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Kyoto University
PARTIAL STATIONARY REFLECTION PRINCIPLES

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1. INTRODUCTION

Throughout this paper, $\kappa$ denotes a regular uncountable cardinal and $\lambda$ a cardinal $\geq \kappa^+$, unless otherwise specified.

Partial stationary reflection on $\mathcal{P}_{\omega_1}\omega_2$ was introduced by H. Sakai [2]. First we extend his notion to arbitrary $\kappa$ and $\lambda$.

Definition 1.1. Let $S^*$ be a stationary subset of $\mathcal{P}_\kappa\lambda$. For a stationary set $T \subseteq \mathcal{P}_{\kappa^+}\lambda$, we say that $\text{RP}(S^*, T)$ holds if for every stationary subset $S \subseteq S^*$ there exists $X \in T$ such that $\kappa \subseteq X$ and $S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$. $\text{RP}(S^*)$ means $\text{RP}(S^*, \mathcal{P}_{\kappa^+}\lambda)$.

It is known that total stationary reflection $\text{RP}(\mathcal{P}_\kappa\lambda)$ is a large cardinal property (e.g., see Velickovic [3]), but Sakai [2] showed that partial stationary reflection on $\mathcal{P}_{\omega_1}\omega_2$ is not:

Fact 1.2 ([2]). Suppose CH. If $\square_{\omega_1}$ holds, then there are a stationary set $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ and a $\sigma$-Baire, $\omega_2$-c. poset $\mathbb{P}$ such that $\mathbb{P}$ forces $\text{RP}(S^*)$.

In this paper, we generalize his result as follows:

Theorem 1.3. Suppose $\kappa^{<\kappa} = \kappa$. Let $T \subseteq \mathcal{P}_{\kappa^+}\lambda$ be a stationary set such that $\forall X \in T (\kappa \subseteq X)$. Then there exists a $\kappa$-closed, $\kappa^+$-c. poset which forces the following statements:

1. $T$ is stationary.
2. There exists a stationary set $S^* \subseteq \mathcal{P}_\kappa\lambda$ such that
   (a) $\forall X \in T (S^* \cap \mathcal{P}_\kappa X$ contains a club in $\mathcal{P}_\kappa X)$,
   (b) $\text{RP}(S^*, T)$ holds.
This theorem shows that, even \( \kappa > \omega_1 \) and \( \lambda > \omega_2 \), our partial stationary reflection is not a large cardinal property.

Next we consider a natural strengthening of partial stationary reflection, *simultaneous partial stationary reflection*.

**Definition 1.4.** For stationary sets \( S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda \) and \( T \subseteq \mathcal{P}_{\kappa^+} \lambda \), we say that \( \text{RP}^2(S_0^*, S_1^*, T) \) holds if for every stationary subsets \( S_0 \subseteq S_0^* \) and \( S_1 \subseteq S_1^* \) in \( \mathcal{P}_\kappa \lambda \), there exists \( X \in T \) such that \( \kappa \subseteq X \) and both \( S_0 \cap \mathcal{P}_\kappa X \) and \( S_1 \cap \mathcal{P}_\kappa X \) are stationary in \( \mathcal{P}_\kappa X \). \( \text{RP}^2(S_0^*, S_1^*) \) means \( \text{RP}^2(S_0^*, S_1^*, \mathcal{P}_{\kappa^+} \lambda) \).

We prove that our simultaneous partial stationary reflection is a large cardinal property by showing the following:

**Definition 1.5.** For a regular uncountable cardinal \( \mu \), \( \Box(\mu) \) holds if there exists a sequence \( \langle C_\xi : \xi < \mu \rangle \) satisfying the following:

1. for all \( \xi < \mu \), \( C_\xi \) is club in \( \xi \) and for all \( \eta \in \text{lim}(C_\xi) \), \( C_\eta = C_\xi \cap \eta \),
2. for all club \( C \) in \( \mu \), there exists \( \xi \in \text{lim}(C) \) such that \( C \cap \xi \neq C_\xi \).

Such an sequence \( \langle C_\xi : \xi < \mu \rangle \) is called a \( \Box(\mu) \)-sequence.

**Theorem 1.6.** Suppose \( \text{RP}^2(S_0^*, S_1^*) \) holds for some stationary \( S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda \). Then for every regular \( \mu \) with \( \kappa^+ \leq \mu \leq \lambda \), \( \Box(\mu) \) fails.

We also prove the following:

**Theorem 1.7.** For every stationary \( S_0^*, S_1^* \subseteq \mathcal{P}_\kappa \lambda \) and regular \( \mu \) with \( \kappa^+ \leq \mu \leq \lambda \), \( \text{RP}^2(S_0^*, S_1^*, \{X \in \mathcal{P}_{\kappa^+} \lambda : \text{cf}(X \cap \mu) < \kappa \}) \) fails, where \( \text{cf}(X) = \text{cf}(\text{ot}(X)) \).

Todorcevic showed that \( \text{RP}(\mathcal{P}_{\omega_1} \omega_2) \) implies that \( 2^\omega \leq \omega_2 \). However we prove the following, which shows that our partial stationary reflection does not affect the size of the continuum:

**Theorem 1.8.**
1. Suppose \( \text{RP}(S^*) \) for some stationary \( S^* \subseteq \mathcal{P}_\kappa \lambda \). Then every \( \kappa \)-c.c. forcing preserves \( \text{RP}(S^*) \).
2. Suppose \( \text{PFA}^{++} \). Let \( \lambda \geq \omega_2 \). Then every c.c.c. forcing notion forces \( \text{RP}^2(\mathcal{P}_{\omega_1} \lambda, \mathcal{P}_{\omega_1} \lambda) \).

2. Preliminaries

For a set \( X \) of ordinals, let \( \text{cf}(X) = \text{cf}(\text{ot}(X)) \).

For regular cardinals \( \nu < \mu \), let \( E^\nu_\nu = \{ \alpha < \mu : \text{cf}(\alpha) = \nu \} \) and \( E^\nu_{<\nu} = \{ \alpha < \mu : \text{cf}(\alpha) < \nu \} \).
The proofs of the following lemmata are easy:

**Lemma 2.1.** For a stationary $S \subseteq \mathcal{P}_\kappa \lambda$ and a $\kappa$-c.c. poset $\mathbb{P}$, $\mathbb{P}$ preserves the stationarity of $S$.

