. \qed

**Lemma 3.6.** Let $D$ be as in Lemma 3.5. Let $D' = \{ \langle p, q \rangle \in \mathbb{P}_T \ast \dot{Q} : p \in D, q = r \text{ for some } r \in \mathcal{C} \text{ and } d(r) = \bigcup u(p) \}$. Then $D'$ is a $\kappa$-closed dense subset in $\mathbb{P}_T \ast \dot{Q}$.

**Proof.** Density: Take $\langle p, q \rangle \in \mathbb{P}_T \ast \dot{Q}$. Take $p' \in D$ and $r$ such that $p' \Vdash \langle \check{r} = \check{q} \rangle$ and $\bigcup u(p') \supseteq d(r)$. Now define $r'$ as the following:
(1) $r' : \bigcup u(p') \times \bigcup u(p') \rightarrow \kappa$,
(2) for $a \in \bigcup u(p') \times \bigcup u(p')$, if $a \in d(r) \times d(r)$ the $r'(a) = r(a)$, otherwise $r'(a) = \text{sup}(\bigcup(u(p' \cap \kappa)) + 1$.

It is easy to show that $p' \vDash \text{ "} q\text{'} \in \mathbb{C}(\dot{S})\text{"} \text{ and } \langle p', \check{r}' \rangle \leq \langle p, \check{q} \rangle$.

Next we prove $D'$ is $\kappa$-closed. Let $\gamma < \kappa$ and $(p_i, \check{q}_i) (i < \gamma)$ be a decreasing sequence in $D'$. We show that this sequence has a lower bound. Let $p^* \in \mathbb{P}_T$ be the greatest lower bound of the $p_i$'s. Note that for all $X \in \text{dom}(p^*)$, $p^* \vDash_{\mathbb{P}_T} \text{ "max}(p^*(X))$ is closed under $\check{g}_X\"$.

Let $q^* = \bigcup_{i < \gamma} q_i$. $q^*$ is a function with the domain $d(q^*) \times d(q^*)$, where $d(q^*) = \bigcup_{i < \gamma} d(q_i)$. Notice that $d(q^*) = \bigcup_{i < \gamma} d(q_i) = \bigcup_{i < \gamma} u(p_i) = \bigcup u(p^*)$. We complete the proof by showing the following claim.

**Claim 3.7.** $p^* \vDash \text{ "} q^* \in \mathbb{C}(\dot{S})\text{"}$.\[
\]

**Proof.** Take a $(V, \mathbb{P}_T)$-generic $G$ with $p^* \in G$ and work in $V[G]$. First note that

\{x \in S^* : x \subseteq \bigcup u(p^*)\} = u(p^*)$. To show that $q^* \in \mathbb{C}(S)$, take $x \subseteq d(q^*)$ with $x \in S$. We check that $x$ is not closed under $q^*$. Since $x \subseteq d(q^*) = \bigcup u(p^*)$ and $x \in S \subseteq S^*$, we have $x \in u(p^*)$. Hence there exists $X \in \text{dom}(p^*)$ such that $x \in p^*(X)$. Because $\text{max}(p^*(X))$ is closed under $g_X$, we know $\text{max}(p^*(X)) \notin S$. Thus $x \neq \text{max}(p^*(X))$ and $x \in p_i(X)$ for some $i < \gamma$ with $X \in \text{dom}(p_i)$. Then $x \subseteq \bigcup u(p_i) = d(q_i)$. Since $q_i$ is a condition, $x$ is not closed under $q_i$, and not closed under $q^*$.\[
\]

Note that, in fact, $D'$ is $\kappa$-directed closed.

By an iteration of the above forcing, we can prove Theorem 1.3. Let $\langle \mathbb{P}_\xi, \check{Q}_\eta : \xi < \zeta, \eta < \zeta \rangle$ be a $\kappa$-support iteration such that for every $\xi < \zeta$,

(1) $\check{Q}_0 = \mathbb{P}_T$,
(2) $\mathbb{P}_\xi$ satisfies the $\kappa^+$-c.c. and has a $\kappa$-closed dense subset,
(3) for $\xi > 0$ there exists a $\mathbb{P}_\xi$-name $\dot{S}_\xi$ such that

