<table>
<thead>
<tr>
<th>Title</th>
<th>Iterated proper forcing with side conditions (Forcing extensions and large cardinals)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MIYAMOTO, Tadatoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 数理解析研究所講究録</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195134">http://hdl.handle.net/2433/195134</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Iterated proper forcing with side conditions

Tadatoshi MIYAMOTO

March 25th 2013

Abstract

We formulate an iterated proper forcing with side conditions. We start with a measurable cardinal. Then we construct our iterated proper forcing along a sequence of increasing transitive models. The measurable cardinal is turned into the second uncountable cardinal. This is a rendition of one of Neeman’s original constructions with two types of models.

Introduction

Neeman introduced a new method of iterating proper forcing in [N]. The new method makes use of models of set theory and is of finite in nature. This note is a rendition of a small fraction of [N]. We start with a measurable cardinal $\kappa$. We formulate an iterated forcing along a sequence of transitive models of set theory that cofinally $\in$-increase in $H_\kappa$. Hence it looks like an ordinary iterated forcing with small initial segments. However in this new iteration projections are locally done. Each condition has limited initial segments and so limited access to intermediate stages.

There is a similar construction that adds many reals by iterating a class of proper forcing in [AM]. Their iteration takes notions of forcing stronger than just being proper and uses systems of countable elementary substructures with markers so that the iteration is proper. Their conditions have every initial segment and so access to every intermediate stage.

Since these new methods are of finite in nature, reals are to be added or even intended to add. Hence it appears that preserving, say the Continuum Hypothesis, there remain roles of countable support iterated proper forcing. The iterations in this note preserve $\omega_1$ and automatically collapse the cardinals strictly between $\omega_1$ and $\kappa$. And $\kappa$ is turned into new $\omega_2$. Hence, the idea of forming iterated forcing along a sequence of transitive models, as in this note, does not work in contexts of higher analogues of proper forcing, where both $\omega_1$ and $\omega_2$ are intended to be preserved with new $\omega_3$.

Preliminary

The items in this section are in use throughout this note without being mentioned. Let $\kappa$ be a regular uncountable cardinal. Let $H_\kappa = \{x \mid |TC(x)| < \kappa\}$, where $TC(x)$ denotes the transitive closure of $x$. Let $X \prec H_\kappa$ abbreviate that $(X, \in)$ is an elementary substructure of $(H_\kappa, \in)$. Typically $X$ is countable but not restricted in general. Hence we may consider $X = H_\theta \prec H_\kappa$, where $\theta$ would be a singular cardinal. Let $P$ be a preorder (i.e. reflexive and transitive) and $P \in X \prec H_\kappa$. Then we denote $X[G] = \{\tau \mid \tau \in X\}$ is a $P$-name, where $G$ is any $P$-generic filter over the ground model $V$.

**Lemma A.** Let $P$ be a preorder and $\kappa$ be a regular uncountable cardinal with $P \in H_\kappa$. Let $P \in X \prec H_\kappa$. Then

1. For any $P$-name $\tau$, there exists a $P$-name $\pi \in H_\kappa$ such that $\Vdash P \ \if \ \tau \in (H_\kappa)^{V[G]}, \ \then \ \tau = \pi^n$.
2. And so $\Vdash P \ "H_\kappa[G] = (H_\kappa)^{V[G]}$.
3. For any $\in$-formula $\phi(y, v)$ and any $P$-name $\tau \in X$, there exists a $P$-name $\pi \in X$ such that $\Vdash P "H_\kappa[G] \ |= \ "V y (\phi(y, \tau) \rightarrow \phi(\pi, \tau))"^n$.
4. And so $\Vdash P "X \cup \{G\} \subseteq X[G] \prec H_\kappa[G]^n$.

The following contents may be improved but suffices for this note.

**Lemma B.** Let $P$ be a preorder and $\kappa$ be a regular uncountable cardinal. Let $\theta$ be a cardinal (may or may not be regular). If $\theta$ is singular, then think of $H_\theta = \bigcup\{H_\chi \mid \chi < \theta, \chi$ is regular $\}$. Let $P \in H_\theta \in N \prec H_\kappa$, $P \in N$ and $N$ be countable. Then
(1) $P \in N \cap H_\theta \prec H_\theta$.
(2) $\models \neg \exists H_\theta[G] \in N[G] \prec H_\theta[G] = (H_\theta)^{V[G]}_\theta$.
(3) $\models \neg \exists H_\theta[G] \cap H_\theta[H_\theta[G] = (H_\theta)^{V[H_\theta]}_\theta$.
(4) For any $P$-name $\tau \in N$, there exists a $P$-name $\pi \in N \cap H_\theta$ such that $\models \neg \tau \in (H_\theta)^{V[G]}_\theta$, then $\tau = \pi$.
(5) And so $\models \neg \exists H_\theta[G] \cap H_\theta[H_\theta[G] = (N \cap H_\theta)[G]$.

Furthermore, let $Q \in N \cap H_\theta$ and $\models \neg \exists \bar{Q}$ be a preorder. Let $\theta$ be singular such that $H_\theta \prec H_\theta$. Then

(6) $\models \neg \exists \bar{Q} \in N[G] \prec H_\theta[G]$.
(7) $\models \neg \exists \bar{Q} \in (N \cap H_\theta)[G] = N[G] \cap H_\theta[G] \prec H_\theta[G] \prec H_\theta[G]$.
(8) $\models \neg \exists \bar{q}$ is $(Q, N[G])$-generic iff $q$ is $(Q, (N \cap H_\theta)[G])$-generic$^\ast$.

In the following, we state it with two countable elementary substructures for simplification.

**Lemma C.** Let $\kappa$ be a regular uncountable cardinal and $\theta$ be of uncountable cofinality. Let $H_\theta \in N \in M$, $N \prec H_\theta$, $M \prec H_\theta$ and both $N$ and $M$ be countable. Then $N \subseteq M$ and $N \cap H_\theta \in M \cap H_\theta \in H_\theta$ holds.

