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PARTIAL STATIONARY REFLECTION PRINCIPLES

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1. INTRODUCTION

Throughout this paper, \( \kappa \) denotes a regular uncountable cardinal and \( \lambda \) a cardinal \( \geq \kappa^+ \), unless otherwise specified.

Partial stationary reflection on \( \mathcal{P}_{\omega_1}\omega_2 \) was introduced by H. Sakai [2]. First we extend his notion to arbitrary \( \kappa \) and \( \lambda \).

**Definition 1.1.** Let \( S^* \) be a stationary subset of \( \mathcal{P}_\kappa \lambda \). For a stationary set \( T \subseteq \mathcal{P}_\kappa + \lambda \), we say that \( \text{RP}(S^*, T) \) holds if for every stationary subset \( S \subseteq S^* \) there exists \( X \in T \) such that \( \kappa \subseteq X \) and \( S \cap \mathcal{P}_\kappa X \) is stationary in \( \mathcal{P}_\kappa X \). \( \text{RP}(S^*) \) means \( \text{RP}(S^*, \mathcal{P}_{\kappa^+}\lambda) \).

It is known that total stationary reflection \( \text{RP}(\mathcal{P}_\kappa \lambda) \) is a large cardinal property (e.g., see Velicikovic [3]), but Sakai [2] showed that partial stationary reflection on \( \mathcal{P}_{\omega_1}\omega_2 \) is not:

**Fact 1.2** ([2]). Suppose CH. If \( \square_{\omega_1} \) holds, then there are a stationary set \( S^* \subseteq \mathcal{P}_{\omega_1}\omega_2 \) and a \( \sigma \)-Baire, \( \omega_2 \)-c.c. poset \( \mathbb{P} \) such that \( \mathbb{P} \) forces \( \text{RP}(S^*) \).

In this paper, we generalize his result as follows:

**Theorem 1.3.** Suppose \( \kappa^{<\kappa} = \kappa \). Let \( T \subseteq \mathcal{P}_{\kappa^+}\lambda \) be a stationary set such that \( \forall X \in T (\kappa \subseteq X) \). Then there exists a \( \kappa \)-closed, \( \kappa^+ \)-c.c. poset which forces the following statements:

1. \( T \) is stationary.
2. There exists a stationary set \( S^* \subseteq \mathcal{P}_\kappa \lambda \) such that
   a. \( \forall X \in T (S^* \cap \mathcal{P}_\kappa X \) contains a club in \( \mathcal{P}_\kappa X \),
   b. \( \text{RP}(S^*, T) \) holds.
This theorem shows that, even $\kappa > \omega_1$ and $\lambda > \omega_2$, our partial stationary reflection is not a large cardinal property.

Next we consider a natural strengthening of partial stationary reflection, simultaneous partial stationary reflection.

**Definition 1.4.** For stationary sets $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda$ and $T \subseteq \mathcal{P}_{\kappa^+} \lambda$, we say that $\text{RP}^2(S_0^*, S_1^*, T)$ holds if for every stationary subsets $S_0 \subseteq S_0^*$ and $S_1 \subseteq S_1^*$ in $\mathcal{P}_\kappa \lambda$, there exists $X \in T$ such that $\kappa \subseteq X$ and both $S_0 \cap \mathcal{P}_\kappa X$ and $S_1 \cap \mathcal{P}_\kappa X$ are stationary in $\mathcal{P}_\kappa X$. $\text{RP}^2(S_0^*, S_1^*)$ means $\text{RP}^2(S_0^*, S_1^*, \mathcal{P}_{\kappa^+} \lambda)$.

We prove that our simultaneous partial stationary reflection is a large cardinal property by showing the following:

**Definition 1.5.** For a regular uncountable cardinal $\mu$, $\square(\mu)$ holds if there exists a sequence $\langle C_\xi : \xi < \mu \rangle$ satisfying the following:

1. for all $\xi < \mu$, $C_\xi$ is club in $\xi$ and for all $\eta \in \text{lim}(C_\xi)$, $C_\eta = C_\xi \cap \eta$,
2. for all club $C$ in $\mu$, there exists $\xi \in \text{lim}(C)$ such that $C \cap \xi \neq C_\xi$.

Such an sequence $\langle C_\xi : \xi < \mu \rangle$ is called a $\square(\mu)$-sequence.

**Theorem 1.6.** Suppose $\text{RP}^2(S_0^*, S_1^*)$ holds for some stationary $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda$. Then for every regular $\mu$ with $\kappa^+ \leq \mu \leq \lambda$, $\square(\mu)$ fails.

We also prove the following:

**Theorem 1.7.** For every stationary $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda$ and regular $\mu$ with $\kappa^+ \leq \mu \leq \lambda$, $\text{RP}^2(S_0^*, S_1^*, \{X \in \mathcal{P}_{\kappa^+} \lambda : \text{cf}(X \cap \mu) < \kappa\})$ fails, where $\text{cf}(X) = \text{cf}(\text{ot}(X))$.

Todorcevic showed that $\text{RP}(\mathcal{P}_{\omega_1} \omega_2)$ implies that $2^\omega \leq \omega_2$. However we prove the following, which shows that our partial stationary reflection does not affect the size of the continuum:

**Theorem 1.8.**
1. Suppose $\text{RP}(S^*)$ for some stationary $S^* \subseteq \mathcal{P}_\kappa \lambda$. Then every $\kappa$-c.c. forcing preserves $\text{RP}(S^*)$.
2. Suppose $\text{PFA}^{++}$. Let $\lambda \geq \omega_2$. Then every c.c.c. forcing notion forces $\text{RP}^2(\mathcal{P}_{\omega_1}^\nu \lambda, \mathcal{P}_{\omega_1}^\nu \lambda)$.

2. Preliminaries

For a set $X$ of ordinals, let $\text{cf}(X) = \text{cf}(\text{ot}(X))$.

For regular cardinals $\nu < \mu$, let $E_\nu^\mu = \{\alpha < \mu : \text{cf}(\alpha) = \nu\}$ and $E_{<\nu}^\mu = \{\alpha < \mu : \text{cf}(\alpha) < \nu\}$.
The proofs of the following lemmata are easy:

**Lemma 2.1.** For a stationary $S \subseteq \mathcal{P}_\kappa \lambda$ and a $\kappa$-c.c. poset $\mathbb{P}$, $\mathbb{P}$ preserves the stationarity of $S$.