$\vDash_{\mathbb{P}_\xi} \text{ "} \dot{S}_\xi \subseteq \dot{S}^* \text{ and } \forall X \in T (\mathcal{P}_\kappa X \cap \dot{S}_\xi \text{ is non-stationary in } \mathcal{P}_\kappa X)\"$,
(4) for every $X \in T$, $\check{g}_X^\xi$ is a $\mathbb{P}_\xi$-name such that

$\vDash_{\mathbb{P}_\xi} \text{ "} \check{g}_X^\xi : [X]^{\omega} \rightarrow X \text{ and } \forall x \in \mathcal{P}_\kappa X (x \in \dot{S}_\xi \Rightarrow x \text{ is not closed under } \check{g}_X^\xi)\"$,
(5) $\vDash_{\mathbb{P}_\xi} \text{ "} \check{Q}_\xi = \mathbb{C} (\dot{S}_\xi)\"$ for $\xi > 0$,
(6) let $D_\xi$ is the set of all $p \in \mathbb{P}_\xi$ such that
(a) \( \forall \eta \in \text{supp}(p) \setminus \{0\} \ (p(\eta) = \check{r} \) for some \( r \in C \),
(b) for all \( X \in \text{dom}(p(0)) \) and \( \eta \in \text{supp}(p) \setminus \{0\} \) \( p|_{\eta} \models_{-0} \text{"max}(p(0)(X)) \) is closed under \( \check{g}_{X}^{\eta} \),
(c) \( \bigcup(u(p(0)) = d(p(\eta)) \) for all \( \eta \in \text{supp}(p) \setminus \{0\} \).

Then \( D_\xi \) is a \( \kappa \)-closed dense set in \( P_\xi \).

Let \( P_\xi \) and \( D_\xi \) be as intended. We can check that \( D_\xi \) is a \( \kappa \)-closed dense set in \( P_\xi \), and \( P_\xi \) has the \( \kappa^{+} \)-c.c.

By a standard book keeping method, we can destroy the stationarity of all non-reflecting subset of \( S^* \) by an iteration above. By \( \kappa^{+} \)-c.c., \( T \) remains stationary in \( P_\kappa \) in the generic extension. Thus \( S^* \) is stationary in \( P_\kappa \), and \( \text{RP}(S^*, T) \) holds.

4. Proof of Theorems 1.6 and 1.7

Proposition 4.1. Let \( \mu \) be a regular cardinal with \( \kappa^{+} \leq \mu \leq \lambda \). Let \( T = \{X \in P_{\kappa^{+}} \lambda : \kappa \subseteq X, \text{cf}(X \cap \mu) < \kappa \} \). Then for every stationary sets \( S_0^*, S_1^* \subseteq P_{\kappa^{+}} \lambda \), \( \text{RP}^2(S_0^*, S_1^*, T) \) fails.

Proof. Suppose not. For each \( \xi \in E^*_{\kappa} \), fix an increasing sequence \( \langle \gamma_i^\xi : i < \text{cf}(\xi) \rangle \) with limit \( \xi \). For \( n < 2, i < \kappa \), and \( \delta < \mu \), let
\[
S_{n,i,\delta} = \{x \in S_n^* : \delta = \min(x \setminus \gamma_i^{\sup(x \cap \mu)}) \}.
\]

Claim 4.2. (1) For every \( \xi < \mu \), there exist \( i < \kappa \) and \( \delta < \mu \) such that \( \delta > \xi \) and \( S_{0,i,\delta} \) is stationary.
(2) For every \( i < \kappa \) and \( \delta < \mu \), if \( S_{0,i,\delta} \) is stationary then \( S_{1,i,\delta} \) is stationary.
(3) For every \( i < \kappa \) and \( \delta_0, \delta_1 < \mu \), if \( S_{0,i,\delta_0} \) and \( S_{1,i,\delta_1} \) are stationary then \( \delta_0 = \delta_1 \).

