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Entropy for Unitary Operators

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Abstract

We define the entropy $S(u)$ for an $n \times n$ unitary mairix u, and by using the values of $S(u)$ we characterize the notion of *mutual orthogonality* between two maximal abelian subalgebras of $M_n(\mathbb{C})$. We apply these method to unitaries in type II_1-factors M and characterize the notion of *commuting square condition* between two subfactors of M with the Jones index 2.

1 Introduction

This is a continuation of my following two reports:

(講究 1) 数理解析研究所講究録 1819, *Entropy via partitions of unity* , pp 9–21;

(講究 2) 数理解析研究所講究録 1820, *A representation of unital completely positive maps*, pp 11–24.

There are several notions which describe some relative position between two subalgebras of operator algebras. As one of such notions for relations between two subalgebras of finite von Neumann algebras, Popa introduced the notion of *mutually orthogonal subalgebras* (definition below) in [15]. By the terminology *complementarity*, the same notion is investigated in the theory of quantum systems (see [12] for example).

We are interested in to give numerical characterizations for the notion of mutually orthogonality between two isomorphic subalgebras. The most primary interest would be the case where two subalgebras of some full matrix algebra, both of which are either maximal abelian subalgebras or isomorphic to also some full matrix algebra. In such the cases, two subalgebras are connected by some unitary, and we would like to know how such a unitary plays a key role.

Our motivation for this work arises from the following fact: To give a numerical characterizations for the notion of mutually orthogonality, in the previous paper [1], as one of way, we defined a constant $h(A|B)$ for two subalgebras A and B of a finite von Neumann algebra, and explained the relative position between maximal abelian subalgebras A and B of the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices by using the values of $h(A|B)$. This $h(A|B)$ is a slight modification of Connes-Størmer relative entropy $H(A|B)$ in [4] (cf. [10]).

If A_1 and A_2 are maximal abelian subalgebras of $M_n(\mathbb{C})$, then there exists a unitary u in $M_n(\mathbb{C})$ such that $A_2 = uA_1u^*$, (which we denote by $u(A_1, A_2)$), and we showed that A_1 and A_2 are mutually orthogonal if and if $h(A_1|A_2) = H(b(u(A_1, A_2))) = \log n$ in [1, Corollary 3.2](cf. (講究 1)), where $H(b(u(A_1, A_2)))$ is the entropy defined in [16] for the unistochastic matrix $b(u(A_1, A_2))$ induced by the unitary $u = u(A_1, A_2)$. This means that A_1 and A_2 are mutually orthogonal if and if the value $h(A_1|A_2)$ is maximal and equals to the logarithm of the dimension of the subalgebras. Also we had in [2] related results in the case of subfactors of the type II_1 factors. We remark that it does not hold in general that $H(A_1|A_2) = h(A_1|A_2)$ (see, for example [13, Appendix]).

Next when A_1 and A_2 are subalgebras of $M_n(\mathbb{C})$, both of which are isomorphic to also some full matrix algebra $M_k(\mathbb{C})$, our discussion in (講究 1) (see, also [3, Section 3]) was as the followings: The algebra $M_n(\mathbb{C})$ is decomposed into the tensor product: $M_n(\mathbb{C}) = M_m(\mathbb{C}) \otimes M_k(\mathbb{C})$ for some integers m with $n = mk$, and also A_1 and A_2 are connected by some unitary $u \in M_n(\mathbb{C})$. By decomposing such a unitary u into the tensor product form, we gave a finite set U satisfiing the property called *finite operational partition of unity* so that a density matrix $\rho(U)$ thanks by the method of Lindblad [9]. By using the von Neumann entropy for $\rho(U)$ in place of relative entropy $h(A_1|A_2)$, we sowed that A_1 and A_2 are mutually orthogonal if and only if the von Neumann entropy of the density matrix $\rho(U)$ takes the maximum value $2\log n$, which is the logarithm of the dimension of the subfactors.