**Lemma 2.2.** For $S \subseteq \mathcal{P}_\kappa \lambda$, if $\{X \in \mathcal{P}_{\kappa+} \lambda : S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X\}$ is stationary in $\mathcal{P}_{\kappa+} \lambda$, then $S$ is stationary in $\mathcal{P}_\kappa \lambda$.

**Lemma 2.3.** For stationary sets $S^* \subseteq \mathcal{P}_\kappa \lambda$ and $T \subseteq \mathcal{P}_{\kappa+} \lambda$, suppose $RP(S^*, T)$ holds. Then for every stationary $S \subseteq S^*$, $\{X \in T : S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X\}$ is stationary in $\mathcal{P}_{\kappa+} \lambda$.

We define club shootings into $\mathcal{P}_\kappa \lambda$, which was observed in [2].

**Definition 2.4.** For $S \subseteq \mathcal{P}_\kappa \lambda$, let $\mathbb{C}(S)$ be the poset which consists of all functions $p$ such that:

1. $|p| < \kappa$,
2. $p : d(p) \times d(p) \to \kappa$ for some $d(p) \in \mathcal{P}_\kappa \lambda$, and
3. $\forall x \subseteq d(p) (x \in S \Rightarrow x$ is not closed under $p)$.

For $p, q \in \mathbb{C}(S)$, $p \leq q$ $\iff$ $q \subseteq p$.

Let $\mathbb{C} = \mathbb{C}(\emptyset)$.

**Lemma 2.5.**

1. $\mathbb{C}(S)$ satisfies the $(2^{<\kappa})^+\text{-}c.c.$
2. For every $x \in \mathcal{P}_\kappa \lambda$, $\{p \in \mathbb{C}(S) : x \subseteq d(p)\}$ is a dense open set in $\mathbb{C}(S)$.
3. Whenever $G$ is $(V, \mathbb{C}(S))$-generic, $\bigcup G$ is a function from $\lambda \times \lambda$ to $\kappa$, and every $x \in S$ is not closed under the function.

**Proof.** For (1), take $A \subseteq \mathbb{C}(S)$ with size $(2^{<\kappa})^+$. By $\Delta$-system lemma, we can find $B \subseteq A$ and $a \in \mathcal{P}_\kappa \lambda$ such that $|B| = (2^{<\kappa})^+$ and $d(p) \cap d(q) = a$ for every distinct $p, q \in B$. Moreover we may assume that $p|a \times a = q|a \times a$ for every $p, q \in B$. We check that $B$ is a pairwise compatible set.

Take $p, q \in B$. Pick $\alpha < \kappa$ with $\alpha > \sup(d(p) \cap \kappa) + 1, \sup(d(q) \cap \kappa) + 1$. Then define $r$ as $\text{dom}(r) = (d(p) \cup d(q)) \times (d(p) \cup d(q))$ and

$$r(\xi, \eta) = \begin{cases} p(\xi, \eta) & \text{if } \xi, \eta \in d(p). \\ q(\xi, \eta) & \text{if } \xi, \eta \in d(q). \\ \alpha & \text{otherwise.} \end{cases}$$

We have $r \leq p, q$. (2) follows from a similar argument, and (3) is straightforward. \qed
3. The proof of Theorem 1.3

Suppose $\kappa^{<\kappa} = \kappa$. Fix a stationary set $T \subseteq P_{\kappa^+\lambda}$ such that $\forall X \in T (\kappa \subseteq X)$.

We consider the following poset $\mathbb{P}_T$, which adds a new stationary subset $S^*$ of $P_{\kappa}\lambda$.

Definition 3.1. $\mathbb{P}_T$ is the set of all functions $p$ satisfying the following:

1. $|p| < \kappa$ and $\text{dom}(p) \subseteq T$,
2. for every $X \in \text{dom}(p)$, $p(X)$ is a $\subseteq$-increasing continuous set $\{x_i : i \leq \gamma\}$ in $\mathcal{P}_\kappa X$ such that $\gamma < \kappa$ and $x_i \cap \kappa \in \kappa$ for all $i \leq \gamma$.

For $p \in \mathbb{P}_T$ and $X \in \text{dom}(p)$, $\max(p(X))$ denotes the maximum element of $p(X)$.

Let $u(p) = \bigcup \{p(X) : X \in \text{dom}(P)\}$. Note that $u(p) \subseteq P_{\kappa}\lambda$ and $|u(p)| < \kappa$. For $p, q \in \mathbb{P}_T$, define $p \leq q$ if

- (a) $\text{dom}(p) \supseteq \text{dom}(q)$,
- (b) $\forall X \in \text{dom}(q) (q(X) = \{x \in p(X) : x \subseteq \max(q(X))\})$ (hence $u(p) \supseteq u(q)$),
- (c) $\forall x \in u(p) (x \subseteq \bigcup u(q) \Rightarrow x \in u(q))$,
- (d) $\forall X \in \text{dom}(p) \setminus \text{dom}(q) (\max(p(X)) \not\subseteq \bigcup u(q))$,
- (e) $\forall X \in \text{dom}(q) \forall x \in p(X) \setminus q(X) (x \not\subseteq \bigcup u(q))$.

Lemma 3.2. (1) $\mathbb{P}_T$ is $\kappa$-closed,
(2) $\mathbb{P}_T$ satisfies the $\kappa^+$-c.c. (if $\kappa^{<\kappa} = \kappa$),
(3) for all $X \in T$ and $x \in \mathcal{P}_\kappa X$, $\{p \in \mathbb{P}_T : X \in \text{dom}(p) \text{ and } x \subseteq \max(p(X))\}$ is dense in $\mathbb{P}_T$.

Proof. (1). Let $\gamma < \kappa$ be a limit ordinal and $(p_i : i < \gamma)$ be a decreasing sequence in $\mathbb{P}_T$. Then define the function $p^*$ as the following manner:

- (i) $\text{dom}(p^*) = \bigcup_{i<\gamma} \text{dom}(p_i)$,
- (ii) for $X \in \text{dom}(p^*)$, $p^*(X) = \bigcup \{p_i(X) : i < \gamma, X \in \text{dom}(p_i)\} \cup \{\max(p_i(X)) : i < \gamma, X \in \text{dom}(p_i)\}$.\]

Since the $p_i$'s are decreasing, it is easy to show that $p^* \in \mathbb{P}_T$. For $i < \gamma$, we show $p \leq p_i$. It is easily verified that the conditions (a) and (b) in the definition of the order are satisfied.