Proof. (1). Let \( T' = \{X \in T : S_0^* \cap P_\kappa X \text{ is stationary, } \xi \in X \} \). \( T' \) is stationary in \( P_{\kappa^{+}} \). Take \( X \in T' \). Then \( \text{cf}(X \cap \mu) < \kappa \subseteq X \) and \( \sup(X \cap \mu) > \xi \), hence there exists \( i \in X \) such that \( \gamma_i^{\sup(X \cap \mu)} > \xi \). By applying Fodor’s lemma to \( T' \), there exists \( i < \kappa \) such that \( T'' = \{x \in T' : \gamma_i^{\sup(X \cap \mu)} > \xi \} \) is stationary in \( P_{\kappa^{+}} \). For \( X \in T'' \) let \( \delta_X = \min(X \setminus \gamma_i^{\sup(X \cap \mu)}) \). By Fodor’s lemma again, there is \( \delta < \mu \) such that \( T^* = \{X \in T'' : \gamma_i^{\sup(X \cap \mu)} > \xi, \delta = \min(X \setminus \gamma_i^{\sup(X \cap \mu)}) \} \) is stationary in \( P_{\kappa^{+}} \).

Pick \( X \in T^* \). Since \( \text{cf}(X \cap \mu) < \kappa \), the set \( D_X = \{x \in P_\kappa X : \sup(x \cap \mu) = \sup(X \cap \mu), \delta \in x \} \) contains a club in \( P_\kappa X \). Clearly \( x \in S_{0,i,\delta} \) for each \( x \in D_X \cap S_0^* \). This means that \( S_{0,i,\delta} \) is stationary in \( P_{\kappa^{+}} \).

(2). By \( \text{RP}^2(S_0^*, S_1^*) \), \( T' = \{X \in T : \delta \in X, S_{0,i,\delta} \cap P_\kappa X, S_1^* \cap P_\kappa X \text{ are stationary} \} \) is stationary in \( P_{\kappa^{+}} \). Fix \( X \in T' \). Since \( S_{0,i,\delta} \cap P_\kappa X \) is stationary in \( P_\kappa X \) and...
cf($X \cap \mu < \kappa$, we have that $\delta = \min(X \setminus \gamma_i^{\sup(X \cap \mu)})$. By the same argument as (1), we have that $S_{1,i,\delta}$ is stationary in $\mathcal{P}_\kappa \lambda$.

(3). Let $X \in T$ be such that $\delta_0, \delta_1 \in X$ and $S_{0,i,\delta_0} \cap \mathcal{P}_\kappa X, S_{1,i,\delta_1} \cap \mathcal{P}_\kappa X$ are stationary. Choose $x_0 \in S_{0,i,\delta_0} \cap \mathcal{P}_\kappa X$ and $x_1 \in S_{1,i,\delta_1} \cap \mathcal{P}_\kappa X$ such that $\sup(x_0 \cap \mu) = \sup(x_1 \cap \mu) = \sup(X \cap \mu)$ and $\delta_0, \delta_1 \in x_0 \cap x_1$. By the minimality of $\delta_0$, we have $\delta_0 \leq \delta_1$. Similarly we know $\delta_1 \leq \delta_0$. Therefore $\delta_0 = \delta_1$. 

\[\square\] [Claim]

Hence we have that if $S_{0,i,\delta}$ and $S_{0,i,\delta'}$ are stationary, then $\delta = \delta'$. For each $i < \kappa$, define $\delta_i < \mu$ as follows: if $S_{0,i,\delta}$ is stationary for some $\delta < \mu$, then let $\delta_i$ be a (unique) $\delta < \mu$ such that $S_{0,i,\delta}$ is stationary. If there is no such $\delta$, then let $\delta_i = 0$. Since $\mu = \text{cf}(\mu) > \kappa$, we know $\sup_{i < \kappa} \delta_i < \mu$. But this contradicts (1) of the claim. 

\[\square\]

**Proposition 4.3.** Let $S^*_0, S^*_1 \subseteq \mathcal{P}_\kappa \lambda$ be stationary and suppose $\text{RP}^2(S^*_0, S^*_1)$ holds. Then for every regular $\mu$ with $\kappa^+ \leq \mu \leq \lambda$, $\square(\mu)$ fails.

**Proof.** We prove only the case $\mu = \lambda$. Other cases follow from similar arguments.

Toward the contradiction, suppose $\square(\lambda)$ holds. Let $\langle C_\xi : \xi < \lambda \rangle$ be a $\square(\lambda)$-sequence.

Let $T = \{X \in \mathcal{P}_\kappa + \lambda : \text{cf}(X) = \kappa \subseteq X\}$. We assumed $\text{RP}^2(S^*_0, S^*_1)$, but by the previous proposition, in fact $\text{RP}^2(S^*_0, S^*_1, T)$ holds.