Here, we pick up another kind of decomposition for the algebra $M_n(\mathbb{C})$. That is the crossed product decomposition $M_n(\mathbb{C}) = D_n(\mathbb{C}) \times_\alpha \mathbb{Z}_n$ of the diagonal matrices $D_n(\mathbb{C})$ by the integer group $\mathbb{Z}_n$ with respect to the action α with $\alpha(e_i) = e_{i+1}$ (mod n), where $\{e_1, e_2, \cdots, e_n\}$ are mutually orthogonal minimal projections of the maximal abelian subalgebra $D_n(\mathbb{C})$ of $M_n(\mathbb{C})$.

More generally, we consider the crossed product M of a finite von Neumann algebra N by a finite group G. We decompose a given unitary u in M into the crossed product form. Then a famile of positive operators in N appears as the coefficients of u in the crossed product decomposition, and the family is a finite partition of unity in the sense of Connes-Størmer [4] (see [10, 1.3]). By considering the von Neumann entropy for these positive operators, we introduce the entropy $S(u)$ for the unitary u. We characterize the *mutual orthogonality* for a pair $\{A, B\}$ of maximal abelian subalgebras of $M_n(\mathbb{C})$ by the value $S(u)$ for a unitary u with $B = uAu^*$. We also apply these method to unitaries in type II_1-factors M and characterize the notion of *commuting square condition* between two subfactors of M with the Jones index 2.

2 Preliminaries

2.1 Entropy function η.

The entropy function η is defined on the interval $[0, 1]$ by

$$\eta(t) = -t\log t \quad (0 < t \leq 1) \quad \text{and} \quad \eta(0) = 0. \tag{2.1}$$

Let $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ be a finite family of real numbers. We call the λ a *finite partition of* 1 if $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. The *entropy* $H(\lambda)$ for λ is given by

$$H(\lambda) = \eta(\lambda_1) + \cdots + \eta(\lambda_n).$$

The function η is strictly concave, that is,

$$\sum_{i=1}^{n} t_i\eta(s_i) \leq \eta(\sum_{i=1}^{n} t_i s_i) \tag{2.2}$$

holds whenevr $s_i \in [0, 1]$ and for real numbers $t_i \geq 0$ with $\sum_{i=1}^{n} t_i = 1$, and equality holds if and only if $s_i = s_j$ for all $i = 1, \cdots, n$. This implies that

$$H(\lambda) \leq \log n \tag{2.3}$$

and $H(\lambda) = \log n$ if and only if $\lambda_i = 1/n$ for all $i = 1, \cdots, n$.

2.2 Finite partition of unity

Let A be a unital C^*-algebra. There are two kind of notions for a finite system, which are named by *a finite partition of unity*, as the followings: The first one was given by Connes-Størmer and the second one was given by Lindblad (See [10] or [11]).

A finite subset $\{x_1, ..., x_k\}$ of A is called a *finite partition of unity* if they are nonnegative operators in A which satisfy that

$$\sum_{i=1}^{n} x_i = 1_A.$$

A finite subset $X = \{x_1, ..., x_k\}$ of A is called a *finite operational partition in A of unity of size k* if

$$\sum_{i}^{k} x_i^* x_i = 1_A.$$

If ϕ is a state of A, then the density matrix $\rho_\phi[X]$ for a finite operational partition X associated with ϕ is the matrix such that the each (i, j)-coefficient $\rho_\phi[X](i, j)$ is given by $\rho_\phi[X](i, j) = \phi(x_j^* x_i)$. When L is a finite von Neumann algebra and that τ_L is a fixed faithful normal tracial state of L, to a finite operational partition X in L of unity of size k, we associate a $k \times k$ density matrix $\rho_{\tau_L}[X]$. We denote this matrix simply by $\rho[X]$, that is, the (i, j)-coefficient $\rho[X](i, j)$ of $\rho[X]$ is given by

$$\rho[X](i, j) = \tau_L(x_j^* x_i), \quad i, j = 1, \cdots, k.$$

We gave sevaral examples about this entropy in (講究 1) and (講究 2).