(c). Take $x \in u(p^*)$ such that $x \subseteq \bigcup u(p_i)$. Take $X \in \text{dom}(p^*)$ such that $x \in p^*(X)$. If $x \neq \max(p^*(X))$, then $x \in p_j(X)$ for some $j > i$ with $X \in \text{dom}(p_j)$. Since $p_j \leq p_i$, we have $x \in p_i(X)$. Next suppose $x = \max(p^*(X))$. Take $k < \gamma$ such that $i < k$ and $X \in \text{dom}(p_k)$. Then $\max(p_k(X)) \subseteq \max(p^*(X)) = x \subseteq \bigcup u(p_i)$ holds. Hence $X \in \text{dom}(p_i)$ by (d). For each $j \geq i$, $\max(p_j(X)) \subseteq \max(p^*(X)) = x \subseteq$
\[ \bigcup u(p_i) \] holds. Thus we have \( \max(p_j(X)) \in p_i(X) \) by (e). Therefore \{ \max(p_j(X)) : i \leq j < \gamma \} \subseteq p_i(X), and we have \( \max(p^*(X)) = \bigcup \{ \max(p_j(X)) : i \leq j < \gamma \} \in p_i(X). \)

(d). Take \( X \in \text{dom}(p^*) \setminus \text{dom}(p_i) \). Then there exists \( j > i \) such that \( X \in \text{dom}(p_j) \). We know \( \max(p_j(X)) \notin \bigcup u(p_i) \). Because \( \max(p_j(X)) \subseteq \max(p^*(X)) \), we know \( \max(p^*(X)) \notin \bigcup u(p_i) \).

(e). Take \( X \in \text{dom}(p_i) \) and \( x \in p^*(X) \setminus p_i(X) \). Then there exist \( j \geq i \) and \( y \in \text{dom}(p_j) \) such that \( y \subseteq x \) and \( y \notin p_i(X) \). Hence \( y \notin \bigcup u(p_i) \) and \( x \notin \bigcup u(p_i) \).

(2). Take an arbitrary \( A \subseteq \mathbb{P}_T \) with \( |A| \geq \kappa^+ \). We prove that \( A \) is not an antichain. By \( \Delta \)-system lemma, we can find \( r \in \mathbb{P}_T \), \( s \in \mathbb{P}_\kappa \lambda \), and \( B \subseteq A \) with \( |B| \geq \kappa^+ \) such that \( \forall p, q \in B \ (\text{dom}(p) \cap \text{dom}(q) = r \) and \( \bigcup u(p) \cap \bigcup u(q) = s) \). By our cardinal arithmetic assumption, there exists \( C \subseteq B \) with \( |C| \geq \kappa^+ \) such that \( \forall p, q \in B (\forall X \in r \ (p(X) = q(X)) \) and \( \mathcal{P}_\kappa s \cap u(p) = \mathcal{P}_\kappa s \cap u(q)) \). We check that any two elements of \( C \) are pairwise compatible. Take \( p, q \in C \). For each \( X \in \text{dom}(p) \cup \text{dom}(q) \), fix \( a_X \in \mathcal{P}_\kappa X \) such that \( (\bigcup u(p) \cup \bigcup u(q)) \cap X \not\subseteq a_X \). Define the function \( r \) as the following:

(i) \( \text{dom}(r) = \text{dom}(p) \cup \text{dom}(q), \)

(ii) \( r(X) = p(X) \cup \{ a_X \} \) if \( X \in \text{dom}(p) \), and \( r(X) = q(X) \cup \{ a_X \} \) if \( X \in \text{dom}(q) \).

This is well-defined because \( p(X) = q(X) \) for all \( X \in \text{dom}(p) \cap \text{dom}(q) \). We see that \( r \) is a lower bound of \( p \) and \( q \). \( r \in \mathbb{P}_T \) is easily verified. For \( r \leq p \), the conditions (a) and (b) are clear.

(c). Take \( x \in \text{dom}(r) \) such that \( x \not\in \bigcup u(p) \). Then \( x \not\in a_X \) for all \( X \in \text{dom}(p) \cup \text{dom}(q) \). Hence \( x \in \text{dom}(p) \cup \text{dom}(q) \). If \( x \in \text{dom}(p) \) then we have done. Assume \( x \in \text{dom}(q) \). Then \( x \not\in \bigcup u(q) \). Since \( x \not\in \bigcup u(p) \), we have \( x \not\in \bigcup u(p) \cup \bigcup u(q) = s \) and \( x \in \mathcal{P}_\kappa s \).

Because \( \mathcal{P}_\kappa s \cap u(p) = \mathcal{P}_\kappa s \cap u(q) \), we have \( x \in \mathcal{P}_\kappa s \cap u(p) \) and \( x \in \text{dom}(p) \).

(d). Take \( X \in \text{dom}(r) \setminus \text{dom}(p) \). Then \( \max(r(X)) = a_X \supsetneq \bigcup u(p) \cap X \), thus \( \max(r(X)) \notin \bigcup u(p) \).

(e). Take \( X \in \text{dom}(p) \) and \( x \in r(X) \setminus p(X) \). By the definition of \( r(X) \), we have \( r(X) = p(X) \cup \{ a_X \} \). Hence \( x = a_X \notin \bigcup u(p) \).

\( r \leq q \) can be proved by the same argument.

(3). Take \( X \in T, x \in \mathcal{P}_\kappa X \) and \( q \in \mathbb{P} \). Take \( x^* \in \mathcal{P}_\kappa X \) such that \( \bigcup u(q) \cap X \not\subseteq x^* \). Define \( p \) as \( \text{dom}(p) = \text{dom}(q) \cup \{ X \} \), \( p|\text{dom}(q) = q \) and \( p(X) = \{ x^* \} \) if \( X \notin \text{dom}(q) \), and \( q(X) \cup \{ x^* \} \) if \( X \in \text{dom}(q) \). Then \( p \leq q \) can be verified. \[ \square \]
Note that the following: For \( \gamma < \kappa \) and a decreasing sequence \( \langle p_i : i < \gamma \rangle \) in \( \mathbb{P}_T \), let \( p^* \) be a lower bound of the \( p_i \)'s as constructed in the proof of (1) above. Then \( p^* \) is the largest lower bound of the \( p_i \)'s and \( \bigcup u(p^*) = \bigcup_{i<\gamma}(\bigcup u(p_i)) \).

**Definition 3.3.** For a canonical name of \((V, \mathbb{P}_T)\)-generic filter \( \dot{G} \), let \( \dot{S}^* \) be a \( \mathbb{P}_T \)-name such that

\[
\forces_{\mathbb{P}_T} \langle \dot{S}^* \rangle = \bigcup \{ u(p) : p \in \dot{G} \}.
\]

The following are easily verified by the definition of \( \mathbb{P}_T \).

