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ON FUNDAMENTAL SOLUTIONS FOR FRACTIONAL DIFFUSION EQUATIONS WITH DIVERGENCE FREE DRIFT

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1. Introduction

In this article we are concerned with the following non-local diffusion equations in the presence of a given divergence free drift term:

$$(1.1)\qquad \partial_t\theta + A_K(t)\theta + v\cdot\nabla\theta = 0, \qquad \nabla\cdot v = 0, \qquad t>0,\ x\in\mathbb{R}^d,$$

where $d\geq 2$ is the dimension, $\alpha\in(0,2)$ is a constant, and $A_K(t)$ is the fractional diffusion operator which is formally defined by

$$(1.2)\qquad \big(A_K(t)f\big)(x) = \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon} (f(x)-f(y))K(t,x,y)\,\mathrm{d}y.$$

We assume that there are $\alpha\in(0,2)$ and $C_0>0$ such that

$$(1.3)\qquad K(t,x,y) = K(t,y,x) \qquad \text{for } a.e.\ (t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,$$

$$(1.4)\qquad \sup_{t>0,x\in\mathbb{R}^d}\int_{|x-y|\leq M} |x-y|^2K(t,x,y)\,\mathrm{d}y \leq C_0M^{2-\alpha} \qquad \text{for } M\in(0,\infty),$$

$$(1.5)\qquad \inf_{t>0,x,y\in\mathbb{R}^d} |x-y|^{d+\alpha}K(t,x,y)\geq C_0^{-1}.$$

In (1.4) and (1.5), 'sup' and 'inf' are interpreted as 'ess.sup' and 'ess.inf', respectively. We note that the operator $A_K(t)$ with the index $\alpha\in(0,2)$ is a natural generalization of the usual fractional Laplacian $(-\Delta)^{\alpha/2}$; in that case the kernel $K(t,x,y)$ is given by $C_{d,\alpha}|x-y|^{-d-\alpha}$, where $C_{d,\alpha}$ is a positive constant. The aim of this article is to give a complement result of the author's work [25] with H. Miura (Osaka Univ.) about the pointwise upper bound for fundamental solutions to (1.1).

When there is no drift term (i.e., $v=0$) the equation (1.1) appears in the theory of Dirichlet forms of jump type as a special case, and it has been investigated mainly from the probabilistic viewpoint; see [9, 21, 22, 5, 1, 3, 4]. On the other hand in recent years the case with the drift term has also attracted much attention especially in the field of fluid mechanics, mathematical finance, biology, and so on. Among of them, the two-dimensional dissipative surface quasi-geostrophic equations (QG) for the active scalar in the geophysical fluid introduced by [13] are extensively studied in the last decade; see, e.g., [11, 10, 12, 20, 16, 7, 19]. The equations (QG) are nonlinear equations of the form (1.1) where the velocity in the drift term is related by $v = (-R_2\theta, R_1\theta)$ via the Riesz transform R_i.

In [6, 18] they considered fundamental solutions to (1.1) when $\alpha \in (1,2)$ and v belongs to a suitable Kato class but without assuming the divergence free condition $\nabla \cdot v = 0$. They proved the existence of fundamental solutions and showed pointwise estimates. However, there seems to be still few works on fundamental solutions for $\alpha \in (0,1]$. In such cases the drift term formally becomes the leading term and is no longer regarded as a simple perturbation of the diffusion term. Moreover, for applications to nonlinear problems it is important to study the linear problem of the form (1.1) under weak assumption for v beyond the Kato class. In such situations the interplay between the diffusion term and the drift term makes problems more subtle and the divergence free structure for the velocity plays a crucial role.