The following is related to atomlessness of relevant preorders and hence adding Cohen reals.

**Lemma D.** Let $\kappa$ be a measurable cardinal. Let $f : [H_\kappa]^{<\omega} \to H_\kappa$. Then

(1) There exists a regular uncountable cardinal $\theta < \kappa$ such that $H_\theta$ is closed under $f$.
(2) For any $X \subseteq H_\kappa$ and $a \in X$, there are two countable sets $M_1, M_2$ of $H_\kappa$ such that $M_1, M_2$ are closed under $f$, $a \in M_1 \cap M_2$, $M_1 \cap X = M_2 \cap X$ and $M_1 \cap \kappa \neq M_2 \cap \kappa$.

§ 1. Constructions of Iterations

In the rest of this note, we assume that $\kappa$ is a strongly inaccessible cardinal and denote $K = H_\kappa$. Let

$$T = \{H_\theta \in K \mid \lambda \in H_\theta \prec K, [H_\lambda]^{\omega} \subseteq H_\theta\}$$

We also assume that $(H_\theta \in T \mid \theta$ is regular uncountable) is $\varepsilon$-cofinal in $K$. We indicate when we further assume that $\kappa$ is measurable. Let us denote

$$S = \{N \in K \mid N \text{ is countable, } N \prec K\}.$$ 

The elements of $T$ and $S$ are called of transitive type (or rank type) and of small type (or countable type), respectively. Hence we are concerned with the elementary substructures, of transitive or countable, of $K$.

We construct a preorder $P_\kappa$ that is the direct limit of an iterated forcing $(P_i \mid i < \kappa)$. This $(P_i \mid i < \kappa)$ has associated sequences $(K_i \mid i < \kappa)$ and $(\dot{Q}_i \mid i < \kappa)$. The sequence $(K_i \mid i < \kappa)$ is $\varepsilon$-increasing in $\varepsilon$-cofinal in $T$ and so in $K$. The sequence $(\dot{Q}_i \mid i < \kappa)$ lists names such that for each $i < \kappa$, we have $\models \neg \exists P_i \dot{Q}_i$ is proper and the associated two stage iteration $P_i \ast \dot{Q}_i$ is formed as a preorder in your favorite manner. We have $P_i \subseteq K_i$ and $P_i, Q_i, P_i \ast \dot{Q}_i \subseteq K$ for all $i < \kappa$. The finally constructed $P_\kappa = \bigcup P_i \mid i < \kappa$ does not satisfy the $\kappa$-chain condition. We demand the continuity as in (2) below so that $P_\kappa$ preserves $\kappa$ to be a cardinal. By the properness of $P_\kappa$, the least uncountable cardinal $\omega_1$ is preserved and every cardinal strictly between $\omega_1$ and $\kappa$ is collapsed. Hence, we have $\kappa = (\omega_2)^{V[G_\kappa]}$, where $G_\kappa$ is any $P_\kappa$-generic filter over the ground model $V$.

**Lemma 1.1.** We may construct sequences $I = \langle (K_i, P_i, \dot{Q}_i, P_i \ast \dot{Q}_i) \mid i < \kappa \rangle$ such that for each $i < \kappa$

(0) $I[i] = \langle (K_j, P_j, \dot{Q}_j, P_j \ast \dot{Q}_j) \mid j < i \rangle \in K$, $(K_j \mid j < i)$ is an $\varepsilon$-increasing sequence of elements in $T$,

$$(P_j \mid j < i)$$

is a sequence of preorders and $(\dot{Q}_j \mid j < i)$ is a sequence of names such that for each $j < i$, we have $\models \neg P_j \dot{Q}_j$ is a proper preorder and the associated two stage iteration $P_j \ast \dot{Q}_j \equiv \{p, \tau \mid p \in P_j, \models \neg \tau \in \dot{Q}_j\}$ is formed as a preorder in your favorite manner.
(1) If $i$ is 0, a successor ordinal or a limit ordinal with $\text{cf}(i) = \omega$, then $I[i \in K_i \in T$ and $\theta_i$ is a regular uncountable cardinal, where $K_i = H_{\theta_i}$.

(2) If $i$ is a limit ordinal with $\text{cf}(i) \geq \omega_1$, then $K_i = \bigcup\{K_j \mid j < i\}$.

(3) (Basic Closure) If $m < i$, then $I[(m+1) = \langle(K_{j}, P_{j}, Q_{j}, P_{j} \ast Q_{j}) \mid j \leq m\rangle \in K_i$.

(4) $P_i$ is a preorder such that for each element $p \in P_i$, $p$ is a pair and $P_1 \subset K_1$ holds.

(5) For $p = (f^p, A^p), p \in P_i$ iff

- $A^p$ is a finite $\epsilon$-chain, $A^p = T^p \cup S^p$, $T^p \subset \{j \mid j < i\}$ and $S^p \subset S \cap K_i$.
- (Basic Closure) If $N \in S^p$ and $K_j \in N$ (note that we do not talk about $j \in N$, because $N$ may contain $j$ bigger than $i$ and that $K_j$ may or may not be in $T^p$), then $I[(j+1) \in N$.
- (Intersection Property) For all $X, Y \in A^p, X \cap Y \in A^p$.
- $f^p$ is a finite function with $\text{dom}(f^p) = \{j < i \mid K_j \in A^p\}$.
- (Local Projection) For $j \in \text{dom}(f^p)$, we have $w(p, j) \in P_j, (w(p, j), f^p(j)) \in P_j \ast Q_j$ and $\not\vdash f^p(j) \in \dot{Q}_j$, where $w(p, j) = (f^p(j), A^p \cap K_j)$.
- (Generic) For $j \in \text{dom}(f^p)$ and $N \in S^p$ with $K_j \in N$, $|\not\vdash f^p(j) = (\dot{Q}_j, N[G_j])$-generic $|$.