**Lemma 3.4.** (1) \( \forces_{\mathbb{P}_T} \langle \forall X \in T (\dot{S}^* \cap \mathcal{P}_\kappa X \text{ contains a club in } \mathcal{P}_\kappa X) \rangle \),

(2) for all \( p \in \mathbb{P}_T \), \( p \forces_{\mathbb{P}_T} \langle \{ y \in \dot{S}^* : y \subseteq \bigcup u(p) \} = u(p) \rangle \).

Now fix a name \( \dot{S} \) such that

\[
\forces_{\mathbb{P}_T} \langle \dot{S} \subseteq \dot{S}^* \rangle \text{ and } \forall X \in T (\mathcal{P}_\kappa X \cap \dot{S} \text{ is non-stationary in } \mathcal{P}_\kappa X) \rangle.
\]

We see that \( \mathbb{P}_T \ast \mathcal{C}(\dot{S}) \) has good properties.

For each \( X \in T \), fix a name \( \dot{g}_X \) such that

\[
\forces_{\mathbb{P}_T} \langle \dot{g}_X : [X]^{<\omega} \rightarrow X \text{ and } \forall x \in \mathcal{P}_\kappa X \ (x \text{ is closed under } \dot{g}_X \Rightarrow x \notin \dot{S}) \rangle.
\]

Let \( \dot{Q} \) be a name such that \( \forces \langle \dot{Q} = \mathcal{C}(\dot{S}) \rangle \). We prove that \( \mathbb{P}_T \ast \dot{Q} \) has a \( \kappa \)-closed dense subset.

**Lemma 3.5.** Let \( D = \{ p \in \mathbb{P}_T : \forall X \in \text{dom}(p) \ (p \forces_{\mathbb{P}_T} \langle \max(p(X)) \text{ is closed under } \dot{g}_X \rangle) \} \). Then \( D \) is dense in \( \mathbb{P}_T \).

**Proof.** Take \( p \in \mathbb{P}_T \). We want to find \( q \in D \) such that \( q \leq p \). We take a decreasing sequence \( p_i \ (i < \omega) \) in \( \mathbb{P}_T \) by induction on \( i < \omega \). Let \( p_0 = p \). Suppose \( p_i \) is defined. By the \( \kappa \)-closedness of \( \mathbb{P}_T \), we can choose \( p' \leq p_i \) and \( a \in \mathcal{P}_\kappa \lambda \) such that \( p' \forces_{\mathbb{P}_T} \langle [\max(p_i(X))]^{<\omega} \subseteq a \cap X \rangle \) for all \( X \in \text{dom}(p_i) \). Then choose \( p_{i+1} \leq p' \) such that \( a \cap X \subseteq \max(p_{i+1}(X)) \) for all \( X \in \text{dom}(p_i) \).

Finally let \( q \) be the greatest lower bound of the \( p_i \)'s. By our construction, it is easy to see that \( q \in D \). \( \square \)

**Lemma 3.6.** Let \( D \) be as in Lemma 3.5. Let \( D' = \{ \langle p, q \rangle \in \mathbb{P}_T \ast \dot{Q} : p \in D, \ q = \check{r} \text{ for some } r \in \mathcal{C} \text{ and } d(r) = \bigcup (u(p)) \} \). Then \( D' \) is a \( \kappa \)-closed dense subset in \( \mathbb{P}_T \ast \dot{Q} \).

**Proof.** Density: Take \( \langle p, q \rangle \in \mathbb{P}_T \ast \dot{Q} \). Take \( p' \in D \) and \( r \) such that \( p' \forces_{\mathbb{P}_T} \langle \check{r} = \check{q} \rangle \) and \( \bigcup u(p') \supseteq d(r) \). Now define \( r' \) as the following:
(1) $r': \bigcup u(p') \times \bigcup u(p') \rightarrow \kappa,$
(2) for $a \in \bigcup u(p') \times \bigcup u(p')$, if $a \in d(r) \times d(r)$ the $r'(a) = r(a)$, otherwise
$r'(a) = \sup(\bigcup(u(p') \cap \kappa)) + 1.$

It is easy to show that $p' \vDash "\gamma \in C(\dot{\mathcal{S}})" \text{ and } \langle p', \dot{p}' \rangle \leq \langle p, \dot{q} \rangle.$

Next we prove $D'$ is $\kappa$-closed. Let $\gamma < \kappa$ and $\langle p_i, \dot{q}_i \rangle (i < \gamma)$ be a decreasing sequence in $D'$. We show that this sequence has a lower bound. Let $p^* \in \mathbb{P}_T$ be the greatest lower bound of the $p_i$'s. Note that for all $X \in \text{dom}(p^*)$, $p^* \vDash \text{"max}(p^*(X)) \text{ is closed under } \dot{g}_X."$

Let $q^* = \bigcup_{i<\gamma} q_i$. $q^*$ is a function with the domain $d(q^*) \times d(q^*)$, where $d(q^*) = \bigcup_{i<\gamma} d(q_i)$. Notice that $d(q^*) = \bigcup_{i<\gamma} d(q_i) = \bigcup_{i<\gamma} \bigcup u(p_i) = \bigcup u(p^*)$. We complete the proof by showing the following claim.

**Claim 3.7.** $p^* \vDash "q^* \in C(\dot{\mathcal{S}})".$

**Proof.** Take a $(V, \mathbb{P}_T)$-generic $G$ with $p^* \in G$ and work in $V[G]$. First note that
\[
\{ x \in S^* : x \subseteq \bigcup u(p^*) \} = u(p^*).
\]
To show that $q^* \in C(S)$, take $x \subseteq d(q^*)$ with $x \in S$. We check that $x$ is not closed under $q^*$. Since $x \subseteq d(q^*) = \bigcup u(p^*)$ and $x \in S \subseteq S^*$, we have $x \in u(p^*)$. Hence there exists $X \in \text{dom}(p^*)$ such that
\[
x \in p^*(X).
\]
Because $\text{max}(p^*(X))$ is closed under $\dot{g}_X$, we know $\text{max}(p^*(X)) \notin S$. Thus $x \neq \text{max}(p^*(X))$ and $x \in p_i(X)$ for some $i < \gamma$ with $X \in \text{dom}(p_i)$. Then $x \subseteq \bigcup u(p_i) = d(q_i)$. Since $q_i$ is a condition, $x$ is not closed under $q_i$, and not closed under $q^*$.

\[
\square\text{[Claim]}
\]

Note that, in fact, $D'$ is $\kappa$-directed closed.