Motivated by these background, [24, 25] studied the fundamental solutions to (1.1) for all range of $\alpha \in (0,2)$. To state their results let us recall the definition of the Campanato spaces:

$$(1.6)\quad \mathcal{L}^{p,\lambda}(\mathbb{R}^d) = \{f \in L^p_{loc}(\mathbb{R}^d) \mid \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^d)} = \sup_B \big(R^{-\lambda} \int_B |f(x) - \fint_B f|^p \,dx\big)^{\frac{1}{p}} < \infty\}.$$

Here the supremum is taken over all balls $B = B_R(x)$ (the ball with radius $R > 0$ centered at $x \in \mathbb{R}^d$), the value $\fint_B f$ is the average in B defined by $\fint_B f = |B|^{-1} \int_B f(x)\,dx$, and $\|\cdot\|_{\mathcal{L}^{p,\lambda}}$ becomes a seminorm. It is easy to see that the continuous embedding holds.

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) \hookrightarrow \mathcal{L}^{1,\mu}(\mathbb{R}^d) \qquad \text{if} \quad p \geq 1 \quad \text{and} \quad \mu = \frac{\lambda - d}{p} + d.$$

In the case of $\lambda < d$, the function in $\mathcal{L}^{p,\lambda}$ is uniformly locally integrable, and $\mathcal{L}^{p,\lambda}$ is identified by the Morrey space $L^{p,\lambda}$ modulo constant. Moreover, it is known that the following embeddings hold.

$$\begin{array}{rclr} L_{\mathrm{w}}^{\frac{pd}{d-\lambda}}(\mathbb{R}^d) & \hookrightarrow & \mathcal{L}^{p,\lambda}(\mathbb{R}^d) & \text{if } 0 < \lambda < d, \\ \mathcal{L}^{p,\lambda}(\mathbb{R}^d) & = & BMO(\mathbb{R}^d) & \text{if } \lambda = d, \\ \mathcal{L}^{p,\lambda}(\mathbb{R}^d) & = & \dot{C}^{\frac{\lambda-d}{p}}(\mathbb{R}^d) & \text{if } d < \lambda \leq d+p. \end{array}$$

See, e.g., [17, 28]. Here $L^p_{\mathrm{w}}(\mathbb{R}^d)$ is the weak L^p space and $\dot{C}^\beta(\mathbb{R}^d)$, $\beta \in (0,1]$, is the homogeneous Hölder space of the order β, i.e.,

$$\dot{C}^\beta(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) \mid \|f\|_{\dot{C}^\beta} = \sup_{x,y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty\}.$$

Next we introduce the Morrey type spaces of $\mathcal{L}^{p,\lambda}$-valued functions:

$$L^{p,\lambda_1}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) = \{f \in L^p_{loc}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) \mid$$

$$\|f\|_{L^{p,\lambda_1}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d))} = \sup_{t>0}\sup_{0<s<t}\Big((t-s)^{-\lambda_1}\int_s^t \|f(\tau)\|^p_{\mathcal{L}^{q,\lambda_2}}\, d\tau\Big)^{\frac{1}{p}} < \infty\}.$$

For $\alpha \in (0,2)$ we impose the following conditions on v.

(C) There are $\lambda \in [2d/\alpha - d, 2d/\alpha + d)$ and $q \in (1,\infty]$ such that

(i) if $\lambda \in [\frac{2d}{\alpha} - d, d]$ then

$$v \in L^{2,\frac{2}{\alpha}-\frac{\lambda}{d}}(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d) \cap L^q_{loc}(0,\infty;(L^1_{loc}(\mathbb{R}^d))^d),$$

(ii) if $\lambda \in (d, \frac{2d}{\alpha} + d)$ then

$$v \in L^{1,\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d) \cap L^q_{loc}(0,\infty;(L^\infty_{loc}(\mathbb{R}^d))^d).$$

For simplicity of notations we set

$$(1.7)\qquad \|v\|_{X_\lambda} = \begin{cases} \|v\|_{L^{2,\frac{2}{\alpha}-\frac{\lambda}{d}}(0,\infty;\mathcal{L}^{\frac{2d}{\alpha},\lambda})} & \text{when } \lambda \in [\frac{2d}{\alpha} - d, d], \\ \|v\|_{L^{1,\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}(0,\infty;\mathcal{L}^{\frac{2d}{\alpha},\lambda})} & \text{when } \lambda \in (d, \frac{2d}{\alpha} + d]. \end{cases}$$