(6) For $p, q \in P_i, q \leq p$ in $P_i$ iff

- $A^q \supset A^p$.
- $\not\vdash w(q, j) \in P_j \ast Q_j$.

(7) $|\not\vdash f^q, \dot{Q}_q \in K[G_i]$ is proper $|$ (whose exact choice depend on the purposes) and the associated two stage iteration $P_1 \ast \dot{Q}_1 \in K$ formed.

Proof. Here is a recursive construction of $K_i, P_i, \dot{Q}_i$ and $P_i \ast \dot{Q}_i$. Suppose $i < \kappa$ and we have constructed $I[i = \langle(K_j, P_j, Q_j, P_1 \ast Q_1) \mid j < i\rangle$.

To construct $K_i$, pick any sufficiently large $K_i \in T$ as in (1), unless $i = 0$ or $\text{cf}(i) \geq \omega_1$. As far as $K_0$ is concerned, we are free to choose any $K_0 = H_{\theta_0} \in T$ with a regular uncountable cardinal $\theta_0$. To construct $P_i$, let $p = (f^p, A^p) \in P_i$, if either the following (1) or (II) holds.

(I) $f^p = \emptyset$ and $A^p$ is a finite $\epsilon$-chain such that

- $A^p \subset S \cap K_i$.
- (Basic Closure) If $N \in A$ and $K_m \in N$, then $I[(m+1) \in N$.

(II) There exist $q \in P_j, j < i$ and $A$ such that

- $(q, \tau) \in P_j \ast \dot{Q}_j$ and $|\not\vdash \tau \in \dot{Q}_j$.
- $A$ is a finite $\epsilon$-chain such that for all $N \in A, K_j \in N \in S \cap K_i$.
- (Basic Closure) If $N \in A$ and $K_m \in N$, then $I[(m+1) \in N$.
- (Weak Intersection Property) For all $N \in A, K_j \cap N \in A^q$.
- (Weak Generic) For all $N \in A, |\not\vdash \tau$ is $(\dot{Q}_j, N[G_j])$-generic $|$.
- $f^q = f^p \cup \{(j, \tau)\}$ and $A^p = A^q \cup \{K_j\} \cup A$.

For $q, p \in P_i$, define $q \leq p$ in $P_i$, if

- $A^q \supset A^p$.
- For all $j \in \text{dom}(f^q)$, $w(q, j) \leq w(p, j)$ in $P_j$ and $w(q, j) \not\vdash f^q(j) \leq f^p(j)$ in $Q_j$.

Namely,

$$(w(q, j), f^q(j)) \leq (w(p, j), f^p(j))$$

in $P_j \ast \dot{Q}_j$.
To construct $\hat{Q}_i$ and $P_i * \hat{Q}_i$, let $\hat{Q}_i$ and $P_i * \hat{Q}_i$ be any as in (7). This completes the recursive construction of $K_i, P_i, \hat{Q}_i$ and $P_i * \hat{Q}_i$. We have to show that they satisfy the induction hypotheses (0) through (7). Since it is quite routine, we make remarks rather than putting details.

For (0), (1) and (2): Due to the assumption on $\kappa$ and $T$, these are satisfied. Notice that there exists a lot of freedom in choosing $K_i$ for $i = 0$, successors or limits with countable cofinality. Also notice that $\theta_i$ has no choice other than specified and may not be regular, when $\text{cf}(i) \geq \omega_1$ but $K_i \in T$ holds.

For (3): If $i$ is a limit with $\text{cf}(i) \geq \omega_1$ and $m < i$, then $\theta^{(m+1)} \in K_{m+1} \in K_i$ by induction.

For (4): We have to wait for (5). But $P_i \subseteq K_i \subseteq P_{i+1}$. Hence the initial segments are all small.

For (5): Item (3) (Basic Closure) is a prerequisite to (Basic Closure) on $N$ in (5). (Weak Intersection Property) entails (Intersection Property) by induction. We liked to consider $\text{dom}(f^p) = \{j < i \mid K_j \in A^p\}$ over $\text{dom}(f^p) = \{j < i \mid K_j \in A^p\}$, the whole indices of transitive type in $A^p$. We also demand $\forall P_i ^- fP(j)$ is $(Q_j, N[Q_j])$-generic" with the Boolean value one. These two for simplification. The witness $w(p,j)$ defined if and only if $K_j \in T^p$.

For (6): This equivalence is to be used to establish $q \leq p$.

For (7): $P_i \in H_{\text{TC}(P_i)} \in K_{i+1}$ and $\forall P_i ^- \hat{Q}_i \in (H_{\text{TC}(Q_i)})^V[G_i] \in K_{i+1}[G_i] < [G_i]^\omega$. The minimal spaces to talk about generic conditions are available as points in $K_{i+1}$ and $K_{i+1}[G_i]$ respectively. □

**Lemma 1.2.** Let $I = \langle(K_i, P_i, \hat{Q}_i, P_i * \hat{Q}_i) \mid i < \kappa\rangle$ as above. Then for each $i < \kappa$

1. If $j < i$, then $P_j$ is a suborder of $P_i$.
2. If $\text{cf}(i) \geq \omega_1$, then $P_i = \bigcup\{P_j \mid j < i \}$.
3. (Local Projection) Let $p \in P_i$ and $K_j \in A^p$. Then

$$P_j * \hat{Q}_j \langle (w(p,j), f^p(j)) \rangle \longleftarrow P_i[p]$$

defined by $x \mapsto (w(x,j), f^x(j))$ is a projection. Namely,

- (Order) If $x \leq y \leq p$ in $P_i$, then $(w(x,j), f^x(j)) \leq (w(y,j), f^y(j)) \leq (w(p,j), f^p(j))$ in $P_j * \hat{Q}_j$.