For each $\alpha < \lambda$ and $n < 2$, let

$$S_{n,\alpha} = \{x \in S^*_n : C_{\sup(x)} \cap \sup(x \cap \alpha) = C_\alpha \cap \sup(x \cap \alpha)\}.$$ 

Let $A_n = \{\alpha < \lambda : S_{n,\alpha}$ is stationary\}.

**Claim 4.4.** For each $n < 2$, $A_n$ is unbounded in $\lambda$.

**Proof.** Fix $n < 2$. By shrinking $S^*_n$ by a club in $\mathcal{P}_\kappa \lambda$, we may assume that the following:

1. For all $x \in S^*_n$ and $\alpha \in x$, if $x \cap \alpha$ is bounded in $\alpha$ then $\text{cf}(\alpha) \geq \kappa$.
2. For all $x \in S^*_n$ and $\alpha \in x \cap E^{2\kappa}$, $\sup(x \cap \alpha) \in \lim(C_\alpha)$ holds.

Let $T' = \{X \in T : S^*_n \cap \mathcal{P}_\kappa X$ is stationary\}. Then $T'$ is stationary in $\mathcal{P}_\kappa + \lambda$. To show that $A_n$ is unbounded, take $\xi < \lambda$. Fix $X \in T'$ with $\sup(X) > \xi$. Since $\text{cf}(X) = \kappa$, the set $\{\beta < \sup(X) : \beta \in \lim(C_{\sup(X)})\}$ contains a club in $\sup(X)$. Note that $C_{\sup(X)} \cap \beta = C_\beta$ for each $\beta$ from the club. Hence we know $S_X = \{x \in S^*_n \cap \mathcal{P}_\kappa X : C_{\sup(x)} = C_{\sup(X)} \cap \sup(x)\}$ is stationary in $\mathcal{P}_\kappa X$. Since $\text{cf}(\sup(X)) = \kappa$, $\lim(X) \cap \lim(C_{\sup(X)})$ is unbounded in $\sup(X)$. Take $\beta \in \lim(X) \cap \lim(C_{\sup(X)})$
with \( \beta > \xi \) and cf(\( \beta \)) < \( \kappa \). Note that \( \{ x \in \mathcal{P}_\kappa X : x \cap \beta \) is unbounded in \( \beta \} \) contains a club. Since \( \beta \in \lim(\sup(X)) \), \( \sup(X) \cap \beta = C_\beta \) holds. For each \( x \in S_X \) such that \( x \cap \beta \) is unbounded in \( \beta \) and \( \sup(x) > \beta \), let \( \beta_x = \min(x \setminus \beta) \).

**Case 1.** \( \beta_x = \beta \). Then \( C_{\beta_x} \cap \sup(x \cap \beta_x) = C_\beta = C_{\sup(X)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x) \).

**Case 2.** \( \beta_x > \beta \). Then \( \sup(x \cap \beta_x) = \beta \) and \( \beta = \sup(x \cap \beta) \in \lim(\beta_x) \), hence \( C_{\beta_x} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x) \).

Hence for each \( x \in S_X \) such that \( x \cap \beta \) is unbounded in \( \beta \) and \( \sup(x) > \beta \), we have \( C_{\sup(x)} \cap \sup(x \cap \beta_x) = C_{\beta_x} \cap \sup(x \cap \beta_x) \). By applying Fodor’s lemma to \( S_X \), we can find \( \beta_x \in X \) such that \( \{ x \in S_X : \beta x = \beta_x \} \) is stationary. Thus \( \{ x \in S^* \cap \mathcal{P}_\kappa X : C_{\sup(x)} \cap \sup(x \cap \beta_x) = C_{\sup(x)} \cap \sup(x \cap \beta_x) \} \)

By applying Fodor’s lemma to \( T' \), we have \( \beta_x < \lambda \) such that \( \{ X \in T' : \beta_x = \beta_X \} \) is stationary. Then \( S_n, \beta_x \) is stationary and \( \beta_x > \xi \). □[Claim]

**Claim 4.5.** For each \( \alpha \in A_0 \) and \( \beta \in A_1 \) with \( \alpha < \beta \), \( C_\alpha = C_\beta \cap \alpha \).