Note that the norm $\|\cdot\|_{X_\lambda}$ is invariant under the scaling

$$(1.8)\qquad v_\lambda(x,t) = \lambda^{\alpha-1}v(\lambda^\alpha t, \lambda x).$$

This scaling is natural in the following sense: If $\theta(t,x)$ is a solution to (1.1) then the rescaled function $\theta(\lambda^\alpha t, \lambda x)$ satisfies (1.1) with the velocity v_λ, instead of v. Heuristically, in order to ensure a smoothing effect by the diffusion term it is essential to assume that v belongs to a scale-invariant function space; see, e.g., [8, 7, 19, 27, 29]. The space X_λ covers the following classes as special cases: $L^\infty(0,\infty;(BMO(\mathbb{R}^d))^d)$ for $\alpha = 1$; $L^\infty(0,\infty;(\dot{C}^{1-\alpha}(\mathbb{R}^d))^d)$ for $\alpha \in (0,1)$. Moreover it also allows a singularity at some $t_0 \geq 0$: $|t-t_0|^{\frac{\lambda}{2d}+\frac{1}{2}-\frac{1}{\alpha}}v(t) \in L^\infty(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d)$. One of the advantages to use the Campanato spaces (1.6) is that they contain certain homogeneous functions. This fact is important for the study of the self-similar solutions in some nonlinear problems. Another advantage is that in the case of $\lambda \geq d$ they contain growing functions at spatial infinity. Except some special cases, e.g., the fractional Ornstein-Uhlenbeck operators, such velocity fields seem not to be studied.

In [24, 25] the fundamental solutions associated with (1.1), denoted by $P_{K,v}(t,x;s,y)$, are studied in details. Because of the weak regularity of K and v the definition of the fundamental solutions have to be given through the weak formulation; see [24, 25] for details. As for the existence and regularity of fundamental solutions, we have the following

Theorem 1.1 ([24]). *Suppose that (1.3) - (1.5) and* **(C)** *hold. Then there exists a fundamental solution $P_{K,v}(t,x;s,y)$ for (1.1) satisfying the following properties.*

$$\int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}x = \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}y = 1,$$

$$0 \le P_{K,v}(t,x;s,y) \le C(t-s)^{-\frac{d}{\alpha}},$$

$$P_{K,v}(t,x;s,y) = \int_{\mathbb{R}^d} P_{K,v}(t,x;\tau,z)P_{K,v}(\tau,z;s,y)\,\mathrm{d}z, \qquad t>\tau>s\ge 0,$$

$$|P_{K,v}(t,x_1;s,y_1) - P_{K,v}(t,x_2;s,y_2)| \le \frac{C_1\big(|x_1-x_2|^\beta + |y_1-y_2|^\beta\big)}{(t-s)^c}.$$

Here the positive constant C depends only on d, α, and C_0, the positive constants C_1, c, β depend only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$.

Remark 1.1. We also have the Hölder continuity of $P_{K,v}(t,x;s,y)$ with respect to the time variables; see [24].

In [25] the pointwise upper bound of $P_{K,v}(t,x;s,y)$ is established.

Theorem 1.2 ([25]). *Under the assumptions of Theorem 1.1 we have*

(1.9)

$$P_{K,v}(t,x;s,y) \le C_2(t-s)^{-\frac{d}{\alpha}}\left(1+\frac{(|x-y|-CF[v](t,s,x,y))_+}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha} + C_3(t-s)^{-\frac{d}{\alpha}}\big(1+\frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\big)^{-\alpha},$$

where

$$F[v](t,s,x,y) := \sup_{s<r<t}\Big|\int_s^r \fint_{B_{|x-y|}(x)} v(\tau)\,\mathrm{d}\tau\Big|. \tag{1.10}$$

Here C_2 depend only on d and α, C_3 depends only on d, α, and $\|v\|_{X_\lambda}$, and $C>1$ is some absolute constant. Moreover, if in addition $K(t,x,y)$ satisfies the stronger condition

$$C_0^{-1}|x-y|^{-d-\alpha} \le K(t,x,y) \le C_0|x-y|^{-d-\alpha}, \tag{1.11}$$

then we can take $C_3 = 0$ in (1.9).