- ("Reduction") If $y \leq p$ in $P_i$ and $(h, \pi) \leq (w(y,j), f^y(j))$ in $P_j * \hat{Q}_j$, then there exists $x \leq y$ in $P_i$ such that $((w(x,j), f^x(j)) \leq (h, \pi))$.

**Proof.** For (1): Directly by the recursive definition. Let $j < i < \kappa$ and $p \in P_j$.

**Case (I):** $p = (\emptyset, A^p)$. Since $A^p \subseteq S \cap K_j \subseteq S \cap K_i$, we are done.

**Case (II):** $p \in P_j$ is constructed from some $r \in P_i, l < j, \tau$ and $A$. Since $l < j < i$ and so $l < i$, we conclude $p \in P_i$.

Hence $P_j \subseteq P_i$. Let $p, q \in P_j$. It is straightforward to observe that $q \leq p$ in $P_j$ iff $q \leq p$ in $P_i$.

For (2): Let $p \in P_i$ and $\text{cf}(i) \geq \omega_1$. Assume we are in case (II). Then $p$ is constructed from some $q \in P_j, j < i$, $\tau$ and $A$. Since $A$ is finite and $S \cap K_i = S \cap \bigcup\{K_j \mid j < i\}$, we conclude $p \in P_m$ for some $m < i$.

For (3): (Well-def) Let $x \leq p$ in $P_i$. Then $K_j \subseteq A^p \subseteq A^x$. Since $w(x,j) \in P_j, (w(x,j), f^x(j)) \in P_j * \hat{Q}_j$ and $\forall P_i ^- f^x(j) \in \hat{Q}_j$". We also have $\langle w(x,j), f^x(j) \rangle \leq (w(p,j), f^p(j))$ in $P_j * \hat{Q}_j$.

(Order) Let $x \leq y$ in $P_i[p]$. Then by definition, $w(x,j) \leq w(y,j)$ in $P_j$ and $w(x,j) \models \forall P_i ^- f^x(j) \leq f^y(j)$ in $\hat{Q}_j$". This means $\langle w(x,j), f^x(j) \rangle \leq (w(y,j), f^y(j))$ in $P_j * \hat{Q}_j$.

("Reduction") Let $(h, \pi) \leq (w(y,j), f^y(j))$ in $P_j * \hat{Q}_j$. Then let $\pi^*$ be a $P_j$-name such that $\models P_i ^- \pi^*$ "if $h \in G_j$, then $\pi^* = \pi$, else $\pi^* = f^y(j)$" and so $\models P_i ^- \pi^* \in \hat{Q}_j$". Then for all $M \in (K_j, K_i) \cap S \cap A^p$, we have $\models P_i ^- \pi^* \in (Q_j, M[G_j])$-generic". Let $x = (f^x, A^x)$, where

- $A^x = A^h \cup A^p$. 
\[ f^e = f^h \cup \{(j, \pi^*)\} \cup f^\mathfrak{r}(j, i). \]

Then we may confirm \( x \in P_i \) and \( x \leq y \) in \( P_i \) by possible repeated uses of (II). We also have \( (w(x, j), f^e(j)) = (h, \pi^*) \leq (h, \pi) \).

\[ \square \]

**Definition 1.3.** Let \( I \) be as above. Let us write \( K_\kappa = K \). Let \( P_\kappa = \bigcup \{ P_i \mid i < \kappa \} \) and for \( p, q \in P_\kappa \), let \( q \leq p \) in \( P_\kappa \), if \( q \leq p \) in some (all) \( P_i \) with \( p, q \in P_i \).

We may view \( P_\kappa \) constructed from \( I = I/\kappa \) as in the recursive construction of \( I \) with \( i = \kappa \). But \( \hat{Q}_\kappa \) is not yet. We study the preorder \( P_\kappa \) in the next section.

§ 2. Amalgamations

In this section, whenever we write \( M \prec H_\chi \), we mean \( M \) is countable and \( \chi \) is a regular uncountable cardinal. Given \( p \in P_i \), we may add new \( N \in S \to S^p \).

**Lemma 2.1.** Let \( i \leq \kappa \) and let \( I[i, K_i, P_i] \in M \prec H_\chi \) so that \( M \cap K_i \in S \cap K_i \) satisfies (Basic Closure). Let \( p \in M \cap P_i \). Then there exists \( q \in P_i \) such that \( q \leq p \) in \( P_i \) and \( M \cap K_i \in A^q \).

**Proof.** We define \( q = (f^q, A^q) \) and \( A^q = T^q \cup S^q \), where

1. \( T^q = T^p \).
2. \( S^q = S^p \cup \{ M \cap K_i \mid K_i \in T^p \cup \{ K_i \} \} \).
3. \( \langle l \mapsto \pi \mid K_i \in T^p \rangle \) satisfy
4. \( i \leq p \) and \( M \cap K_i \in A^q \).

Then it is routine to check that \( q = (f^q, A^q) \in P_i \), \( q \leq p \) in \( P_i \) and \( M \cap K_i \in A^q \).

\[ \square \]

Given \( q \in P_i \), we consider an appropriate copy of \( q \) in \( M \cap P_i \) and possibly an extension \( r \) of it in \( M \cap P_i \), where \( P_i \in M \prec H_\chi \) and \( M \cap K_i \in S^q \). Then we may cook a common extension of \( q \) and \( r \) in \( P_i \), which we call an amalgamation. There are four cases depending on how \( M \cap K_i \) is listed in \( A^q \).

**Lemma** (Amalgamation 00). Let \( i \leq \kappa \) and let \( I[i, K_i, P_i] \in M \prec H_\chi \) and let \( q \in P_i \) with \( M \cap K_i = N \in A^q \). Let

1. \( A^q \cap T = \emptyset \).
2. \( S_0 = N \cap A^q \) and \( S_1 = [N, K_i] \cap A^q \).
3. \( r \in M \cap P_i \) and \( S_0 \subseteq A^\mathfrak{r} \).

Then \( r \) and \( q \) are compatible in \( P_i \).