2.3 The von Neumann entropy.

Let $\rho \in M_n(\mathbb{C})$ be a positive semidefinite matrix with $\rho \leq 1$, where 1 is the identity matrix. The von Neumann entropy $S(\rho)$ is given by

$$S(\rho) = \mathrm{Tr}(\eta(\rho)), \tag{2.4}$$

that is $S(\rho) = \sum_{i=1}^{n} \eta(\lambda_i)$, where $\lambda = \{\lambda_i\}_{1=1}^{n}$ is the eigenvalues of ρ. By a density matrix, we mean a positive semidefinite matrix ρ such that $\mathrm{Tr}(\rho) = 1$. If ρ is a density matrix, then the eigenvalues of ρ is a finite partition of 1.

2.4 Representation that $M_n(\mathbb{C}) = D_n(\mathbb{C}) \times_\alpha \mathbb{Z}_n$

Let x be a $n \times n$ complex marix. We denote by x_{ij} the $\{i, j\}$-component of x. Let $v \in M_n(\mathbb{C})$ be the unitay marix such that $v_{ij} = \delta_{i,j-1}1$ (mod n), for all $i, j = 1, \cdots, n$. The conditional expectation E_D of $M_n(\mathbb{C})$ on the algebra of the diagonal matrices $D_n(\mathbb{C})$ is given by the following form:

$$E(x) = \sum_{i=1}^{n} \varepsilon_i x \varepsilon_i, \quad (x \in M_n(\mathbb{C}),$$

where ε_i is the diagonal matrix whose $\{j, j\}$-component is $\delta_{i,j}1$.

For a given $u \in M_n(\mathbb{C})$, we denote by u_j the diagonal matrix such that each $\{i, i\}$-component is $u_{i\ j+i}$ (mod n) for all $j = 0, 1, \cdots, n-1$ and $i = 1, \cdots, n$. Then u is represented as

$$u = u_0 + u_1 v + u_2 v^2 + u_3 v^3 + \cdots + u_{n-1} v^{n-1}.$$

This means that the algebra $M_n(\mathbb{C})$ is decomposed into the crossed product $D_n(\mathbb{C}) \times_\alpha \mathbb{Z}_n$ of $D_n(\mathbb{C})$ by the cyclic group $\mathbb{Z}_n$ with respect to the action α with $\alpha(\varepsilon_i) = \varepsilon_{i+1}$. Each u_j is also determined by the form that $u_j = E_D(uv^{-j})$ for all $j = 0, 1, \cdots, n-1$.

2.4.1 Example.

As an example, we can see the above fact as the followings:

$$v = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 1 \\ 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 1 & 0 \end{bmatrix}$$

and

$$
\begin{aligned}
u &= \begin{bmatrix} u_{11} & u_{12} & \cdot & \cdot & u_{1n} \\ u_{21} & u_{22} & \cdot & \cdot & u_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \cdot & \cdot & u_{nn} \end{bmatrix} \\
&= \begin{bmatrix} u_{11} & 0 & \cdot & \cdot & 0 \\ 0 & u_{22} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & u_{nn} \end{bmatrix} + \begin{bmatrix} u_{12} & 0 & \cdot & \cdot & 0 \\ 0 & u_{23} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & u_{n1} \end{bmatrix} v + \\
&+ \cdots + \begin{bmatrix} u_{1n} & 0 & \cdot & \cdot & 0 \\ 0 & u_{21} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & u_{n\ n-1} \end{bmatrix} v^{n-1} \\
&= u_0 + u_1 v + u_2 v^2 + \cdots + u_{n-1} v^{n-1}.
\end{aligned}
$$

Here

$$
u_j = \begin{bmatrix} u_{1\ j+1} & 0 & \cdot & \cdot & 0 \\ 0 & u_{2\ j+2} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & u_{nj} \end{bmatrix}, \quad (j = 0, 1, \cdots, n-1).
$$