**Lemma 2.2.** For $S \subseteq \mathcal{P}_\kappa \lambda$, if $\{ X \in \mathcal{P}_{\kappa^+} \lambda : S \cap \mathcal{P}_\kappa X \text{ is stationary in } \mathcal{P}_\kappa X \}$ is stationary in $\mathcal{P}_{\kappa^+} \lambda$, then $S$ is stationary in $\mathcal{P}_\kappa \lambda$.

**Lemma 2.3.** For stationary sets $S^* \subseteq \mathcal{P}_\kappa \lambda$ and $T \subseteq \mathcal{P}_{\kappa^+} \lambda$, if $\{ X \in \mathcal{P}_{\kappa^+} \lambda : S \cap \mathcal{P}_\kappa X \text{ is stationary in } \mathcal{P}_\kappa X \}$ is stationary in $\mathcal{P}_{\kappa^+} \lambda$, then $S$ is stationary in $\mathcal{P}_\kappa \lambda$.

We define club shootings into $\mathcal{P}_\kappa \lambda$, which was observed in [2].

**Definition 2.4.** For $S \subseteq \mathcal{P}_\kappa \lambda$, let $\mathbb{C}(S)$ be the poset which consists of all functions $p$ such that:

1. $|p| < \kappa$,
2. $p : d(p) \times d(p) \to \kappa$ for some $d(p) \in \mathcal{P}_\kappa \lambda$, and
3. $\forall x \subseteq d(p) (x \in S \Rightarrow x$ is not closed under $p)$.

For $p, q \in \mathbb{C}(S)$, $p \leq q \iff q \subseteq p$.

Let $\mathbb{C} = \mathbb{C}(\emptyset)$.

**Lemma 2.5.**

1. $\mathbb{C}(S)$ satisfies the $(2^{<\kappa})^+\text{-c.c.}$

2. For every $x \in \mathcal{P}_\kappa \lambda$, $\{ p \in \mathbb{C}(S) : x \subseteq d(p) \}$ is a dense open set in $\mathbb{C}(S)$.

3. Whenever $G$ is $(V, \mathbb{C}(S))$-generic, $\bigcup G$ is a function from $\lambda \times \lambda$ to $\kappa$, and every $x \in S$ is not closed under the function.

*Proof.* For (1), take $A \subseteq \mathbb{C}(S)$ with size $(2^{<\kappa})^+$. By $\Delta$-system lemma, we can find $B \subseteq A$ and $a \in \mathcal{P}_\kappa \lambda$ such that $|B| = (2^{<\kappa})^+$ and $d(p) \cap d(q) = a$ for every distinct $p, q \in B$. Moreover we may assume that $p|a \times a = q|a \times a$ for every $p, q \in B$. We check that $B$ is a pairwise compatible set.

Take $p, q \in B$. Pick $\alpha < \kappa$ with $\alpha > \sup(d(p) \cap \kappa) + 1, \sup(d(q) \cap \kappa) + 1$. Then define $r$ as $\text{dom}(r) = (d(p) \cup d(q)) \times (d(p) \cup d(q))$ and

$$r(\xi, \eta) = \begin{cases} p(\xi, \eta) & \text{if } \xi, \eta \in d(p), \\ q(\xi, \eta) & \text{if } \xi, \eta \in d(q), \\ \alpha & \text{otherwise.} \end{cases}$$

We have $r \leq p, q$. (2) follows from a similar argument, and (3) is straightforward. \(\square\)
3. The proof of Theorem 1.3

Suppose $\kappa^{<\kappa} = \kappa$. Fix a stationary set $T \subseteq \mathcal{P}_\kappa + \lambda$ such that $\forall X \in T (\kappa \subseteq X)$. We consider the following poset $\mathbb{P}_T$, which adds a new stationary subset $S^*$ of $\mathcal{P}_\kappa \lambda$.

**Definition 3.1.** $\mathbb{P}_T$ is the set of all functions $p$ satisfying the following:

1. $|p| < \kappa$ and $\text{dom}(p) \subseteq T$,
2. for every $X \in \text{dom}(p)$, $p(X)$ is a $\subseteq$-increasing continuous set $\{x_i : i \leq \gamma\}$ in $\mathcal{P}_\kappa X$ such that $\gamma < \kappa$ and $x_i \cap \kappa \in \kappa$ for all $i \leq \gamma$.

For $p \in \mathbb{P}_T$ and $X \in \text{dom}(p)$, $\max(p(X))$ denotes the maximum element of $p(X)$. Let $u(p) = \bigcup \{p(X) : X \in \text{dom}(P)\}$. Note that $u(p) \subseteq \mathcal{P}_\kappa \lambda$ and $|u(p)| < \kappa$. For $p, q \in \mathbb{P}_T$, define $p \leq q$ if

1. $\text{dom}(p) \supseteq \text{dom}(q)$,
2. $\forall X \in \text{dom}(q) (q(X) = \{x \in p(X) : x \subseteq \max(q(X))\})$ (hence $u(p) \supseteq u(q)$),
3. $\forall x \in u(p) (x \subseteq \cup u(q) \Rightarrow x \in u(q))$,
4. $\forall X \in \text{dom}(p) \setminus \text{dom}(q) (\max(p(X)) \not\subseteq \cup u(q))$,
5. $\forall X \in \text{dom}(q) \forall x \in p(X) \setminus q(X) (x \not\subseteq \cup u(q))$.

**Lemma 3.2.**

1. $\mathbb{P}_T$ is $\kappa$-closed,
2. $\mathbb{P}_T$ satisfies the $\kappa^+$-c.c. (if $\kappa^{<\kappa} = \kappa$),
3. for all $X \in T$ and $x \in \mathcal{P}_\kappa X$, $\{p \in \mathbb{P}_T : X \in \text{dom}(p)$ and $x \subseteq \max(p(X))\}$ is dense in $\mathbb{P}_T$.

**Proof.** (1). Let $\gamma < \kappa$ be a limit ordinal and $\langle p_i : i < \gamma\rangle$ be a decreasing sequence in $\mathbb{P}_T$. Then define the function $p^*$ as the following manner:

1. $\text{dom}(p^*) = \bigcup_{i<\gamma} \text{dom}(p_i)$,
2. for $X \in \text{dom}(p^*)$, $p^*(X) = \bigcup \{p_i(X) : i < \gamma, X \in \text{dom}(p_i)\} \cup \{\max(p_i(X)) : i < \gamma, X \in \text{dom}(p_i)\}$.

Since the $p_i$'s are decreasing, it is easy to show that $p^* \in \mathbb{P}_T$. For $i < \gamma$, we show $p \leq p_i$. It is easily verified that the conditions (a) and (b) in the definition of the order are satisfied.