**Proof.** We define \( e = (f^e, A^e) \) with \( A^e = T^e \cup S^e \). Let

- \( T^e = T^r \).
- \( S^e = S^p \cup \{ N \cap K_i \mid N \in S_1, K_i \in T^p \} \cup S_1 \).
- \( \langle l \mapsto \pi \mid K_i \in T^r \rangle \) such that
  - \( i \leq p \) and \( N \in S_1, \pi \) is \( (\hat{Q}_I, \overline{N}[G_i]) \)-generic.
  - \( f^e = \{(l, \pi) \mid K_i \in T^r \} \).

Then it is routine to check that \( e = (f^e, A^e) \in P_i \) and that \( e \leq r, q \) in \( P_i \).
Lemma (Amalgamation 01). Let $i \leq \kappa$ and let $I[i, K_i, P_i \in M < H_\chi$ and let $q \in P_i$ with $M \cap K_i = N \in A^q$. Let

1. $K_j \in A^q$, $K_j \cap A^q \cap T = \emptyset$ and $N \in (K_j, K_i)$.
2. $S_0 = N \cap A^q$ and $S_1 = [N, K_j) \cap A^q$.
3. $r \in M \cap P_i$ and $S_0 \subseteq A^r$.

Then $r$ and $q$ are compatible in $P_i$.

Proof. We may use (Amalgamation 00). Let us consider $w(q, j^*)$ and $r$. Apply (Amalgamation 00) to get $e \in P_i$ with $e \leq w(q, j^*)$, $r$ in $P_i$. Since $e \leq w(q, j^*)$ in $P_i$ holds, we may take $e' \in P_i$ with $e' \leq q$ in $P_i$ and $w(e', j^*) = e$. Then $e' \leq w(e', j^*)$ in $P_i$ and so $e' \leq q, r$ in $P_i$ holds.

Lemma (Amalgamation 10). Let $i \leq \kappa$ and let $I[i, K_i, P_i \in M < H_\chi$ and let $q \in P_i$ with $M \cap K_i = N \in A^q$. Let

1. $K_j \in A^q$, $(K_j, K_i) \cap A^q \cap T = \emptyset$ and $N \in (K_j, K_i)$.
2. $S_0 = (K_j, N) \cap A^q$ and $S_1 = [N, K_i) \cap A^q$.
3. $r \in M \cap P_i$ and $\{K_j\} \cup S_0 \subseteq A^r$.
4. $(\text{Head}) (w(q, j), f^q(j))$ and $(w(r, j), f^r(j))$ are compatible in $P_j * \dot{Q}_j$.

Then $r$ and $q$ are compatible in $P_i$.

Proof. We define $e = (f^e, A^e)$ with $A^e = T^e \cup S^e$. Let

- $(h, \pi) \leq (w(q, j), f^q(j)), (w(r, j), f^r(j))$ in $P_j * \dot{Q}_j$.
- $T^e = T^h \cup (T^r \setminus K_j)$.
- $S^e = S^h \cup S^r \cup \{\overline{N} \cap K_l | \overline{N} \in S_1, K_l \in T^r \setminus (\{K_j\} \cup K_j)\} \cup S_1$.
- Let $\parallel_{P_j} "\pi^* \in \dot{Q}_j$ and $\pi^* \leq f^q(j)^n$.
- $h \parallel_{P_j} "\pi = \pi^*$.
- And so for all $\overline{N} \in S_1$, $\parallel_{P_j} "\pi^* is (\dot{Q}_j, \overline{N}[G_j])$-generic".
- Fix $(l \mapsto \tau_l | K_l \in T^r \setminus (\{K_j\} \cup K_j))$ such that
- $\parallel_{P_j} \tau_l \leq f^r(l)$ in $\dot{Q}_l$ and for all $\overline{N} \in S_1$, $\tau_l is (\dot{Q}_l, \overline{N}[G_l])$-generic".
- $f^e = f^h \cup \{(j, \pi^*)\} \cup \{(l, \tau_l) | K_l \in T^r \setminus (\{K_j\} \cup K_j)\}$.

Then it is routine to check that $e = (f^e, A^e) \in P_i$ and that $e \leq r, q$ in $P_i$.

Lemma (Amalgamation 11). Let $i \leq \kappa$ and let $I[i, K_i, P_i \in M < H_\chi$ and let $q \in P_i$ with $M \cap K_i = N \in A^q$. Let

1. $K_j, K_{j^*} \in A^q$, $(K_j, K_{j^*}) \cap A^q \cap T = \emptyset$ and $N \in (K_j, K_{j^*}) \cap A^q$.
2. $S_0 = (K_j, N) \cap A^q$ and $S_1 = [N, K_{j^*}) \cap A^q$.
3. $r \in M \cap P_i$ and $\{K_j\} \cup S_0 \subseteq A^r$.
3.1. We are compatible in $P_j \ast \dot{Q}_j$.

**Proof.** We may use (Amalagamation 10). Let us consider $w(q,j^*)$ and $r$. Apply (Amalagamation 10) to get $e \in P_i$ with $e \leq w(q,j^*), r$ in $P_i$. Since $e \leq w(q,j^*)$ in $P_j$, we may take $e' \in P_i$ with $e' \leq q$ in $P_i$ and $w(e',j^*) = e$. Then $e' \leq w(e',j^*)$ in $P_i$ and so $e' \leq q, r$ in $P_i$ holds.

\[\square]\]