3 Entropy for unitary operators via crossed product decomposition

In this section, we assume that M is given as the the crossed product $M = N \times_\alpha G$ of a finite von Neumann algebra N by a finite group G with respect to a freely acting τ_N-preserving (i.e., $\tau_N \circ \alpha_g = \tau_N$, for all $g \in G$) action α of G on N. Here τ_N is a fixed normal faithful tracial state of N. We regard N as a von Neumann subalgebra of M, then we have a unitary representation

v of G to M such that $\alpha_g(y) = v_g y v_g^*$ for all $g \in G$, $y \in N$. Each $x \in M$ is uniquely written as

$$x = \sum_{g \in G} x_g v_g, \quad x_g \in N \tag{3.1}$$

The conditional expectation E_N of M onto N is given by $E_N(x) = x_1$, where 1 is the unit of G, and it holds, for each $x \in M$, that

$$x_g = E_N(x{v_g}^*) \quad \text{for all} \quad g \in G. \tag{3.2}$$

The trace τ_N is extended to the trace τ_M of M by $\tau_M = \tau_N \circ E_N$.

If $u \in M$ is a unitary, then the family $\{u_g; g \in G\} \subset N$ satisfies that

$$\sum_{g \in G} u_g u_g^* = 1_N \quad \text{and} \quad \sum_{g \in G} u_g \alpha_k({u_{hk^{-1}}}^*) = 0, \ (k \neq 1). \tag{3.3}$$

By means of the family $\{u_g u_g^*; g \in G\}$, which is a finite partition of unity in N, we define the entropy $S(u)$ as the followings:

3.1 Definition.

For a unitary $u \in M = N \times_\alpha G$, let

$$S(u) = \sum_{g \in G} \tau_N \eta(u_g u_g^*). \tag{3.4}$$

3.1.1 Case of type I_n factors.

First, we take up the case of the type I_n factor M. Let A be a maximal abelian subalgebra of M. Then M is isomorphic to the matix algebra $M_n(\mathbb{C})$ and A is isomorphic to the algebra of diagonal matrices $D_n(\mathbb{C})$. The M is represented as the the crossed product of A by the group $\mathbb{Z}_n$ with respect to α:

$$M = A \times_\alpha \mathbb{Z}_n.$$

Here, the automorphism α of A is given by

$$\alpha(e_i) = e_{i+1}, \quad (\text{mod } n)$$

for a mutually orthogonal minimal projections $\{e_1, e_2, \cdots, e_n\}$ of A. Let $\{e_{ij}; i, j = 1, 2, \cdots, n\}$ be a system of a matrix units of M with $e_{ii} = e_i, (i =$

$1, 2, \cdots, n)$. Then the unitary $v = \sum_{i=1}^{n} e_{i\ i-1}$ in M satisfies that $\alpha(a) = vav^*$ for all $a \in A$. For the decomposition of a unitary $u \in M$:

$$u = \sum_{j=0}^{n-1} u_j v^j.$$

the entropy $S(u)$ is nothing else but the average of the von Neumann entropy for $\{S(u_j u_j^*) : j = 0, 1, \cdots, n-1\}$, that is

$$S(u) = \sum_{j=0}^{n-1} \tau_A \eta(u_j u_j^*) = \frac{1}{n} \sum_{j=0}^{n-1} \mathrm{Tr}\eta(u_j u_j^*) = \frac{1}{n} \sum_{j=0}^{n-1} S(u_j u_j^*).$$

3.2 Mutually orthogonal maximal abelian subalgebras

Let A and B be two maximal abelian subalgebras of $M = M_n(\mathbb{C})$. Then there exists a unitary u with $B = uAu^*$. By using the crossed product decomposition $M = A \times_\alpha \mathbb{Z}_n$, we gave a characterization in [3] for the mutually orthogonality of A and B via the von Neumann entropy $S(u)$.