(c). Take $x \in u(p^*)$ such that $x \subseteq \cup u(p_i)$. Take $X \in \text{dom}(p^*)$ such that $x \in p^*(X)$. If $x \neq \max(p^*(X))$, then $x \in p_j(X)$ for some $j > i$ with $X \in \text{dom}(p_j)$. Since $p_j \leq p_i$, we have $x \in p_i(X)$. Next suppose $x = \max(p^*(X))$. Take $k < \gamma$ such that $i < k$ and $X \in \text{dom}(p_k)$. Then $\max(p_k(X)) \subseteq \max(p^*(X)) = x \subseteq \cup u(p_i)$ holds. Hence $X \in \text{dom}(p_i)$ by (d). For each $j \geq i$, $\max(p_j(X)) \subseteq \max(p^*(X)) = x \subseteq$...
\( \bigcup u(p_i) \) holds. Thus we have \( \max(p_j(X)) \in p_i(X) \) by (e). Therefore \( \{ \max(p_j(X)) : i \leq j < \gamma \} \subseteq p_i(X) \), and we have \( \max(p^*(X)) = \bigcup \{ \max(p_j(X)) : i \leq j < \gamma \} \in p_i(X) \).

(d). Take \( X \in \text{dom}(p^*) \setminus \text{dom}(p_i) \). Then there exists \( j > i \) such that \( X \in \text{dom}(p_j) \). We know \( \max(p_j(X)) \notin \bigcup u(p_i) \). Because \( \max(p_j(X)) \subseteq \max(p^*(X)) \), we know \( \max(p^*(X)) \notin \bigcup u(p_i) \).

(e). Take \( X \in \text{dom}(p_i) \) and \( x \in p^*(X) \setminus p_i(X) \). Then there exist \( j \geq i \) and \( y \in \text{dom}(p_j) \) such that \( y \subseteq x \) and \( y \notin p_i(X) \). Hence \( y \notin \bigcup u(p_i) \) and \( x \notin \bigcup u(p_i) \).

(2). Take an arbitrary \( A \subseteq \mathbb{P}_T \) with \( |A| \geq \kappa^+ \). We prove that \( A \) is not an antichain. By \( \Delta \)-system lemma, we can find \( r \in \mathbb{P}_s, s \in \mathbb{P}_s \), and \( B \subseteq A \) with \( |B| \geq \kappa^+ \) such that \( \forall p, q \in B \ (\text{dom}(p) \cap \text{dom}(q) = r \) and \( \bigcup u(p) \cap \bigcup u(q) = s \). By our cardinal arithmetic assumption, there exists \( C \subseteq B \) with \( |C| \geq \kappa^+ \) such that \( \forall p, q \in B(\forall X \in r (p(X) = q(X)) \) and \( \mathcal{P}_s \cap \bigcup u(p) = \mathcal{P}_s \cap \bigcup u(q) \). We check that any two elements of \( C \) are pairwise compatible. Take \( p, q \in C \). For each \( X \in \text{dom}(p) \cup \text{dom}(q) \), fix \( a_X \in \mathbb{P}_s \) such that \( (\bigcup u(p) \cup \bigcup u(q)) \cap X \not\subsetneq a_X \). Define the function \( r \) as the following:

(i) \( \text{dom}(r) = \text{dom}(p) \cup \text{dom}(q) \),

(ii) \( r(X) = p(X) \cup \{ a_X \} \) if \( X \in \text{dom}(p) \), and \( r(X) = q(X) \cup \{ a_X \} \) if \( X \in \text{dom}(q) \).

This is well-defined because \( p(X) = q(X) \) for all \( X \in \text{dom}(p) \cap \text{dom}(q) \). We see that \( r \) is a lower bound of \( p \) and \( q \). \( r \in \mathbb{P}_T \) is easily verified. For \( r \leq p \), the conditions (a) and (b) are clear.

(c). Take \( x \in \text{dom}(r) \) such that \( x \subseteq \bigcup u(p) \). Then \( x \neq a_X \) for all \( X \in \text{dom}(p) \cup \text{dom}(q) \). Hence \( x \in \text{dom}(p) \cup \text{dom}(q) \). If \( x \in \text{dom}(p) \) then we have done. Assume \( x \in \text{dom}(q) \). Then \( x \subseteq \bigcup u(q) \). Since \( x \subseteq \bigcup u(q) \), we have \( x \subseteq \bigcup u(p) \cap u(q) = s \) and \( x \in \mathcal{P}_s \). Because \( \mathcal{P}_s \cap \bigcup u(p) = \mathcal{P}_s \cap \bigcup u(q) \), we have \( x \in \mathcal{P}_s \cap \bigcup u(p) \) and \( x \in \text{dom}(p) \).

(d). Take \( X \in \text{dom}(r) \setminus \text{dom}(p) \). Then \( \max(r(X)) = a_X \supsetneq \bigcup u(p) \cap X \), thus \( \max(r(X)) \notin \bigcup u(p) \).

(e). Take \( X \in \text{dom}(p) \) and \( x \in r(X) \setminus p(X) \). By the definition of \( r(X) \), we have \( r(X) = p(X) \cup \{ a_X \} \). Hence \( x = a_X \notin \bigcup u(p) \).

\( r \leq q \) can be proved by the same argument.

(3). Take \( X \in T, x \in \mathcal{P}_s X \) and \( q \in \mathbb{P} \). Take \( x^* \in \mathcal{P}_s X \) such that \( \bigcup u(q) \cap X \not\subset x^* \). Define \( p \) as \( \text{dom}(p) = \text{dom}(q) \cup \{ X \}, p|\text{dom}(q) = q \) and \( p(X) = \{ x^* \} \) if \( X \notin \text{dom}(q) \), and \( q(X) \cup \{ x^* \} \) if \( X \in \text{dom}(q) \). Then \( p \leq q \) can be verified. \( \square \)
Note that the following: For $\gamma < \kappa$ and a decreasing sequence $\langle p_i : i < \gamma \rangle$ in $\mathbb{P}_T$, let $p^*$ be a lower bound of the $p_i$’s as constructed in the proof of (1) above. Then $p^*$ is the largest lower bound of the $p_i$’s and $\bigcup u(p^*) = \bigcup_{i<\gamma} (\bigcup u(p_i))$.

**Definition 3.3.** For a canonical name of $(V, \mathbb{P}_T)$-generic filter $\dot{G}$, let $\dot{S}^*$ be a $\mathbb{P}_T$-name such that

$$\vdash \dot{S}^* = \bigcup \{ u(p) : p \in \dot{G} \}.$$

The following are easily verified by the definition of $\mathbb{P}_T$.