### § 3. Preservation of Properness

Let $p \in P_i$. Then the proper initial segments $w(p,i)$ defined in $P_i$ are $(P_i, M)$-generic for right $M \in A^\kappa$. For the finite sequence $w(p,i)$ we may assume that there are $K_i, K_j \in A^\kappa$ as in (Amalagamation 11). Other cases are similar. Let $G_j \ast H_j$ be a $P_3 \ast \dot{Q}_j$-generic filter over $V$ with $(w(q,j), f^q(j)) \in G_j \ast H_j$. We have $M[G_j \ast H_j] \cap H_j = \{H_j\} [G_j \ast H_j]$ in $V[G_j \ast H_j]$. Since $i < l, q \in P_i, \{K_i, M \cap K_i\} \subseteq A^\kappa$ and $K_j \in M \cap K_i$, we may apply induction hypothesis to conclude $w(q,j)$ is $(P_i, M)$-generic. Hence $(w(q,j), f^q(j))$ is $(P_i \ast \dot{Q}_i, M)$-generic. Hence in $V[G_j \ast H_j], M[G_j \ast H_j] \cap V = M$ holds. Therefore there exists $r \in D \cap M$ such that $(K_j \cup \delta_0 \subseteq A\ast$ and $(w(r,j), f^r(j)) \in G_j \ast H_j$ due to $q$. Since $(w(q,j), f^q(j)), (w(r,j), f^r(j)) \in G_j \ast H_j$, they are compatible in $P_j \ast \dot{Q}_j$. Hence by (Amalagamation 11), we know that $q$ and $r$ are compatible in $P_i$. Hence we are done.

\[\square]\]

We show $(P_i \mid i \leq \kappa)$ is a sequence of proper preorders. Recall we set $I = \langle (K_i, P_i, \dot{Q}_i, P_i \ast \dot{Q}_i) \mid i < \kappa \rangle$. For $m \leq \kappa$, we write $I \upharpoonright m$ to denote the initial segment of $I$ by $m$. Namely, $I \upharpoonright m = \langle (K_i, P_i, \dot{Q}_i, P_i \ast \dot{Q}_i) \mid i < m \rangle$.

**Lemma 3.2.** Let $i \leq \kappa$ and $(K_i, P_i, \dot{Q}_i) \subseteq M \ast H_\kappa$. Then for all $p \in P_i \cap M$, there exists $q \in P_i$ such that $q \leq p$ in $P_i$ and $M \cap K_i \in A^\kappa$. This $q$ is $(P_i, M)$-generic and $P_i$ is proper.

**Proof.** Given $p$, we construct $q$ by Lemma 2.1. We know that the proper initial segments $w(q,j)$ and $(w(q,j), f^q(j))$ are $(P_j, N)$-generic and $(P_j \ast \dot{Q}_j, N)$-generic, respectively, for the right $N \in S^\kappa$ by Lemma 3.1. We have to show $q$ itself is $(P_i, M)$-generic. To this end we repeat the same argument. Let $D \subseteq P_i$ be dense and $D \in M$. We want to show $D \cap M$ is dense below $q$ in $P_i$. Let $d \leq q$ in $P_i$. We assume that $d \in D$. Since $M \cap K_i \in S^\kappa$, we have four cases depending on how $M \cap K_i$ is listed in $A^\kappa$. We assume that there are $K_j$ and $K_j^*$ as in (Amalagamation 11). Other cases are similar. Since $P_j \ast \dot{Q}_j \subseteq K_j$ and $(w(d,j), f^d(j))$ is $(P_j \ast \dot{Q}_j, M \cap K_j)$-generic, it is also $(P_j \ast \dot{Q}_j, M)$-generic. Let $G_j \ast H_j$ be $P_j \ast \dot{Q}_j$-generic with $(w(d,j), f^d(j)) \in G_j \ast H_j$. We have

\[M[G_j \ast H_j] \cap V = M.\]

We consider a suitable copy $r$ of $d$ in $M \cap D$. By (Amalagamation 11), we know that $r$ and $d$ are compatible in $P_i$.

\[\square]\]

The cardinals strictly between $\omega_1$ and $\kappa$ are collapsed.

**Lemma 3.3.**

(1) Let $i < \kappa$ and let $p \in P_{i+1}$ with $K_i \in T^p$. Then $p \forces_{P_{i+1}} \lceil K_{i+1} \rceil = \omega_1$.

(2) For all $i < \kappa$, we have $\lceil \dot{p}_i \rceil \cap \omega_1 = \omega_1$.
Proof. For any \( q \leq p \) in \( P_{\kappa+1} \) and any \( a \in K_{i+1} \), there exists \( r \in P_{\kappa+1} \) such that \( r \leq q \) in \( P_{\kappa+1} \) and that there exists \( N \in S^r \) with \( a \in N \). Hence \( K_{i+1} = \bigcup \{ N \mid N \in \langle K_i, K_{i+1} \rangle \cap S^r, r \in G_{i+1} \} \) holds, where \( G_{i+1} \) is any \( P_{i+1} \)-generic filter with \( p \in G_{i+1} \) over the ground model \( \mathcal{V} \). But \( \langle N \rightarrow N \cap \omega_1 \mid N \in \langle K_i, K_{i+1} \rangle \cap S^r, r \in G_{i+1} \rangle \) is one-to-one. Hence \( K_{i+1} = \mathcal{V} \) holds.

\[ \square \]

\( \mathcal{P}_\kappa \) does not have the \( \kappa \)-chain condition. To see this, let \( \langle N_\eta \mid \eta < \kappa \rangle \) be a sequence of different elements in \( S \) such that \( N_\eta \cap \omega_1 \) are constant. Then this gives rise to an antichain of size \( \kappa \) in \( \mathcal{P}_\kappa \).

**Lemma 3.4.** \( \mathcal{P}_\kappa \) preserves \( \kappa \) to be a cardinal. Hence \( \lnot \exists \mathcal{P}_\kappa \mathcal{V}^\omega_{\kappa+1} \).

**Proof.** Let \( \lnot \exists \mathcal{P}_\kappa \mathcal{V}^\omega_{\kappa+1} \). Since all \( \kappa \)-many \( P_{\kappa+1} \)-Cohen reals based on \([\mathcal{U}Y] \). Let \( i < \kappa \), we define a suborder of \( P_{\kappa+1} \) in \( \mathcal{V}[G_i \ast H_i] \),

\[ P_{\kappa+1}/G_i \ast H_i = \{ y \in P_{\kappa+1} \mid K_i \in T^9, (w(y,i), f^q(i)) \in G_i \ast H_i \} \]

where \( G_i \ast H_i \) is any \( P_i \ast \dot{Q}_i \)-generic filter over \( V \).