3.2.1 Theorem.

Let A and B be maximal abelian subalgebras of M. Let u be a unitary in M with $uAu^* = B$. Then the following are equivalent:

(1) *A and B are mutually orthogonal;*

(2) $u_j u_j^* = \frac{1}{n} 1_A, \quad j = 0, 1, 2, \cdots, n-1;$

(3) *the entropy $S(u)$ takes the maximal value:*

$$S(u) = \log n = \max\{S(w) \mid w \in M, \text{unitary}\}.$$

3.2.2 Complex Hadamard matrix.

A uniary matrix $u \in M_n(\mathbb{C})$ is called a *complex Hadamard matrix* in [7] if all entries $u(i, j)$ of u have the same modulus, that is $|u(i, j)| = 1/\sqrt{n}$ for all $i, j = 1, \cdots, n$. For a $u \in M_n(\mathbb{C})$, Sunder and Jones gave the characterization in [7, 5.2.2] that $D_n(\mathbb{C})$ and $uD_n(\mathbb{C})u^*$ are mutually orthogonal if and only if u is a complex Hadamard matrix.

It is clear that if u is a complex Hadamard matrix then $S(u) = \log n$ and the above proof for (3) $\Rightarrow$ (2) of Theorem 4.2.1 shows that the converse is true. Hence we have that u is a complex Hadamard matrix if and only if $S(u) = \log n$. See [7, 5.2.2] for examples of complex Hadamard matrices.

3.3 Commuting squares and the entropy for unitaries

In this section, let us see how the discussion in the section 3 develops in the framework of II_1 factors.

For a subfactor N of a finite factor M, Jones ([6]) introduced the index $[M : N]$, the set of all values of which is $\{4\cos^2 \frac{\pi}{n}; n = 3, 4, 5, \cdots\} \cup [4, \infty)$.

The most typical example of orthogonal pairs of subfactors in II_1 factors will be the followings:

3.3.1 Example of orthogonal pair of subfactors in II_1 factors.

Let λ be a real number such that

$$\lambda^{-1} \in \{4\cos^2 \frac{\pi}{n}; n = 3, 4, 5, \cdots\} \cup [4, \infty),$$

and let $\{\cdots, e_{-1}, e_0, e_1, e_2, \cdots\}$ be a sequence of projections with the properties:

$$e_i e_{i=\pm 1} e_i = \lambda e_i, \quad \text{and} \quad e_i e_j = e_j e_i \quad \text{if} |i - j| \geq 2.$$

Such the sequences of projections appeared in the step of Jones construction of subfactors ([6]).

Let M_∞ be the von Neumann algebra generated by $\{e_i | i \in \mathbb{Z}\}$, then M_∞ is the hyperfinite II_1 factor, and the unique trace τ of M_∞ is given by the property:

$$\tau(we_k) = \lambda\tau(w), \quad (w \in \mathrm{alg}\{e_k | j < k\}.$$

Let N be the von Neumann subalgebra of M_∞ generated by $\{e_i | i < 0\}$, and let L be the von Neumann subalgebra of M_∞ generated by $\{e_i | i \geq 0\}$. Then $\{N, L\}$ is a mutually orthogonal pair of subfactors in M_∞.

We remark that $[M_\infty : N] = [M_\infty : L] = \infty$, and we would like to discuss in the framework of subfactors with finite index.

3.3.2 Commuting square condition

From now, assume that M is a type II_1-factor. If N is a von Neumann subalgebra of M, then we have always the unique faithful normal conditional expectation $E_N : M \to N$. Let N_1 and N_2 be von Neumann subalgebras of M. In a connection with Jones index theory([6]), the notion of a *mutual orthogonal pair* was generalized by Goodman-Harpe-Jones ([5]) to the notion of a pair satisfying the *commuting square condition*. The diagram

$$\begin{array}{ccc} N_1 & \subset & M \\ \cup & & \cup \\ N_3 & \subset & N_2 \end{array}$$

is said to be a commuting square if

$$E_{N_1}E_{N_2} = E_{N_2}E_{N_1} \quad \text{and} \quad N_3 = N_1 \cap N_2.$$

We say that a pair $\{N_1, N_2\}$ satisfies the commuting square condition if $E_{N_1}E_{N_2} = E_{N_2}E_{N_1}$. Of course, the pair $\{N_1, N_2\}$ satisfies the commuting square condition if $N_1 = N_2$. We say a pair $\{N_1, N_2\}$ is nontrivial if $N_1 \neq N_2$, and we are interested in non-trivial pairs of subfactors which satisfy the commuting square condition.