**Lemma 3.4.**

1. $\vdash_{\mathbb{P}_T} \forall X \in T (\dot{S}^* \cap \mathcal{P}_\kappa X$ contains a club in $\mathcal{P}_\kappa X)$”,
2. for all $p \in \mathbb{P}_T$, $p \vdash_{\mathbb{P}_T} \{ y \in \dot{S}^* : y \subseteq \bigcup u(p) \} = u(p)$”.

Now fix a name $\dot{S}$ such that

$$\vdash_{\mathbb{P}_T} \dot{S} \subseteq \dot{S}^* \text{ and } \forall X \in T (\mathcal{P}_\kappa X \cap \dot{S} \text{ is non-stationary in } \mathcal{P}_\kappa X).$$

We see that $\mathbb{P}_T * \mathbb{C}(\dot{S})$ has good properties.

For each $X \in T$, fix a name $\dot{g}_X$ such that

$$\vdash_{\mathbb{P}_T} \dot{g}_X : [X]^{\omega} \rightarrow X \text{ and } \forall x \in \mathcal{P}_\kappa X \ (x \text{ is closed under } \dot{g}_X \Rightarrow x \notin \dot{S}).$$

Let $\dot{Q}$ be a name such that $\vdash \dot{Q} = \mathbb{C}(\dot{S})$”. We prove that $\mathbb{P}_T * \dot{Q}$ has a $\kappa$-closed dense subset.

**Lemma 3.5.** Let $D = \{ p \in \mathbb{P}_T : \forall X \in \text{dom}(p) \ (p \vdash_{\mathbb{P}_T} \text{max}(p(X)) \text{ is closed under } \dot{g}_X) \}$. Then $D$ is dense in $\mathbb{P}_T$.

**Proof.** Take $p \in \mathbb{P}_T$. We want to find $q \in D$ such that $q \leq p$. We take a decreasing sequence $p_i$ ($i < \omega$) in $\mathbb{P}_T$ by induction on $i < \omega$. Let $p_0 = p$. Suppose $p_i$ is defined. By the $\kappa$-closedness of $\mathbb{P}_T$, we can choose $p' \leq p_i$ and $a \in \mathcal{P}_\kappa \lambda$ such that $p' \vdash \dot{g}_X^{\omega} \in \text{max}(p_i(X))^{\omega} \subseteq a \cap X$ for all $X \in \text{dom}(p_i)$. Then choose $p_{i+1} \leq p'$ such that $a \cap X \subseteq \text{max}(p_{i+1}(X))$ for all $X \in \text{dom}(p_i)$.

Finally let $q$ be the greatest lower bound of the $p_i$’s. By our construction, it is easy to see that $q \in D$. \qed

**Lemma 3.6.** Let $D$ be as in Lemma 3.5. Let $D' = \{ \langle p, q \rangle \in \mathbb{P}_T * \dot{Q} : p \in D, \ q = \check{r} \ \text{for some } r \in \mathbb{C} \text{ and } d(r) = \bigcup (u(p)) \}$. Then $D'$ is a $\kappa$-closed dense subset in $\mathbb{P}_T * \dot{Q}$.

**Proof.** Density: Take $\langle p, \check{q} \rangle \in \mathbb{P}_T * \dot{Q}$. Take $p' \in D$ and $r$ such that $p' \vdash \check{r} = \check{q}$ and $\bigcup u(p') \supseteq d(r)$. Now define $r'$ as the following:
(1) \( r' : \bigcup u(p') \times \bigcup u(p') \rightarrow \kappa \),
(2) for \( a \in \bigcup u(p') \times \bigcup u(p') \), if \( a \in d(r) \times d(r) \) the \( r'(a) = r(a) \), otherwise
\( r'(a) = \sup(\bigcup (u(p') \cap \kappa)) + 1 \).

It is easy to show that \( p' \models " r' \in \mathbb{C}(\dot{S})" \) and \( \langle p', \dot{r}' \rangle \leq \langle p, \dot{q} \rangle \).

Next we prove \( D' \) is \( \kappa \)-closed. Let \( \gamma < \kappa \) and \( \langle p_i, \dot{q}_i \rangle (i < \gamma) \) be a decreasing sequence in \( D' \). We show that this sequence has a lower bound. Let \( p^* \in \mathbb{P}_T \) be the greatest lower bound of the \( p_i \)'s. Note that for all \( X \in \text{dom}(p^*) \), \( p^* \models \text{"max}(p^*(X)) \) is closed under \( \dot{g}_X " \).

Let \( q^* = \bigcup_{i<\gamma} q_i \). \( q^* \) is a function with the domain \( d(q^*) \times d(q^*) \), where \( d(q^*) = \bigcup_{i<\gamma} d(q_i) \). Notice that \( d(q^*) = \bigcup_{i<\gamma} d(q_i) = \bigcup_{i<\gamma} \bigcup u(p_i) = \bigcup u(p^*) \). We complete the proof by showing the following claim.

**Claim 3.7.** \( p^* \models " q^* \in \mathbb{C}(\dot{S}) " \).

**Proof.** Take a \((V, \mathbb{P}_T)\)-generic \( G \) with \( p^* \in G \) and work in \( V[G] \). First note that \( \{ x \in S^* : x \subseteq \bigcup u(p^*) \} = u(p^*) \). To show that \( q^* \in \mathbb{C}(S) \), take \( x \subseteq d(q^*) \) with \( x \in S \). We check that \( x \) is not closed under \( q^* \). Since \( x \subseteq d(q^*) = \bigcup u(p^*) \) and \( x \in S \subseteq S^* \), we have \( x \in u(p^*) \). Hence there exists \( X \in \text{dom}(p^*) \) such that \( x \in p^*(X) \). Because \( \text{max}(p^*(X)) \) is closed under \( g_X \), we know \( \text{max}(p^*(X)) \notin S \).

Thus \( x \neq \text{max}(p^*(X)) \) and \( x \in p_i(X) \) for some \( i < \gamma \) with \( X \in \text{dom}(p_i) \). Then \( x \subseteq \bigcup u(p_i) = d(q_i) \). Since \( q_i \) is a condition, \( x \) is not closed under \( q_i \), and not closed under \( q^* \).

\( \square \)[Claim]

Note that, in fact, \( D' \) is \( \kappa \)-directed closed.