By an iteration of the above forcing, we can prove Theorem 1.3. Let $\langle \mathbb{P}_\xi, \dot{Q}_\eta : \xi < \zeta, \eta < \zeta \rangle$ be a $\kappa$-support iteration such that for every $\xi < \zeta$,

(1) $\dot{Q}_0 = \mathbb{P}_T$,
(2) $\mathbb{P}_\xi$ satisfies the $\kappa^+$-c.c. and has a $\kappa$-closed dense subset,
(3) for $\xi > 0$ there exists $\mathbb{P}_\xi$-name $\dot{S}_\xi$ such that
\[
\vDash \xi \text{"} \dot{S}_\xi \subseteq \dot{\mathcal{S}} \text{ and } \forall X \in T (\mathcal{P}_\kappa X \cap \dot{S}_\xi \text{ is non-stationary in } \mathcal{P}_\kappa X),"\]
(4) for every $X \in T$, $\dot{g}_X^\xi$ is a $\mathbb{P}_\xi$-name such that
\[
\vDash \xi \text{"} \dot{g}_X^\xi : [X]^{<\omega} \rightarrow X \text{ and } \forall x \in \mathcal{P}_\kappa X (x \in \dot{S}_\xi \Rightarrow x \text{ is not closed under } \dot{g}_X^\xi,"\]
(5) $\vDash \xi \text{"} \dot{Q}_\xi = C(\dot{\mathcal{S}}_\xi)"$ for $\xi > 0$,
(6) let $D_\xi$ is the set of all $p \in \mathbb{P}_\xi$ such that
(a) \( \forall \eta \in \text{supp}(p) \setminus \{0\} \ (p(\eta) = \hat{r} \) for some \( r \in C \),
(b) for all \( X \in \text{dom}(p(0)) \) and \( \eta \in \text{supp}(p) \setminus \{0\} \) (\( p|_\eta \vdash \text{"max}(p(0)(X)) \) is closed under \( \bar{g}^\eta_X \)),
(c) \( \bigcup(u(p(0))) = d(p(\eta)) \) for all \( \eta \in \text{supp}(p) \setminus \{0\} \).

Then \( D_\xi \) is a \( \kappa \)-closed dense set in \( \mathbb{P}_\xi \).

Let \( \mathbb{P}_\xi \) and \( D_\xi \) be as intended. We can check that \( D_\xi \) is a \( \kappa \)-closed dense set in \( \mathbb{P}_\xi \), and \( \mathbb{P}_\xi \) has the \( \kappa^+ \)-c.c.

By a standard book keeping method, we can destroy the stationarity of all non-reflecting subset of \( S^* \) by an iteration above. By \( \kappa^+ \)-c.c., \( T \) remains stationary in \( \mathcal{P}_\kappa \lambda \) in the generic extension. Thus \( S^* \) is stationary in \( \mathcal{P}_\kappa \lambda \), and \( \text{RP}(S^*, T) \) holds.

4. PROOF OF THEOREMS 1.6 AND 1.7

Proposition 4.1. Let \( \mu \) be a regular cardinal with \( \kappa^+ \leq \mu \leq \lambda \). Let \( T = \{X \in \mathcal{P}_\kappa \lambda : \kappa \subseteq X, \text{cf}(X \cap \mu) < \kappa \} \). Then for every stationary sets \( S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda \), \( \text{RP}^2(S_0^*, S_1^*, T) \) fails.

Proof. Suppose not. For each \( \xi \in E_{\kappa}^\lambda \), fix an increasing sequence \( \langle \gamma_i^\xi : i < \text{cf}(\xi) \rangle \) with limit \( \xi \). For \( n < 2, i < \kappa, \) and \( \delta < \mu \), let
\[
S_{n,i,\delta} = \{x \in S_n^* : \delta = \min(x \setminus \gamma_i^{\sup(x \cap \mu)})\}.
\]

Claim 4.2. (1) For every \( \xi < \mu \), there exist \( i < \kappa \) and \( \delta < \mu \) such that \( \delta > \xi \) and \( S_{0,i,\delta} \) is stationary.

(2) For every \( i < \kappa \) and \( \delta < \mu \), if \( S_{0,i,\delta} \) is stationary then \( S_{1,i,\delta} \) is stationary.

(3) For every \( i < \kappa \) and \( \delta_0, \delta_1 < \mu \), if \( S_{0,i,\delta_0} \) and \( S_{1,i,\delta_1} \) are stationary then \( \delta_0 = \delta_1 \).

Proof. (1). Let \( T' = \{X \in T : S_0^* \cap \mathcal{P}_\kappa X \) is stationary, \( \xi \in X \} \). \( T' \) is stationary in \( \mathcal{P}_\kappa \lambda \). Take \( X \in T' \). Then \( \text{cf}(X \cap \mu) < \kappa \subseteq X \) and \( \sup(X \cap \mu) > \xi \), hence there exists \( i \in X \) such that \( \gamma_i^{\sup(X \cap \mu)} > \xi \). By applying Fodor’s lemma to \( T' \), there exists \( i < \kappa \) such that \( T'' = \{x \in T' : \gamma_i^{\sup(X \cap \mu)} > \xi \} \) is stationary in \( \mathcal{P}_\kappa \lambda \). For \( X \in T'' \) let \( \delta_X = \min(X \setminus \gamma_i^{\sup(X \cap \mu)}) \). By Fodor’s lemma again, there is \( \delta < \mu \) such that \( T^* = \{X \in T'' : \gamma_i^{\sup(X \cap \mu)} > \xi, \delta = \min(X \setminus \gamma_i^{\sup(X \cap \mu)}) \} \) is stationary in \( \mathcal{P}_\kappa \lambda \).

Pick \( X \in T^* \). Since \( \text{cf}(X \cap \mu) < \kappa \), the set \( D_X = \{x \in \mathcal{P}_\kappa X : \sup(x \cap \mu) = \sup(X \cap \mu), \delta \in x \} \) contains a club in \( \mathcal{P}_\kappa X \). Clearly \( x \in S_{0,i,\delta} \) for each \( x \in D_X \cap S_0^* \).

This means that \( S_{0,i,\delta} \) is stationary in \( \mathcal{P}_\kappa \lambda \).