**Lemma 4.1.** Let \( i < \kappa \). Let \( I(i+1), K_{i+1}, P_{i+1} \in M \prec H_\chi \). Let \( p \in P_{i+1} \) with \( \langle K_i, M \cap K_{i+1} \rangle \subseteq A^P \). Then \( p \vdash_{P_{i+1}} \forall \mathcal{D} \in \mathcal{V}[G_i \ast H_i], D \subseteq M \cap \langle P_{i+1}/G_i \ast H_i \rangle \) dense, \( D \cap G_{i+1} \neq \emptyset \). And so if \( (w(q,i), f^q(i)) \vdash_{P_{i+1}} \langle \exists y \leq r \rangle \in G_{i+1} \ast H_i \) is atomless”, then \( p \vdash_{P_{i+1}} \langle \exists \rangle \in \mathcal{V}[G_i \ast H_i] \).

**Proof.** Let \( p' \leq p \) in \( P_{i+1} \) and let \( \mathcal{D} \) be a \( P_i \ast \dot{Q}_i \)-name with \( p' \vdash_{P_{i+1}} \langle \exists r \rangle \in \mathcal{V}[G_i \ast H_i], D \subseteq M \cap \langle P_{i+1}/G_i \ast H_i \rangle \) dense in \( M \cap \langle P_{i+1}/G_i \ast H_i \rangle \). We show \( E = \{ y \in P_{\kappa+1} \mid \exists y \leq r \in P_{i+1} \in \mathcal{V}[G_i \ast H_i] \) is dense below \( p' \) in \( P_{i+1} \). Hence \( p' \vdash_{P_{i+1}} \langle \exists r \rangle \in \mathcal{V}[G_i \ast H_i] \).

Let \( q \leq p' \) in \( P_{i+1} \). Let \( G_i \ast H_i \) be \( P_i \ast \dot{Q}_i \)-generic over \( V \) with \( (w(q,i), f^q(i)) \in G_i \ast H_i \). Since \( (w(q,i), f^q(i)) \) is \( \langle \exists r \rangle \in \mathcal{V}[G_i \ast H_i] \), we have

\[ \mathcal{M}[G_i \ast H_i] \cap V = M. \]

In \( \mathcal{V}[G_i \ast H_i] \), there exists \( q' \in P_{i+1} \cap M \) such that \( \langle K_i, M \cap K_{i+1} \rangle \cap S^q \subseteq A^9 \) and \( (w(q',i), f^{q'}(i)) \in G_i \ast H_i \). This holds due to \( q \). Since \( q' \in M \cap \langle P_{i+1}/G_i \ast H_i \rangle \) and \( D = \mathcal{D}_{G_i \ast H_i} \) is dense, we have \( r \subseteq M \cap \langle P_{i+1}/G_i \ast H_i \rangle \) with \( r \leq q' \) in \( P_{i+1}/G_i \ast H_i \) (in \( P_{i+1} \)). Take \( (x,\pi) \leq (w(q,i), f^q(i)) \in P_i \ast \dot{Q}_i \) and \( (x,\pi) \in G_i \ast H_i \) and \( (x,\pi) \vdash_{P_i \ast \dot{Q}_i} \langle \exists r \rangle \in \mathcal{V}[G_i \ast H_i] \). We may apply (Amalgamation 10) to conclude \( r \) and a condition formed from \( q \) whose head is strengthened by \( (x,\pi) \) are compatible in \( P_{i+1} \). Hence we may take \( y \leq r, q \) in \( P_{i+1} \) such that \( y \vdash_{P_{i+1}} \langle \exists r \rangle \in \mathcal{V}[G_i \ast H_i] \).

\[ \square \]

To have that \( P_{i+1}/G_i \ast H_i \) is atomless, we assume that \( \kappa \) is a measurable cardinal.
Lemma 4.2. Let $\kappa$ be measurable. Then we may assume that $\models_{P_{1}}^{*}Q_{1}, "P_{i+1}/G_{i} \ast H_{i} is atomless"$.

Proof. Given $K_{i}$, we construct $K_{i+1}$ closed under an appropriate function so that the pair of $M_{1}$ and $M_{2}$ as in Lemma D are found in $K_{i+1}$. These $M_{1}$ and $M_{2}$ provide incompatible conditions $p_{1}$ and $p_{2}$ in $P_{i+1}$ with the common head $(w(p_{1}, i), f^{p_{1}}(i)) = (w(p_{2}, i), f^{p_{2}}(i))$ in $G_{i} \ast H_{i}$.

Here is some details: Let $(q, \tau) \in P_{i} \ast Q_{i}$ and $A \subset (K_{i}, K) \cap S$ be such that

- $\models_{P_{i}}^{*} \tau \in Q_{i}$ and $A$ is a finite $\epsilon$-chain.
- (Explicit Basic Closure) If $N \in A$, $m \leq i$ and $K_{m} \in N$, then $I[(m+1) \in N]$.
- For all $N \in A$, $K_{i} \in N$ and $\models_{P_{i}}^{*} \tau \in Q_{i}$, $N[G_{i}]$ is generic. (souslin)

Then by Lemma D, there exist $(h, \pi) \in P_{i} \ast Q_{i}$, $M_{1}$ and $M_{2}$ such that

- $h \leq q$ in $P_{i}$,
- $I[(i+1), (q, \tau), A \in M_{1} \cap M_{2}$ and $M_{1}, M_{2} \in S$.