By an iteration of the above forcing, we can prove Theorem 1.3. Let \( \langle \mathbb{P}_\xi, \dot{Q}_\eta : \xi < \zeta, \eta < \zeta \rangle \) be a \( < \kappa \)-support iteration such that for every \( \xi < \zeta \),

1. \( \dot{Q}_0 = \mathbb{P}_T \),
2. \( \mathbb{P}_\xi \) satisfies the \( \kappa^+ \)-c.c. and has a \( \kappa \)-closed dense subset,
3. for \( \xi > 0 \) there exists \( \mathbb{P}_\xi \)-name \( \dot{S}_\xi \) such that

\[ \models \xi " \dot{S}_\xi \subseteq \dot{S}^* \) and \( \forall X \in T (\mathbb{P}_\kappa X \cap \dot{S}_\xi \) is non-stationary in \( \mathbb{P}_\kappa X " \),

4. for every \( X \in T \), \( \dot{g}_X^\xi \) is a \( \mathbb{P}_\xi \)-name such that

\[ \models \xi " \dot{g}_X^\xi : [X]^{<\omega} \rightarrow X \) and \( \forall x \in \mathbb{P}_\kappa X (x \in \dot{S}_\xi \Rightarrow x \) is not closed under \( \dot{g}_X^\xi " \),

5. \( \models \xi " \dot{Q}_\xi = \mathbb{C}(\dot{S}_\xi) " \) for \( \xi > 0 \),
6. let \( D_\xi \) is the set of all \( p \in \mathbb{P}_\xi \) such that
\begin{align*}
(a) \quad & \forall \eta \in \text{supp}(p) \setminus \{0\} \ (p(\eta) = \check{r} \text{ for some } r \in C), \\
(b) \quad & \text{for all } X \in \text{dom}(p(0)) \text{ and } \eta \in \text{supp}(p) \setminus \{0\} \ (p|\eta \vDash \eta \ "\max(p(0)(X)) \text{ is closed under } \check{g}_X^\eta"), \\
(c) \quad & \bigcup\{u(p(0)) \mid d(p(\eta)) \text{ for all } \eta \in \text{supp}(p) \setminus \{0\}.
\end{align*}

Then \(D_\xi\) is a \(\kappa\)-closed dense set in \(\mathbb{P}_\xi\).

Let \(\mathbb{P}_\zeta\) and \(D_\zeta\) be as intended. We can check that \(D_\zeta\) is a \(\kappa\)-closed dense set in \(\mathbb{P}_\zeta\), and \(\mathbb{P}_\zeta\) has the \(\kappa^+\)-c.c.

By a standard book keeping method, we can destroy the stationarity of all non-reflecting subset of \(S^*\) by an iteration above. By \(\kappa^+\)-c.c., \(T\) remains stationary in \(\mathcal{P}_\kappa\lambda\) in the generic extension. Thus \(S^*\) is stationary in \(\mathcal{P}_\kappa\lambda\), and \(\text{RP}(S^*, T)\) holds.

4. Proof of Theorems 1.6 and 1.7

**Proposition 4.1.** Let \(\mu\) be a regular cardinal with \(\kappa^+ \leq \mu \leq \lambda\). Let \(T = \{X \in \mathcal{P}_\kappa\lambda : \kappa \subseteq X, \text{ cf}(X \cap \mu) < \kappa\}\). Then for every stationary sets \(S_0^*, S_1^* \subseteq \mathcal{P}_\kappa\lambda\), \(\text{RP}(S_0^*, S_1^*, T)\) fails.

**Proof.** Suppose not. For each \(\xi \in E^\kappa_{\kappa}\), fix an increasing sequence \(\langle \gamma_i^\xi : i < \text{cf}(\xi) \rangle\) with limit \(\xi\). For \(n < 2, i < \kappa\), and \(\delta < \mu\), let

\[ S_{n,i,\delta} = \{ x \in S_n^* : \delta = \min(x \setminus \gamma_i^{\sup(x \cap \mu)}) \}. \]

**Claim 4.2.**

1. For every \(\xi < \mu\), there exist \(i < \kappa\) and \(\delta < \mu\) such that \(\delta > \xi\) and \(S_{0,i,\delta}\) is stationary.
2. For every \(i < \kappa\) and \(\delta < \mu\), if \(S_{0,i,\delta}\) is stationary then \(S_{1,i,\delta}\) is stationary.
3. For every \(i < \kappa\) and \(\delta_0, \delta_1 < \mu\), if \(S_{0,i,\delta_0}\) and \(S_{1,i,\delta_1}\) are stationary then \(\delta_0 = \delta_1\).

**Proof.** (1). Let \(T' = \{X \in T : S_0^* \cap \mathcal{P}_\kappa X \text{ is stationary, } \xi \in X\}\). \(T'\) is stationary in \(\mathcal{P}_\kappa\lambda\). Take \(X \in T'\). Then \(\text{cf}(X \cap \mu) < \kappa \subseteq X\) and \(\sup(X \cap \mu) > \xi\), hence there exists \(i \in X\) such that \(\gamma_i^{\sup(X \cap \mu)} > \xi\). By applying Fodor’s lemma to \(T'\), there exists \(i < \kappa\) such that \(T'' = \{ x \in T' : \gamma_i^{\sup(X \cap \mu)} > \xi \}\) is stationary in \(\mathcal{P}_\kappa\lambda\). For \(X \in T''\) let \(\delta_X = \min(X \setminus \gamma_i^{\sup(X \cap \mu)})\). By Fodor’s lemma again, there is \(\delta < \mu\) such that \(T^* = \{ X \in T'' : \gamma_i^{\sup(X \cap \mu)} > \xi, \delta = \min(X \setminus \gamma_i^{\sup(X \cap \mu)}) \}\) is stationary in \(\mathcal{P}_\kappa\lambda\).

Pick \(X \in T^*\). Since \(\text{cf}(X \cap \mu) < \kappa\), the set \(D_X = \{ x \in \mathcal{P}_\kappa X : \sup(x \cap \mu) = \sup(X \cap \mu), \delta < x \}\) contains a club in \(\mathcal{P}_\kappa X\). Clearly \(x \in S_{0,i,\delta}\) for each \(x \in D_X \cap S_0^*\).

This means that \(S_{0,i,\delta}\) is stationary in \(\mathcal{P}_\kappa\lambda\).