(2). By \( \text{RP}^2(S_0^*, S_1^*) \), \( T' = \{X \in T : \delta \in X, S_{0,i,\delta} \cap \mathcal{P}_\kappa X, S_1^* \cap \mathcal{P}_\kappa X \) are stationary \} \) is stationary in \( \mathcal{P}_\kappa \lambda \). Fix \( X \in T' \). Since \( S_{0,i,\delta} \cap \mathcal{P}_\kappa X \) is stationary in \( \mathcal{P}_\kappa X \) and
cf($X \cap \mu < \kappa$, we have that \( \delta = \min(X \setminus \gamma_{i}^{\sup(X \cap \mu)}) \). By the same argument as (1), we have that \( S_{1,i,\delta} \) is stationary in \( \mathcal{P}_{\kappa}\lambda \).

(3) Let \( X \in T \) be such that \( \delta_{0}, \delta_{1} \in X \) and \( S_{0,i,\delta_{0}} \cap \mathcal{P}_{\kappa}X, S_{1,i,\delta_{1}} \cap \mathcal{P}_{\kappa}X \) are stationary. Choose \( x_{0} \in S_{0,i,\delta_{0}} \cap \mathcal{P}_{\kappa}X \) and \( x_{1} \in S_{1,i,\delta_{1}} \cap \mathcal{P}_{\kappa}X \) such that \( \sup(x_{0} \cap \mu) = \sup(x_{1} \cap \mu) = \sup(X \cap \mu) \) and \( \delta_{0}, \delta_{1} \in x_{0} \cap x_{1} \). By the minimality of \( \delta_{0} \), we have \( \delta_{0} \leq \delta_{1} \). Similarly we know \( \delta_{1} \leq \delta_{0} \). Therefore \( \delta_{0} = \delta_{1} \). \[\square\text{[Claim]}\]

Hence we have that if \( S_{0,i,\delta} \) and \( S_{0,i,\delta'} \) are stationary, then \( \delta = \delta' \).

For each \( i < \kappa \), define \( \delta_{i} < \mu \) as follows: if \( S_{0,i,\delta} \) is stationary for some \( \delta < \mu \), then let \( \delta_{i} \) be a (unique) \( \delta < \mu \) such that \( S_{0,i,\delta} \) is stationary. If there is no such \( \delta \), then let \( \delta_{i} = 0 \). Since \( \mu = \cf(\mu) > \kappa \), we know \( \sup_{i<\kappa} \delta_{i} < \mu \). But this contradicts (1) of the claim. \[\square\]

**Proposition 4.3.** Let \( S_{0}^{\ast}, S_{1}^{\ast} \subseteq \mathcal{P}_{\kappa}\lambda \) be stationary and suppose \( \mathrm{RP}^{2}(S_{0}^{\ast}, S_{1}^{\ast}) \) holds. Then for every regular \( \mu \) with \( \kappa^{+} \leq \mu \leq \lambda \), \( \square(\mu) \) fails.

**Proof.** We prove only the case \( \mu = \lambda \). Other cases follow from similar arguments.

Toward the contradiction, suppose \( \square(\lambda) \) holds. Let \( \langle C_{\xi} : \xi < \lambda \rangle \) be a \( \square(\lambda) \)-sequence.

Let \( T = \{ X \in \mathcal{P}_{\kappa}X : \cf(X) = \kappa \subseteq X \} \). We assumed \( \mathrm{RP}^{2}(S_{0}^{\ast}, S_{1}^{\ast}) \), but by the previous proposition, in fact \( \mathrm{RP}^{2}(S_{0}^{\ast}, S_{1}^{\ast}, T) \) holds.

For each \( \alpha < \lambda \) and \( n < 2 \), let

\[ S_{n,\alpha} = \{ x \in S_{n}^{\ast} : C_{\sup(x)} \cap \sup(x \cap \alpha) = C_{\alpha} \cap \sup(x \cap \alpha) \}. \]

Let \( A_{n} = \{ \alpha < \lambda : S_{n,\alpha} \text{ is stationary} \} \).

**Claim 4.4.** For each \( n < 2 \), \( A_{n} \) is unbounded in \( \lambda \).

**Proof.** Fix \( n < 2 \). By shrinking \( S_{n}^{\ast} \) by a club in \( \mathcal{P}_{\kappa}\lambda \), we may assume that the following:

1. For all \( x \in S_{n}^{\ast} \) and \( \alpha \in x \), if \( x \cap \alpha \) is bounded in \( \alpha \) then \( \cf(\alpha) \geq \kappa \).
2. For all \( x \in S_{n}^{\ast} \) and \( \alpha \in x \cap E_{2}^{X} \), \( \sup(x \cap \alpha) \in \lim(C_{\alpha}) \) holds.

Let \( T' = \{ X \in T : S_{n}^{\ast} \cap \mathcal{P}_{\kappa}X \text{ is stationary} \} \). Then \( T' \) is stationary in \( \mathcal{P}_{\kappa+\lambda} \).

To show that \( A_{n} \) is unbounded, take \( \xi < \lambda \). Fix \( X \in T' \) with \( \sup(X) > \xi \). Since \( \cf(X) = \kappa \), the set \( \{ \beta < \sup(X) : \beta \in \lim(C_{\sup(X)}) \} \) contains a club in \( \sup(X) \).

Note that \( C_{\sup(X)} \cap \beta = C_{\beta} \) for each \( \beta \) from the club. Hence we know \( S_{X} = \{ x \in S_{n}^{\ast} \cap \mathcal{P}_{\kappa}X : C_{\sup(x)} = C_{\sup(X)} \cap \sup(x) \} \) is stationary in \( \mathcal{P}_{\kappa}X \). Since \( \cf(\sup(X)) = \kappa \), \( \lim(X) \cap \lim(C_{\sup(X)}) \) is unbounded in \( \sup(X) \). Take \( \beta \in \lim(X) \cap \lim(C_{\sup(X)}) \).
with $\beta > \xi$ and $\text{cf}(\beta) < \kappa$. Note that $\{x \in \mathcal{P}_\kappa X : x \cap \beta \text{ is unbounded in } \beta\}$ contains a club. Since $\beta \in \text{lim}(C_{\sup(X)})$, $C_{\sup(X)} \cap \beta = C_\beta$ holds. For each $x \in S_X$ such that $x \cap \beta$ is unbounded in $\beta$ and $\sup(x) > \beta$, let $\beta_x = \min(x \setminus \beta)$.

**Case 1.** $\beta_x = \beta$. Then $C_{\beta_x} \cap \sup(x \cap \beta_x) = C_{\beta} = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x)$.

**Case 2.** $\beta_x > \beta$. Then $\sup(x \cap \beta_x) = \beta$ and $\beta = \sup(x \cap \beta) \in \text{lim}(C_{\beta_x})$, hence $C_{\beta_x} \cap \beta = C_{\beta} = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x)$.