**Proof.** Let \( T^* = \{ X \in T : S_{0,\alpha} \cap \mathcal{P}_\kappa X, S_{1,\beta} \cap \mathcal{P}_\kappa X \) are stationary in \( \mathcal{P}_\kappa X \} \). Take \( X \in T^* \). Since \( D_X = \{ x \in \mathcal{P}_\kappa X : C_{\sup(X)} \cap \sup(x) = C_{\sup(x)} \} \) contains a club in \( \mathcal{P}_\kappa X \), \( D_X \cap S_{0,\alpha} \) is stationary in \( \mathcal{P}_\kappa X \). For \( x \in C_X \cap S_{0,\alpha} \), \( C_\alpha \cap \sup(x \cap \alpha) = C_{\sup(x)} \cap \sup(x \cap \alpha) \) holds. Since \( \{ \sup(x \cap \alpha) : x \in C_X \cap S_{0,\alpha} \} \) is unbounded in \( \sup(x \cap \alpha) \), we have \( C_{\sup(x)} \cap \sup(X \cap \alpha) = C_\alpha \cap \sup(X \cap \alpha) \).

Similarly, we have \( C_\beta \cap \sup(X \cap \beta) = C_{\sup(x)} \cap \sup(X \cap \beta) \). Therefore we have \( C_\alpha \cap \sup(X \cap \alpha) = C_\beta \cap \sup(X \cap \alpha) \).

Because \( \{ \sup(X \cap \alpha) : X \in T^* \} \) is unbounded in \( \alpha \), we have \( C_\alpha = C_\beta \cap \alpha \).

□[Claim]

Now, let \( C = \{ C_\beta : \beta \in A_0 \} \). Since \( A_0 \) is unbounded, \( C \) is unbounded. Furthermore, \( C_\alpha = C_\beta \cap \alpha \) for all \( \alpha < \beta \in A_1 \). For \( \alpha, \beta \in A_0 \) with \( \alpha < \beta \), choose \( \gamma \in A_1 \) with \( \beta < \gamma \). Then \( C_\alpha = C_\gamma \cap \alpha \) and \( C_\beta = C_\gamma \cap \alpha \). Thus \( C_\alpha = C_\beta \cap \alpha \).

Hence \( C \) forms a club in \( \lambda \). Take \( \alpha \in \lim(C) \). Then there exists \( \beta \in A_0 \) such that \( C \cap \alpha = C_\beta \cap \alpha \). Since \( \alpha \in \lim(C) \), we know \( \alpha \in \lim(C_\beta) \) and \( C_\alpha = C_\beta \cap \alpha = C \cap \alpha \). Thus \( \forall \alpha \in \lim(C) \) \( (C \cap \alpha = C_\alpha) \), this is a contradiction. □

Baumgartner[1] showed that if a weakly compact cardinal \( \kappa \) is collapsed to \( \omega_2 \) by Levy-collapse with countable conditions, then \( \text{RP}(\mathcal{P}_\omega \omega_2) \) holds, and it is known that in fact \( \text{RP}^2(\mathcal{P}_\omega \omega_2, \mathcal{P}_\omega \omega_2) \) holds in the generic extension. Conversely, Velickovic [3] showed that if \( \text{RP}(\mathcal{P}_\omega \omega_2) \) holds, then \( \omega_2 \) is weakly compact in \( L \). Consequently, we have the following equiconsistency:
Corollary 4.6. The following are equiconsistent:

1. ZFC + “there exists a weakly compact cardinal”.
2. ZFC + “RP($\mathcal{P}_{\omega_1}\omega_2$) holds”.
3. ZFC + “RP2($\mathcal{P}_{\omega_1}\omega_2$) holds”.
4. ZFC + “RP2($S_0^*, S_1^*$) holds for some stationary sets $S_0^*, S_1^* \subseteq \mathcal{P}_{\omega_1}\omega_2$”.

5. Proof of Theorem 1.8

Proposition 5.1. Suppose RP($S^*$) for some stationary $S^* \subseteq \mathcal{P}_\kappa\lambda$. Then every $\kappa$-c.c. forcing preserves RP($S^*$).

Proof. First note that every $\kappa$-c.c. forcing preserves the stationarity of $S^*$.