Remark 1.2. For the endpoint case $\lambda = 2d/\alpha+d$ in **(C)**, the estimate (1.9) holds if $\|v\|_{X_{2d/\alpha+d}}$ or $|t-s|$ is sufficiently small. We note that $\mathcal{L}^{2d/\alpha,2d/\alpha+d}(\mathbb{R}^d)$ coincides with $\mathrm{Lip}(\mathbb{R}^d)$, the space of all Lipschitz functions. For simplicity we do not deal with this endpoint case $\lambda = 2d/\alpha + d$ in this article.

Remark 1.3. We note that the extra assumptions $v \in L^q_{loc}(0,\infty;(L^1_{loc}(\mathbb{R}^d))^d)$ or $v \in L^q_{loc}(0,\infty;(L^\infty_{loc}(\mathbb{R}^d))^d)$ in **(C)** is used only to guarantee the existence of the fundamental solution in [24]. It is weaker than the assumption $v \in X_\lambda$ in view of the scaling.

Because of the weak regularity of K and v the uniqueness of weak solutions to (1.1) seems to be unknown so far, especially in the case $\alpha \in (0,1]$. In this sense even the semigroup property of $P_{K,v}(t,.x;s,y)$ in Theorem 1.1 is not trivial, and we are forced to perform a careful limiting procedure to establish it; cf. [24]. Theorem 1.2 shows that if (1.11) holds then the fundamental solution $P_{K,v}(t,x;s,y)$ is bounded by the modification of $C(t-s)^{-d/\alpha}(1+|x-y|(t-s)^{-1/\alpha})^{-d-\alpha}$, which means that $P_{K,v}(t,x;s,y)$ possesses the similar decay estimate for the fractional heat equations

$$\partial_t\theta + (-\Delta)^{\frac{\alpha}{2}}\theta = 0, \qquad t>0,\ x\in\mathbb{R}^d. \tag{1.12}$$

The modification $F[v]$ in (1.9) represents the transport effect by the drift term. Since $\mathcal{L}^{p,\lambda}$ includes some growing functions, the term $F[v]$ is not necessarily bounded in space variables. More precisely, from the condition **(C)** one can see that $F[v]$ grows no faster than linearly, thus (1.9) shows that the fundamental solution decays with order $-d-\alpha$ when $|x-y|$ is large. On the other hand, in the case of $\alpha \in [1,2)$ if we assume $v \in L^{1,1/\alpha}(0,\infty;(L^\infty(\mathbb{R}^d))^d)$ and (1.11), then it is easy to see from Theorem 1.2 that $P_{K,v}(t,x;s,y)$ is bounded by a constant multiple of the fundamental solution to (1.12).

After the pioneering work of [26, 2], there are a lot of results on the pointwise upper bounds for the fundamental solutions of the second order parabolic equations. In particular, for the drift diffusion equation (1.1) with $\alpha = 2$, the Gaussian upper bounds are obtained in [27, 8] under the scale-invariant assumptions; see also [30, 23] for recent related works. On the other hand, the fundamental solution for $\alpha < 2$ is expected to decay only with polynomial order: In the case $v = 0$ a standard Fourier analysis shows that the fundamental solution satisfies the estimate (1.9) with $C_3 = 0$. If v is regarded as a simple perturbation of the diffusion term, it is possible to obtain the same upper bound as well. However, under our assumptions for v (and α), the perturbation argument is no longer applicable to handle with our problem. To overcome the difficulty the articles [24, 25] applied the idea of Carlen-Kusuoka-Stroock [9], where they derived pointwise upper bounds for the fundamental solution for certain non-local diffusion equations without the drift term based on Davies' method [15]. The key idea to take the transport effect into account is the introduction of a trajectory determined by a local average of v. This idea is motivated by the work of [7, 19], where the authors studied the regularity of the weak solution of the equation (QG). Another ingredient of the proof is the use of the logarithmic

Sobolev inequality of the fractional order recently proved in [14], which plays a crucial role to estimate the diffusion term.