Hence for each $x \in S_X$ such that $x \cap \beta$ is unbounded in $\beta$ and $\sup(x) > \beta$, we have $C_{\sup(x)} \cap \sup(x \cap \beta_x) = C_{\beta_x} \cap \sup(x \cap \beta_x)$. By applying Fodor’s lemma to $S_X$, we can find $\beta_X \in X$ such that $\{x \in S_X : \beta_X = \beta_x\}$ is stationary. Thus $\{x \in S^* \cap \mathcal{P}_\kappa X : C_{\sup(x)} \cap \sup(x \cap \beta_x) = C_{\beta_X} \cap \sup(x \cap \beta_X)\}$ is stationary.

By applying Fodor’s lemma to $T'$, we have $\beta_* < \lambda$ such that $\{x \in T' : \beta_* = \beta_X\}$ is stationary. Then $S_{n, \beta_*}$ is stationary and $\beta_* > \xi$. \[\square\] [Claim]

**Claim 4.5.** For each $\alpha \in A_0$ and $\beta \in A_1$ with $\alpha < \beta$, $C_\alpha = C_\beta \cap \alpha$ holds.

**Proof.** Let $T^* = \{X \in T : S_{0, \alpha} \cap \mathcal{P}_\kappa X, S_{1, \beta} \cap \mathcal{P}_\kappa X \text{ are stationary in } \mathcal{P}_\kappa X\}$. Take $X \in T^*$. Since $D_X = \{x \in \mathcal{P}_\kappa X : C_{\sup(x)} \cap \sup(x) = C_{\sup(x)}\}$ contains a club in $\mathcal{P}_\kappa X$, $D_X \cap S_{0, \alpha}$ is stationary in $\mathcal{P}_\kappa X$. For $x \in C_X \cap S_{0, \alpha}$, $C_\alpha \cap \sup(x \cap \alpha) = C_{\sup(x)} \cap \sup(x \cap \alpha)$ holds. Since $\{x \in C_X \cap S_{0, \alpha} : \beta \in \text{lim}(C_{\sup(X)}\cap \beta) \in \text{lim}(C_{\sup(x)}\cap \beta)\}$ is unbounded in $\sup(x \cap \alpha)$, we have $C_{\sup(x)} \cap \sup(x \cap \alpha) = C_\alpha \cap \sup(x \cap \alpha)$.

Similarly, we have $C_\beta \cap \sup(X \cap \beta) = C_{\sup(x)} \cap \sup(X \cap \beta)$. Therefore we have $C_\alpha \cap \sup(X \cap \alpha) = C_\beta \cap \sup(X \cap \alpha)$.

Because $\{\sup(X \cap \alpha) : X \in T^*\}$ is unbounded in $\alpha$, we have $C_\alpha = C_\beta \cap \alpha$. \[\square\] [Claim]

Now, let $C = \{C_\beta : \beta \in A_0\}$. Since $A_0$ is unbounded, $C$ is unbounded. Furthermore, $C_\alpha = C_\beta \cap \alpha$ for all $\alpha < \beta \in A$; For $\alpha, \beta \in A_0$ with $\alpha < \beta$, choose $\gamma \in A_1$ with $\beta < \gamma$. Then $C_\alpha = C_\gamma \cap \alpha$ and $C_\beta = C_\gamma \cap \alpha$. Thus $C_\alpha = C_\beta \cap \alpha$. Hence $C$ forms a club in $\lambda$. Take $\alpha \in \text{lim}(C)$. Then there exists $\beta \in A_0$ such that $C \cap \alpha = C_\beta \cap \alpha$. Since $\alpha \in \text{lim}(C)$, we know $\alpha \in \text{lim}(C_\beta)$ and $C_\alpha = C_\beta \cap \alpha = C \cap \alpha$. Thus $\forall \alpha \in \text{lim}(C) (C \cap \alpha = C_\alpha)$, this is a contradiction. \[\square\]

Baumgartner[1] showed that if a weakly compact cardinal $\kappa$ is collapsed to $\omega_2$ by Levy-collapse with countable conditions, then $\text{RP}(\mathcal{P}_{\omega_1}\omega_2)$ holds, and it is known that in fact $\text{RP}^2(\mathcal{P}_{\omega_1}\omega_2, \mathcal{P}_{\omega_1}\omega_2)$ holds in the generic extension. Conversely, Velickovic [3] showed that if $\text{RP}(\mathcal{P}_{\omega_1}\omega_2)$ holds, then $\omega_2$ is weakly compact in $L$. Consequently, we have the following equiconsistency:
Corollary 4.6. The following are equiconsistent:

(1) ZFC + "there exists a weakly compact cardinal".
(2) ZFC + "RP(\mathcal{P}_{\omega_1\omega_2}) holds".
(3) ZFC + "RP^2(\mathcal{P}_{\omega_1\omega_2}, \mathcal{P}_{\omega_1\omega_2}) holds".
(4) ZFC + "RP^2(S_0^*, S_1^*) holds for some stationary sets S_0^*, S_1^* \subseteq \mathcal{P}_{\omega_1\omega_2}".

5. Proof of Theorem 1.8

Proposition 5.1. Suppose RP(S^*) for some stationary S^* \subseteq \mathcal{P}_\kappa\lambda. Then every \kappa-c.c. forcing preserves RP(S^*).

Proof. First note that every \kappa-c.c. forcing preserves the stationarity of S^*.

Let \mathbb{P} be a poset which satisfies the \kappa-c.c. Let \dot{S} be a \mathbb{P}-name such that \forces "\dot{S} \subseteq S^* is stationary". It is enough to show that there are some p \in \mathbb{P} and X \subseteq \mathcal{P}_\kappa X such that p \forces "\dot{S} \cap \mathcal{P}_\kappa X is stationary in \mathcal{P}_\kappa X".

Let S' = \{x \in S^*: \exists p \in \mathbb{P} (p \forces \text{"x \in } \dot{S}"\text{)}\}. It is easy to check that S' is a stationary subset of S^*. By RP(S'), there is X \in \mathcal{P}_\kappa X such that \|X| = \kappa \subseteq X and S' \cap \mathcal{P}_\kappa X is stationary in \mathcal{P}_\kappa X. We see that p \forces "\dot{S} \cap \mathcal{P}_\kappa X is stationary" for some p \in \mathbb{P}. Suppose to the contrary that \forces "\dot{S} \cap \mathcal{P}_\kappa X is non-stationary". Since \|X| = \kappa and \mathbb{P} satisfies the \kappa-c.c., we can find a club C \subseteq \mathcal{P}_\kappa X such that \forces "\dot{S} \cap \mathcal{P}_\kappa X = \emptyset". S' \cap \mathcal{P}_\kappa X is stationary, hence there is x \in S' \cap C. Pick p \in \mathbb{P} with p \forces "x \in \dot{S}". Then p \forces "x \in \dot{S} \cap C\text{"}, this is a contradiction.