3.3.3 Index 2 subfactors

Here, we replace the notion of *mutual orthogonality* to that of *commuting square condition*, and show that, for a pair of finite index subfactors, the entropy $S(u)$ plays a key role in our characterization similarly in the section 3.

As the first non-trivial subfactor N of M, index 2 subfactors appear. The index 2 subfactor is unique up to the conjugacy and it is the biggest subfactor from the point of view of the index theory. By replacing the mutual orthogonality to the commuting square condition, we study the pairs of the biggest subfactos, that is the index 2 subfactors.

Jones picked up the index 2 subfactors N of M in [8, Chapter 3] and invesgated properties of $N \cap uNu^*$, where u is a unitary in M satisfying some conddition. For such a pair $\{N, u\}$ of the index 2 subfactor N and u, we characterize the commuting square condition for the pair $\{N, uNu^*\}$.

Let N be a subfactor of M such that $[M : N] = 2$. As Jones showed in [6], M is decomposed into the the crossed product of N by the group of an outer automorphism α of N with the periond 2:

$$M = N \times_\alpha \mathbb{Z}_2.$$

Then there exists a self-adjoint unitary v in M such that

$$\alpha(x) = vxv^* \quad \text{for all} \quad x \in N$$

and M is represented as $M = N \oplus Nv$. Also we have a projection e in N such that

$$\alpha(e) = e^\perp = 1 - e.$$

The uniqueness of the trace τ of a II_1 factor implies that $\tau(e) = 1/2$.

Let A be the von Neumann subalgebra of N generated by e:

$$A = \mathbb{C}e \oplus \mathbb{C}e^\perp = \mathbb{C}e \oplus \mathbb{C}(1-e). \tag{3.5}$$

Each unitary u in M is decomposed into the form

$$u = u_0 + u_1 v, \quad u_0, u_1 \in N \tag{3.6}$$

and

$$u_0 = E_N(u), \quad u_1 = E_N(uv). \tag{3.7}$$

We call the $\{u_0, u_1\}$ the coefficients of u.

We restrict our attention to the unitaries $u \in M$ such that the coefficients of u are contained in the abelian subalgebra A of M, and we characterized in [3] the commuting square condition for $\{N, uNu^*\}$ via the entropy $S(u)$ as follows:

3.3.4 Theorem.

Let N be a II_1 factor and α be an outer automorphism of N with the periond 2. Let M be the crossed product $M = N \times_\alpha \mathbb{Z}_2$. Assume that A is the above 2-dimensional subalgebra of N. Then for a unitary $u \in M$ whose coefficients u_0 and u_1 are contained in A, the following conditions are equivalent:

1. *N and uNu^* satisfy the non-trivial commuting square condition;*
2. $u_j u_j^* = \frac{1}{2} 1_N, \quad j = 0, 1;$
3. $S(u) = \log 2.$

3.3.5 Remark.

1. The notion of the Jones index for subfactors are generalized to some constant for subalgebras by of Pimsner-Popa([14]). Since the above A is a 2-dimensional *-subalgebra of N and $\alpha(A) = A$, it implies by [8, Lemma 3.10] that under the above assumption for the coefficients $\{u_0, u_1\}$ of u, that is $\{u_0, u_1\} \subset A$, the von Neumann subalgebra $N \cap uNu^*$ of M is of finite index in the sense of Pimsner-Popa([14]).

2. If N_1 and N_2 are subfactors with finite index of a type II_1 factor M and if N_1 and N_2 are mutually orthogonal, then $N_1 \cap N_2 = \mathbb{C}1$ so that the index is infinite: $[M : N_1 \cap N_2] = \infty$.

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