(2). By \(\text{RP}(S_0^*, S_1^*), T' = \{ X \in T : \delta \in X, S_{0,i,\delta} \cap \mathcal{P}_\kappa X, S_1^* \cap \mathcal{P}_\kappa X \text{ are stationary} \}\) is stationary in \(\mathcal{P}_\kappa\lambda\). Fix \(X \in T'\). Since \(S_{0,i,\delta} \cap \mathcal{P}_\kappa X\) is stationary in \(\mathcal{P}_\kappa X\) and
cf($X \cap \mu < \kappa$, we have that $\delta = \min(X \setminus \gamma_i^{sup(X \cap \mu)})$. By the same argument as (1), we have that $S_{1,i,\delta}$ is stationary in $\mathcal{P}_{\kappa}\lambda$.

(3). Let $X \in T$ be such that $\delta_0, \delta_1 \in X$ and $S_{0,i,\delta_0} \cap \mathcal{P}_{\kappa}X, S_{1,i,\delta_1} \cap \mathcal{P}_{\kappa}X$ are stationary. Choose $x_0 \in S_{0,i,\delta_0} \cap \mathcal{P}_{\kappa}X$ and $x_1 \in S_{1,i,\delta_1} \cap \mathcal{P}_{\kappa}X$ such that $sup(x_0 \cap \mu) = sup(x_1 \cap \mu) = sup(X \cap \mu)$ and $\delta_0, \delta_1 \in x_0 \cap x_1$. By the minimality of $\delta_0$, we have $\delta_0 \leq \delta_1$. Similarly we know $\delta_1 \leq \delta_0$. Therefore $\delta_0 = \delta_1$. \hfill $\Box$[Claim]

Hence we have that if $S_{0,i,\delta}$ and $S_{0,i,\delta'}$ are stationary, then $\delta = \delta'$.

For each $i < \kappa$, define $\delta_i < \mu$ as follows: if $S_{0,i,\delta}$ is stationary for some $\delta < \mu$, then let $\delta_i$ be a (unique) $\delta < \mu$ such that $S_{0,i,\delta}$ is stationary. If there is no such $\delta$, then let $\delta_i = 0$. Since $\mu = cf(\mu) > \kappa$, we know $sup_{i<\kappa} \delta_i < \mu$. But this contradicts (1) of the claim. \hfill $\Box$

**Proposition 4.3.** Let $S_n^0, S_n^1 \subseteq \mathcal{P}_{\kappa}\lambda$ be stationary and suppose $RP^2(S_n^0, S_n^1)$ holds. Then for every regular $\mu$ with $\kappa^+ \leq \mu \leq \lambda$, $\Box(\mu)$ fails.

**Proof.** We prove only the case $\mu = \lambda$. Other cases follow from similar arguments.

Toward the contradiction, suppose $\Box(\lambda)$ holds. Let $\langle C_{\xi} : \xi < \lambda \rangle$ be a $\Box(\lambda)$-sequence.

Let $T = \{X \in \mathcal{P}_{\kappa+\lambda} : cf(X) = \kappa \subseteq X\}$. We assumed $RP^2(S_n^0, S_n^1)$, but by the previous proposition, in fact $RP^2(S_n^0, S_n^1, T)$ holds.

For each $\alpha < \lambda$ and $n < 2$, let

$$S_{n,\alpha} = \{x \in S_n^* : C_{sup(x) \cap sup(x \cap \alpha)} = C_{\alpha} \cap sup(x \cap \alpha)\}.$$

Let $A_n = \{\alpha < \lambda : S_{n,\alpha}$ is stationary\}.

**Claim 4.4.** For each $n < 2$, $A_n$ is unbounded in $\lambda$.

**Proof.** Fix $n < 2$. By shrinking $S_n^*$ by a club in $\mathcal{P}_{\kappa}\lambda$, we may assume that the following:

1. For all $x \in S_n^*$ and $\alpha \in x$, if $x \cap \alpha$ is bounded in $\alpha$ then $cf(\alpha) \geq \kappa$.

2. For all $x \in S_n^*$ and $\alpha \in x \cap \mathcal{E}_{\lambda^+}$, $sup(x \cap \alpha) \in lim(C_{\alpha})$ holds.

Let $T' = \{X \in T : S_n^* \cap \mathcal{P}_{\kappa}X$ is stationary\}.

Then $T'$ is stationary in $\mathcal{P}_{\kappa+\lambda}$. To show that $A_n$ is unbounded, take $\xi < \lambda$. Fix $X \in T'$ with $sup(X) > \xi$. Since $cf(X) = \kappa$, the set $\{\beta < sup(X) : \beta \in lim(C_{sup(X)})\}$ contains a club in $sup(X)$. Note that $C_{sup(X) \cap \beta} = C_{\beta}$ for each $\beta$ from the club. Hence we know $S_X = \{x \in S_n^* \cap \mathcal{P}_{\kappa}X : C_{sup(x) = C_{sup(X) \cap sup(x)}\}$ is stationary in $\mathcal{P}_{\kappa}X$. Since $cf(sup(X)) = \kappa$, $lim(X) \cap lim(C_{sup(X)})$ is unbounded in $sup(X)$. Take $\beta \in lim(X) \cap lim(C_{sup(X)})$
with $\beta > \xi$ and $\text{cf}(\beta) < \kappa$. Note that $\{x \in \mathcal{P}_\kappa X : x \cap \beta \text{ is unbounded in } \beta\}$ contains a club. Since $\beta \in \lim(C_{\sup(X)})$, $C_{\sup(X)} \cap \beta = C_\beta$ holds. For each $x \in S_X$ such that $x \cap \beta$ is unbounded in $\beta$ and $\sup(x) > \beta$, let $\beta_x = \min(x \setminus \beta)$.

Case 1. $\beta_x = \beta$. Then $C_{\beta_x} \cap \sup(x \cap \beta_x) = C_{\beta} = C_{\sup(X)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x)$.

Case 2. $\beta_x > \beta$. Then $\sup(x \cap \beta_x) = \beta$ and $\beta = \sup(x \cap \beta) \in \lim(C_{\beta_x})$, hence $C_{\beta_x} \cap \beta = C_{\beta} = C_{\sup(X)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x)$.

Hence for each $x \in S_X$ such that $x \cap \beta$ is unbounded in $\beta$ and $\sup(x) > \beta$, we have $C_{\sup(x)} \cap \sup(x \cap \beta_x) = C_{\beta_x} \cap \sup(x \cap \beta_x)$. By applying Fodor's lemma to $S_X$, we can find $\beta_X \in X$ such that $\{x \in S_X : \beta_X = \beta_x\}$ is stationary. Thus $\{x \in S^* \cap \mathcal{P}_\kappa X : C_{\sup(x)} \cap \sup(x \cap \beta_x) = C_{\beta_x} \cap \sup(x \cap \beta_X)\}$ is stationary.