Recall that PFA^{++} is the assertion that for every proper forcing notion \mathbb{P}, every dense subsets D_i (i < \omega_1) of \mathbb{P}, and every \mathbb{P}-names \dot{S}_i (i < \omega_1) for stationary subsets of \omega_1, there is a filter F on \mathbb{P} such that:

(1) D_i \cap F \neq \emptyset for every i < \omega_1.
(2) S_i = \{\alpha < \omega_1: \exists p \in F (p \forces \text{"}\alpha \in \dot{S}_i\"}\} is stationary in \omega_1 for i < \omega_1.

Proposition 5.2. Suppose PFA^{++}. Let \lambda \geq \omega_2. Then every c.c.c. forcing notion forces RP^2(\mathcal{P}_{\omega_1\lambda}, \mathcal{P}_{\omega_2\lambda}).

Proof. Let \mathbb{P} be a poset which satisfies the c.c.c. Let \dot{S}_0, \dot{S}_1 be \mathbb{P}-names so that \forces "\dot{S}_0, \dot{S}_1 \subseteq \mathcal{P}_{\omega_1\lambda} are stationary". We will find p \in \mathbb{P} and X \in \mathcal{P}_{\omega_2\lambda} such that p \forces "\dot{S}_0 \cap \mathcal{P}_{\omega_1} X, \dot{S}_1 \cap \mathcal{P}_{\omega_1} X are stationary".

Let \dot{\mathbb{Q}} be a \mathbb{P}-name for a \sigma-closed poset which adds a bijection from \omega_1 to \lambda. We know that \forces_{\mathbb{P} \ast \dot{\mathbb{Q}}} "\dot{S}_0, \dot{S}_1 remain stationary". Fix a \mathbb{P} \ast \dot{\mathbb{Q}}-name \pi for a bijection from \omega_1 to \lambda. Let \dot{E}_0, \dot{E}_1 be \mathbb{P} \ast \dot{\mathbb{Q}}-names such that \forces_{\mathbb{P} \ast \dot{\mathbb{Q}}} "\dot{E}_i = \{\alpha < \omega_1: \pi \text{"}\alpha \in \dot{S}_i, \pi \text{"}\alpha \cap \omega_1 = \alpha\}" for i = 0, 1. We know \forces_{\mathbb{P} \ast \dot{\mathbb{Q}}} "\dot{E}_i is stationary in \omega_1".
Now fix a sufficiently large regular cardinal $\theta$ and take $M \prec H_\theta$ such that $|M| = \omega_1 \subseteq M$ and $M$ contains all relevant objects.

$\mathbb{P} \ast \dot{\mathbb{Q}}$ is proper, hence we can apply PFA++ to $\mathbb{P} \ast \dot{\mathbb{Q}}$ and $E_i$. By PFA++ we can find a filter $F$ on $\mathbb{P} \ast \dot{\mathbb{Q}}$ such that:

1. $F \cap \mathbb{D} \neq \emptyset$ for all dense $\mathbb{D} \in \mathbb{P} \ast \dot{\mathbb{Q}}$.
2. $E_i = \{ \alpha < \omega_1 : \exists p \in F(p \Vdash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \alpha \in \dot{E}_i)\}$ is stationary in $\omega_1$ for $i = 0, 1$.

Let $X = \{ \beta < \lambda : \exists p \in F \exists \alpha < \omega_1 (p \Vdash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \dot{\pi}(\alpha) = \beta)\}$. We can check that $|X| = \omega_1 \subseteq X$.

Since $\dot{S}_0, \dot{S}_1$ are names for subsets of $\mathcal{P}_{\omega_1}^{\omega_1} \lambda$, for each $\alpha \in E_i$, we can find $x \in \mathcal{P}_{\omega_1} \lambda$ and $p \in F$ such that $x \cap \omega_1 = \alpha$ and $p \Vdash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \dot{\pi}(\alpha) = x$. Moreover it is easy to see that $x \in \mathcal{P}_{\omega_1} X$.

For $i < 2$ and $\alpha \in E_i$, take $x_{i,\alpha} \in \mathcal{P}_{\omega_1} X$ such that there is $p \in F$ with $p \Vdash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \dot{\pi}(\alpha) = x_{i,\alpha}$. Let $S_i = \{ x_{i,\alpha} : \alpha \in E_i \}$. The following are easy to check for $i < 2$:

1. $x_{i,\alpha} \subseteq x_{i,\beta}$ holds for $\alpha, \beta \in E_i$ with $\alpha < \beta$.
2. If $\alpha \in \lim(E_i) \cap E_i$, then $x_{i,\alpha} = \bigcup_{\beta \in E_i \cap \alpha} x_{i,\beta}$.
3. $\bigcup S_i = X$.

Furthermore, since $E_i = \{ x_{i,\alpha} \cap \omega_1 : \alpha \in E_i \}$ is stationary in $\omega_1$, we can check that each $S_i$ is stationary in $\mathcal{P}_{\omega_1} X$.

Now we see that $p \Vdash_{\mathbb{P}} \dot{S}_0 \cap \mathcal{P}_{\omega_1} X, \dot{S}_1 \cap \mathcal{P}_{\omega_1} X$ are stationary” for some $p \in \mathbb{P}$. Suppose otherwise. Since $\mathbb{P}$ satisfies the c.c.c. and $|X| = \omega_1$, we can find a club $C$ in $\mathcal{P}_{\omega_1} X$ such that $\mathbb{P} \Vdash \dot{C} \cap \dot{S}_0 = \emptyset$ or $C \cap \dot{S}_1 = \emptyset$.

Since $S_0$ and $S_1$ are stationary in $\mathcal{P}_{\omega_1} X$, we can find $x_0 \in S_0 \cap C$ and $x_1 \in S_1 \cap C$. Then there is $q \in F$ such that $q \Vdash_{\mathbb{P} \ast \dot{\mathbb{Q}}} x_0 \in \dot{S}_0$ and $x_1 \in \dot{S}_1$. Thus $q \Vdash_{\mathbb{P} \ast \dot{\mathbb{Q}}} C \cap \dot{S}_0 \neq \emptyset$ and $C \cap \dot{S}_1 \neq \emptyset$, this is a contradiction.

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**References**


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