Let $\mathbb{P}$ be a poset which satisfies the $\kappa$-c.c. Let $\dot{S} \in \mathbb{P}$-name such that $\Vdash \"\dot{S} \subseteq S^*\"$ is stationary”. It is enough to show that there are some $p \in \mathbb{P}$ and $X \subseteq \mathcal{P}_\kappa^+\lambda$ such that $p \Vdash \"\dot{S} \cap \mathcal{P}_\kappa^+X \text{ is stationary in } \mathcal{P}_\kappa^+X\"$.

Let $S' = \{ x \in S^* : \exists p \in \mathbb{P} (p \Vdash \"x \in \dot{S}\") \}$. It is easy to check that $S'$ is a stationary subset of $S^*$. By RP($S'$), there is $X \subseteq \mathcal{P}_\kappa^+\lambda$ such that $|X| = \kappa \subseteq X$ and $S' \cap \mathcal{P}_\kappa^+X$ is stationary in $\mathcal{P}_\kappa^+X$. We see that $p \Vdash \"\dot{S} \cap \mathcal{P}_\kappa^+X \text{ is stationary}\"$ for some $p \in \mathbb{P}$. Suppose to the contrary that $p \Vdash \"\dot{S} \cap \mathcal{P}_\kappa^+X \text{ is non-stationary}\"$.

Since $|X| = \kappa$ and $\mathbb{P}$ satisfies the $\kappa$-c.c., we can find a club $C \subseteq \mathcal{P}_\kappa^+X$ such that $p \Vdash \"\dot{S} \cap C = \emptyset\"$. $S' \cap \mathcal{P}_\kappa^+X$ is stationary, hence there is $x \in S' \cap C$. Pick $p \in \mathbb{P}$ with $p \Vdash \"x \in \dot{S}\"$. Then $p \Vdash \"x \in \dot{S} \cap C\"$, this is a contradiction. \qed

Recall that PFA++ is the assertion that for every proper forcing notion $\mathbb{P}$, every dense subsets $D_i$ ($i < \omega_1$) of $\mathbb{P}$, and every $\mathbb{P}$-names $\dot{S}_i$ ($i < \omega_1$) for stationary subsets of $\omega_1$, there is a filter $F$ on $\mathbb{P}$ such that:

1. $D_i \cap F \neq \emptyset$ for every $i < \omega_1$.
2. $S_i = \{ \alpha < \omega_1 : \exists p \in F (p \Vdash \"\alpha \in \dot{S}_i\") \}$ is stationary in $\omega_1$ for $i < \omega_1$.

Proposition 5.2. Suppose PFA++. Let $\lambda \geq \omega_2$. Then every c.c.c. forcing notion forces RP2($\mathcal{P}_{\omega_1}^\lambda, \mathcal{P}_{\omega_2}^\lambda$).

Proof. Let $\mathbb{P}$ be a poset which satisfies the c.c.c. Let $\dot{S}_0, \dot{S}_1$ be $\mathbb{P}$-names so that $\Vdash \"\dot{S}_0, \dot{S}_1 \subseteq \mathcal{P}_{\omega_1}^\lambda \text{ are stationary}\"$. We will find $p \in \mathbb{P}$ and $X \subseteq \mathcal{P}_{\omega_2}^\lambda$ such that $p \Vdash \"\dot{S}_0 \cap \mathcal{P}_{\omega_1}X, \dot{S}_1 \cap \mathcal{P}_{\omega_1}X \text{ are stationary}\"$.

Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a $\sigma$-closed poset which adds a bijection from $\omega_1$ to $\lambda$. We know that $\Vdash_{\mathbb{P}\ast\dot{\mathbb{Q}}} \"\dot{S}_0, \dot{S}_1 \text{ remain stationary}\"$. Fix a $\mathbb{P}\ast\dot{\mathbb{Q}}$-name $\dot{\pi}$ for a bijection from $\omega_1$ to $\lambda$. Let $E_0, E_1$ be $\mathbb{P}\ast\dot{\mathbb{Q}}$-names such that $\Vdash_{\mathbb{P}\ast\dot{\mathbb{Q}}} \"E_i = \{ \alpha < \omega_1 : \dot{\pi}\alpha \in \dot{S}_i, \dot{\pi}\alpha \cap \omega_1 = \alpha \}\"$ for $i = 0, 1$. We know $\Vdash_{\mathbb{P}\ast\dot{\mathbb{Q}}} \"E_i \text{ is stationary in } \omega_1\"$.\qed


Now fix a sufficiently large regular cardinal \( \theta \) and take \( M \prec H_\theta \) such that \( |M| = \omega_1 \subseteq M \) and \( M \) contains all relevant objects.