And so

- (Explicit Basic Closure) If $m \leq i$ and $K_{m} \in M_{i}$, then $I[(m+1) \in M_{i}]$ ($l = 1, 2$).
- $M_{1} \cap K_{i} = M_{2} \cap K_{i} \in A^{h}$
- $\models_{P_{i}}^{*} \pi \leq \tau$ in $Q_{i}$ and $\pi$ is $(Q_{i}, M_{i}[G_{i}])$ generic. (souslin)
- $M_{1} \cap \kappa \neq M_{2} \cap \kappa$.

Since $\kappa$ is measurable, there is a regular uncountable cardinal $\theta < \kappa$ such that

- $I[(i+1) \in H_{\theta}$.
- $H_{\theta} \in T$.
- For any $(q, \tau) \in P_{i} \ast Q_{i}$ and any $A \subset (K_{i}, H_{\theta}) \cap S$ as above, there exist $(h, \pi), M_{1}$ and $M_{2}$ as above with $M_{1}, M_{2} \in H_{\theta}$.

Let $K_{i+1} = H_{\theta}$. Then for any $p = (f^{g} \cup \{(i, \tau), A^{g} \cup \{K_{i}\} \cup A) \in P_{i+1}$, there exist

\[ p_{1} = (f^{h} \cup \{(i, \pi), A^{h} \cup \{K_{i}\} \cup A \cup \{M_{1}\}) \]
\[ p_{2} = (f^{h} \cup \{(i, \pi), A^{h} \cup \{K_{i}\} \cup A \cup \{M_{2}\}) \]

in $P_{i+1}$ such that $p_{1}, p_{2} \leq p$ in $P_{i+1}$ and that $p_{1}$ and $p_{2}$ are incompatible in $P_{i+1}$. Hence $\models_{P_{1}}^{*}Q_{1}, "P_{i+1}/G_{i} \ast H_{i}$ is atomless".

\[ \square \]

We show that Souslin trees may be preserved by $P_{\kappa}$.

Theorem 4.3. Let $T$ be a Souslin tree. If for all $i < \kappa$, $\models_{P_{i}}^{*}Q_{i}$ preserves $T$, if $T$ were Souslin". Then $\models_{P_{\kappa}}^{*}T$ remains Souslin".

Proof. Let $T$ be a Souslin tree in $V$. By induction on $i \leq \kappa$, we show that $\models_{P_{i}}^{*}T$ remains Souslin. Let $p \models_{P_{i}}^{*}X \subset T$ be a maximal antichain. We want $q \leq p$ such that $q \models_{P_{i}}^{*}X$ is countable.

Let $p, T, X, I[i, K_{i}, P_{i} \in M \prec H_{X}$. Let $q \leq p$ be $(P_{i}, M)$-generic. Let $(s_{n} \mid n < \omega)$ enumerate the elements of $T_{\mathcal{M} \cap \omega_{1}}$, the $(\mathcal{M} \cap \omega_{1})$-th level of $T$. We show that for all $n < \omega$, $q \models_{P_{i}}^{*} \exists t < s_{n} \in X$ and so $q \models_{P_{i}}^{*}X$ is countable. To this end let $r \leq q$ be $P_{i}$. We find $t < s_{n}$ and $y \leq r$ with $y \models_{P_{i}}^{*}t \in X$. We have four cases depending on how $M \cap K_{i}$ is listed in $A^{r}$. Assume we are as in (Amalgamation 11). Let $G_{j} \ast H_{j}$ be $P_{i} \ast Q_{j}$- generic over $V$ with $(w(r, j), f^{r}(j)) \in G_{j} \ast H_{j}$ in $V[G_{j} \ast H_{j}], T$ remains Souslin by induction.

Let us take a copy

\[ r' \models M[G_{j} \ast H_{j} \cap (P_{i}/G_{j} \ast H_{j}) = M \cap (P_{i}/G_{j} \ast H_{j}) \]

of $r$. We may assume $r' \leq p$ in $P_{i}$ and $\{K_{j}\} \cup S \subseteq A^{r}$. Let $E = \{s \in T \mid \exists t < s \in T \exists x \leq r' \in P_{i}/G_{j} \ast H_{j} \}

Then this is a dense subset of $T$ and $E \in M[G_{j} \ast H_{j}]$. Since $s_{n}$ is $(T, M[G_{j} \ast H_{j}])$-generic,
we have $s, t, x \in M$ such that $t < s < s_n$ in $T$, $x \leq r'$ in $P_i / G_j \ast H_j$ and $x \models_{P_i} \text{	extquotedblright} t \in X$. By (Amalgamation 11), $x$ and $r$ are compatible in $P_i$. Hence there exists $y \in P_i$ such that $y \leq r$ and $y \models_{P_i} \text{	extquotedblright} t \in X$.

\[ \square \]

There are two operations to form proper preorders. We want a new theory of iterated forcing with local projections that puts forward these two operations together with the direct limit.

**Note 4.4.** (1) Let $P$ be a proper preorder and let $\chi$ be a sufficiently large regular uncountable cardinal. We define an associated preorder $Q$ such that $(p, A) \in Q$, if $A$ is a finite $\epsilon$-chain of countable elementary substructures $N$ of $H_\chi$ with $P \in N$ and $p \in P$ is $(P, N)$-generic for all $N \in A$. For $(p_2, A_2), (p_1, A_1) \in Q$, let $(p_2, A_2) \leq (p_1, A_1)$ in $Q$, if $A_2 \supseteq A_1$ and $p_2 \leq p_1$ in $P$. Then we may show that $Q$ is proper and there exists a natural projection $P \leftarrow Q$.

(2) Let $(P_n \mid n < \omega)$ be an iterated forcing such that all $P_n$ are proper. Then we may associate a proper preorder $Q$ that is in a similar situation as (1). However, projections from $Q$ to $P_n$ are formed locally.

**References**


miyamoto@nanzan-u.ac.jp

Mathematics

Nanzan University

18 Yamazato-cho, Showa-ku, Nagoya

466-8673 Japan

南山大学 数学 宫元忠敏