By applying Fodor's lemma to $T^*$, we have $\beta_* < \lambda$ such that $\{X \in T^* : \beta_* = \beta_X\}$ is stationary. Then $S_{n, \beta_*}$ is stationary and $\beta_* > \xi$.

Claim 4.5. For each $\alpha \in A_0$ and $\beta \in A_1$ with $\alpha < \beta$, $C_\alpha = C_\beta \cap \alpha$ holds.

**Proof.** Let $T^* = \{X \in T : S_{0, \alpha} \cap \mathcal{P}_\kappa X, S_{1, \beta} \cap \mathcal{P}_\kappa X \text{ are stationary in } \mathcal{P}_\kappa X\}$. Take $X \in T^*$. Since $D_X = \{x \in \mathcal{P}_\kappa X : C_{\sup(x)} \cap \sup(x) = C_{\sup(x)}\}$ contains a club in $\mathcal{P}_\kappa X$, $D_X \cap S_{0, \alpha}$ is stationary in $\mathcal{P}_\kappa X$. For $x \in C_X \cap S_{0, \alpha}$, $C_\alpha \cap \sup(x \cap \alpha) = C_{\sup(x)} \cap \sup(x \cap \alpha) = C_{\sup(x)} \cap \sup(x \cap \alpha)$ holds. Since $\sup(x \cap \alpha) : x \in C_X \cap S_{0, \alpha}$ is unbounded in $\sup(x \cap \alpha)$, we have $C_{\sup(x)} \cap \sup(x \cap \alpha) = C_\alpha \cap \sup(x \cap \alpha)$.

Similarly, we have $C_\beta \cap \sup(X \cap \beta) = C_{\sup(X)} \cap \sup(X \cap \beta)$. Therefore we have $C_\alpha \cap \sup(X \cap \alpha) = C_\beta \cap \sup(X \cap \alpha)$.

Because $\{\sup(X \cap \alpha) : X \in T^*\}$ is unbounded in $\alpha$, we have $C_\alpha = C_\beta \cap \alpha$.

[][Claim]

Now, let $C = \{C_\beta : \beta \in A_0\}$. Since $A_0$ is unbounded, $C$ is unbounded. Furthermore, $C_\alpha = C_\beta \cap \alpha$ for all $\alpha < \beta \in A$; For $\alpha, \beta \in A_0$ with $\alpha < \beta$, choose $\gamma \in A_1$ with $\beta < \gamma$. Then $C_\alpha = C_\gamma \cap \alpha$ and $C_\beta = C_\gamma \cap \alpha$. Thus $C_\alpha = C_\beta \cap \alpha$.

Hence $C$ forms a club in $\lambda$. Take $\alpha \in \lim(C)$. Then there exists $\beta \in A_0$ such that $C \cap \alpha = C_\beta \cap \alpha$. Since $\alpha \in \lim(C)$, we know $\alpha \in \lim(C_\beta)$ and $C_\alpha = C_\beta \cap \alpha = C \cap \alpha$.

Thus $\forall \alpha \in \lim(C)$ ($C \cap \alpha = C_\alpha$), this is a contradiction.

[][Claim]}

Baumgartner [1] showed that if a weakly compact cardinal $\kappa$ is collapsed to $\omega_2$ by Levy-collapse with countable conditions, then $\text{RP}(\mathcal{P}_{\omega_1}(\omega_2))$ holds, and it is known that in fact $\text{RP}^2(\mathcal{P}_{\omega_1}(\omega_2), \mathcal{P}_{\omega_1}(\omega_2))$ holds in the generic extension. Conversely, Veličković [3] showed that if $\text{RP}(\mathcal{P}_{\omega_1}(\omega_2))$ holds, then $\omega_2$ is weakly compact in $L$. Consequently, we have the following equiconsistency:
Corollary 4.6. The following are equiconsistent:

(1) ZFC + “there exists a weakly compact cardinal”.
(2) ZFC + “$\text{RP}(\mathcal{P}_{\omega_1}\omega_2)$ holds”.
(3) ZFC + “$\text{RP}^2(\mathcal{P}_{\omega_1}\omega_2, \mathcal{P}_{\omega_1}\omega_2)$ holds”.
(4) ZFC + “$\text{RP}^2(S_0^*, S_1^*)$ holds for some stationary sets $S_0^*, S_1^* \subseteq \mathcal{P}_{\omega_1}\omega_2$”.

5. PROOF OF THEOREM 1.8

Proposition 5.1. Suppose $\text{RP}(S^*)$ for some stationary $S^* \subseteq \mathcal{P}_\kappa\lambda$. Then every $\kappa$-c.c. forcing preserves $\text{RP}(S^*)$.

Proof. First note that every $\kappa$-c.c. forcing preserves the stationarity of $S^*$.

Let $\mathbb{P}$ be a poset which satisfies the $\kappa$-c.c. Let $\dot{S}$ be a $\mathbb{P}$-name such that $\forces \"\dot{S} \subseteq S^*\"$ is stationary”. It is enough to show that there are some $p \in \mathbb{P}$ and $X \subseteq \mathcal{P}_\kappa\lambda$ such that $p \forces \"\dot{S} \cap X \text{ is stationary in } \mathcal{P}_\kappa X\"$.

Let $S' = \{x \in S^* : \exists p \in \mathbb{P} (p \forces \"x \in \dot{S}\") \}$. It is easy to check that $S'$ is a stationary subset of $S^*$. By $\text{RP}(S^*)$, there is $X \subseteq \mathcal{P}_\kappa\lambda$ such that $|X| = \kappa \subseteq X$ and $S' \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$. We see that $p \forces \"\dot{S} \cap \mathcal{P}_\kappa X \text{ is stationary}\"$ for some $p \in \mathbb{P}$. Suppose to the contrary that $\forces \"\dot{S} \cap \mathcal{P}_\kappa X \text{ is non-stationary}\"$. Since $|X| = \kappa$ and $\mathbb{P}$ satisfies the $\kappa$-c.c., we can find a club $C \subseteq \mathcal{P}_\kappa X$ such that $\forces \"\dot{S} \cap C = \emptyset\"$. $S' \cap \mathcal{P}_\kappa X$ is stationary, hence there is $x \in S' \cap C$. Pick $p \in \mathbb{P}$ with $p \forces \"x \in \dot{S}\"$. Then $p \forces \"x \in \dot{S} \cap C\",$ this is a contradiction.