\( \mathbb{P} \ast \mathbb{Q} \) is proper, hence we can apply PFA++ to \( \mathbb{P} \ast \mathbb{Q} \) and \( \check{E}_i \). By PFA++ we can find a filter \( F \) on \( \mathbb{P} \ast \mathbb{Q} \) such that:

1. \( F \cap D \neq \emptyset \) for all dense \( D \in M \) in \( \mathbb{P} \ast \mathbb{Q} \).
2. \( E_i = \{ \alpha < \omega_1 : \exists p \in F (p \Vdash_{\mathbb{P} \ast \mathbb{Q}} \alpha \in \check{E}_i) \} \) is stationary in \( \omega_1 \) for \( i = 0, 1 \).

Let \( X = \{ \beta < \lambda : \exists p \in F \exists \alpha < \omega_1 (p \Vdash_{\mathbb{P} \ast \mathbb{Q}} \check{\pi}(\alpha) = \beta) \} \). We can check that \( |X| = \omega_1 \subseteq X \).

Since \( \check{S}_0, \check{S}_1 \) are names for subsets of \( \mathcal{P}_{\omega_1}^V \lambda \), for each \( \alpha \in E_i \), we can find \( x \in \mathcal{P}_{\omega_1} \lambda \) and \( p \in F \) such that \( x \cap \omega_1 = \alpha \) and \( p \Vdash_{\mathbb{P} \ast \mathbb{Q}} \check{\pi} \check{\pi}(\alpha) = x \). Moreover it is easy to see that \( x \in \mathcal{P}_{\omega_1} X \).

For \( i < 2 \) and \( \alpha \in E_i \), take \( x_{i,\alpha} \in \mathcal{P}_{\omega_1} X \) such that there is \( p \in F \) with \( p \Vdash_{\mathbb{P} \ast \mathbb{Q}} \check{\pi} \check{\pi}(\alpha) = x_{i,\alpha} \). Let \( S_i = \{ x_{i,\alpha} : \alpha \in E_i \} \). The following are easy to check for \( i < 2 \):

1. \( x_{i,\alpha} \subseteq x_{i,\beta} \) holds for \( \alpha, \beta \in E_i \) with \( \alpha < \beta \).
2. If \( \alpha \in \text{lim}(E_i) \cap E_i \), then \( x_{i,\alpha} = \bigcup_{\beta \in E_i \cap \alpha} x_{i,\beta} \).
3. \( \bigcup S_i = X \).

Furthermore, since \( E_i = \{ x_{i,\alpha} \cap \omega_1 : \alpha \in E_i \} \) is stationary in \( \omega_1 \), we can check that each \( S_i \) is stationary in \( \mathcal{P}_{\omega_1} X \).

Now we see that \( p \Vdash_{\mathbb{P}} \check{S}_0 \cap \mathcal{P}_{\omega_1} X, \check{S}_1 \cap \mathcal{P}_{\omega_1} X \) are stationary” for some \( p \in \mathbb{P} \). Suppose otherwise. Since \( \mathbb{P} \) satisfies the c.c.c. and \( |X| = \omega_1 \), we can find a club \( C \) in \( \mathcal{P}_{\omega_1} X \) such that \( p \Vdash_{\mathbb{P}} C \cap \check{S}_0 = \emptyset \) or \( C \cap \check{S}_1 = \emptyset \).

Since \( S_0 \) and \( S_1 \) are stationary in \( \mathcal{P}_{\omega_1} X \), we can find \( x_0 \in S_0 \cap C \) and \( x_1 \in S_1 \cap C \).

Then there is \( q \in F \) such that \( q \Vdash_{\mathbb{P} \ast \mathbb{Q}} x_0 \in \check{S}_0 \) and \( x_1 \in \check{S}_1 \). Thus \( q \Vdash_{\mathbb{P} \ast \mathbb{Q}} C \cap \check{S}_0 \neq \emptyset \) and \( C \cap \check{S}_1 \neq \emptyset \), this is a contradiction. \( \square \)

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(T. Usuba) INSTITUTE FOR ADVANCED RESEARCH, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSAKU, NAGOYA, 464-8601, JAPAN

E-mail address: usuba@math.cm.is.nagoya-u.ac.jp