Recall that $\text{PFA}^{++}$ is the assertion that for every proper forcing notion $\mathbb{P}$, every dense subsets $D_i$ ($i < \omega_1$) of $\mathbb{P}$, and every $\mathbb{P}$-names $\dot{S}_i$ ($i < \omega_1$) for stationary subsets of $\omega_1$, there is a filter $F$ on $\mathbb{P}$ such that:

(1) $D_i \cap F \neq \emptyset$ for every $i < \omega_1$.
(2) $S_i = \{ \alpha < \omega_1 : \exists p \in F (p \forces \"\alpha \in \dot{S}_i\") \}$ is stationary in $\omega_1$ for $i < \omega_1$.

Proposition 5.2. Suppose $\text{PFA}^{++}$. Let $\lambda \geq \omega_2$. Then every c.c.c. forcing notion forces $\text{RP}^2(\mathcal{P}_{\omega_1}\lambda, \mathcal{P}_{\omega_1}\lambda)$.

Proof. Let $\mathbb{P}$ be a poset which satisfies the c.c.c. Let $\dot{S}_0, \dot{S}_1$ be $\mathbb{P}$-names so that $\forces \"\dot{S}_0, \dot{S}_1 \subseteq \mathcal{P}_{\omega_1}\lambda \text{ are stationary}\"$. We will find $p \in \mathbb{P}$ and $X \subseteq \mathcal{P}_{\omega_2}\lambda$ such that $p \forces \"\dot{S}_0 \cap \mathcal{P}_{\omega_1} X, \dot{S}_1 \cap \mathcal{P}_{\omega_1} X \text{ are stationary}\"$.

Let $\dot{Q}$ be a $\mathbb{P}$-name for a $\sigma$-closed poset which adds a bijection from $\omega_1$ to $\lambda$. We know that $\forces_{\mathbb{P} \upharpoonright \dot{Q}} \"\dot{S}_0, \dot{S}_1 \text{ remain stationary}\"$. Fix a $\mathbb{P} \ast \dot{Q}$-name $\dot{\pi}$ for a bijection from $\omega_1$ to $\lambda$. Let $\dot{E}_0, \dot{E}_1$ be $\mathbb{P} \ast \dot{Q}$-names such that $\forces_{\mathbb{P} \ast \dot{Q}} \"\dot{E}_i = \{ \alpha < \omega_1 : \dot{\pi} \-
\alpha \cap \omega_1 = \alpha \}\"$ for $i = 0, 1$. We know $\forces_{\mathbb{P} \ast \dot{Q}} \"\dot{E}_i \text{ is stationary in } \omega_1\"$. 


Now fix a sufficiently large regular cardinal $\theta$ and take $M \prec H_\theta$ such that $|M| = \omega_1 \subseteq M$ and $M$ contains all relevant objects.

$\mathbb{P} \ast \dot{\mathbb{Q}}$ is proper, hence we can apply PFA$^{++}$ to $\mathbb{P} \ast \dot{\mathbb{Q}}$ and $\dot{E}_i$. By PFA$^{++}$ we can find a filter $F$ on $\mathbb{P} \ast \dot{\mathbb{Q}}$ such that:

1. $F \cap D \neq \emptyset$ for all dense $D \in M$ in $\mathbb{P} \ast \dot{\mathbb{Q}}$.
2. $E_i = \{ \alpha < \omega_1 : \exists p \in F (p \vDash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \alpha \in \dot{E}_i) \}$ is stationary in $\omega_1$ for $i = 0, 1$.

Let $X = \{ \beta < \lambda : \exists p \in F \exists \alpha < \omega_1 (p \vDash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \dot{\pi}(\alpha) = \beta) \}$. We can check that $|X| = \omega_1 \subseteq X$.

Since $\dot{S}_0, \dot{S}_1$ are names for subsets of $\mathcal{P}_{\omega_1}^V \lambda$, for each $\alpha \in E_i$, we can find $x \in \mathcal{P}_{\omega_1} \lambda$ and $p \in F$ such that $x \cap \omega_1 = \alpha$ and $p \vDash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \dot{\pi} \alpha = x$. Moreover it is easy to see that $x \in \mathcal{P}_{\omega_1} X$.

For $i < 2$ and $\alpha \in E_i$, take $x_{i,\alpha} \in \mathcal{P}_{\omega_1} X$ such that there is $p \in F$ with $p \vDash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \dot{\pi} \alpha = x_{i,\alpha}$. Let $S_i = \{ x_{i,\alpha} : \alpha \in E_i \}$. The following are easy to check for $i < 2$:

1. $x_{i,\alpha} \subseteq x_{i,\beta}$ holds for $\alpha, \beta \in E_i$ with $\alpha < \beta$.
2. If $\alpha \in \text{lim}(E_i) \cap E_i$, then $x_{i,\alpha} = \bigcup_{\beta \in E_i \cap \alpha} x_{i,\beta}$.
3. $\bigcup S_i = X$.

Furthermore, since $E_i = \{ x_{i,\alpha} \cap \omega_1 : \alpha \in E_i \}$ is stationary in $\omega_1$, we can check that each $S_i$ is stationary in $\mathcal{P}_{\omega_1} X$.

Now we see that $p \vDash_{\mathbb{P}} \dot{S}_0 \cap \mathcal{P}_{\omega_1} X, \dot{S}_1 \cap \mathcal{P}_{\omega_1} X$ are stationary" for some $p \in \mathbb{P}$. Suppose otherwise. Since $\mathbb{P}$ satisfies the c.c.c. and $|X| = \omega_1$, we can find a club $C$ in $\mathcal{P}_{\omega_1} X$ such that $p \vDash_{\mathbb{P}} \dot{C} \cap \dot{S}_0 = \emptyset$ or $C \cap \dot{S}_1 = \emptyset$.

Since $S_0$ and $S_1$ are stationary in $\mathcal{P}_{\omega_1} X$, we can find $x_0 \in S_0 \cap C$ and $x_1 \in S_1 \cap C$. Then there is $q \in F$ such that $q \vDash_{\mathbb{P} \ast \dot{\mathbb{Q}}} \dot{x}_0 \in \dot{S}_0$ and $x_1 \in \dot{S}_1$. Thus $q \vDash_{\mathbb{P} \ast \dot{\mathbb{Q}}}$ $C \cap \dot{S}_0 \neq \emptyset$ and $C \cap \dot{S}_1 \neq \emptyset$, this is a contradiction.  

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