The natural question here is that whether or not we may take $C_3 = 0$ in (1.9) without assuming the extra condition (1.11). The aim of this article is to give an affirmative answer to this question. That is, our main result is

Theorem 1.3. *Under the assumptions of Theorem 1.1 the upper bound* (1.9) *is valid with* $C_3 = 0$.

In [25] the Dirichlet form $\mathcal{E}_K^t$ (see Section 2.3) is divided into the singular part and the regular part. But if we use Lemma 3.1 below such decomposition is in fact not necessary, which leads to (1.9) with $C_3 = 0$ for general case. In this article we establish only the a priori estimate. For detailed approximation and limiting procedures the reader is referred to [25].

2. Preliminaries

2.1. **Logarithmic Sobolev Inequality.** The logarithmic Sobolev inequality with fractional order is stated as follows.

Lemma 2.1 ([14]). *Let f be a function in $H^\alpha(\mathbb{R}^d)$ and $\beta > 0$ be any positive number. Then*

$$\Big(\int |f|^2 \log \frac{|f|^2}{\|f\|_{L^2}^2} dx + (d + \log \frac{\alpha \Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2\alpha})} + \frac{d}{2\alpha} \log \beta) \|f\|_{L^2}^2 \Big) \leq \frac{\beta}{\pi^\alpha} \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^2}^2$$

holds.

2.2. **Estimates for the Trajectory.** Next we recall some lemmas for the estimate of the drift term.

Lemma 2.2 ([24, Lemma 2.2]). *Let $f \in \mathcal{L}^{1,\mu}(\mathbb{R}^d)$ for some $\mu \in [0, d+1]$. Let $x_1, x_2 \in \mathbb{R}^d$ and $R_1 \geq R_2 > 0$. Then*

$$\begin{aligned} &\Big| \fint_{B_{R_1}(x_1)} f - \fint_{B_{R_2}(x_2)} f \Big| \\ &\leq \begin{cases} C\|f\|_{\mathcal{L}^{1,\mu}} R_2^{\mu-d} & \text{if} \quad 0 \leq \mu < d, \\ C\|f\|_{\mathcal{L}^{1,\mu}} \big(\log(e + \frac{|x_1 - x_2|}{R_2}) + \log \frac{R_1}{R_2} \big) & \text{if} \quad \mu = d, \\ C\|f\|_{\mathcal{L}^{1,\mu}} \big(|x_1 - x_2|^{\mu-d} + R_1^{\mu-d} \big) & \text{if} \quad d < \mu \leq d+1. \end{cases} \end{aligned} \tag{2.1}$$

Here C depends only on d and μ.

The trajectory generated by the local average of the vector field u is defined as the solution to the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \xi_u(t; x, R) = \displaystyle\fint_{B_R(x + \xi_u(t;x,R))} u(t), \quad 0 \leq t \leq t_0, \\ \xi_u(0; x, R) = 0, \end{cases} \tag{2.2}$$

where $x \in \mathbb{R}^d$ and $R > 0$. Then we have

Lemma 2.3 ([25, Lemma 2.4]). *Let $\xi_u(t; x, R)$ be the solution to (2.2) with $R \geq t_0^{1/\alpha}$. Assume that u satisfies* **(C)** *for $\lambda \in [d, 2d/\alpha + d)$. If $\lambda > d$ then*

$$|\xi_u(t_0; x, R)| \leq C\big(R\|u\|_{X_\lambda} + \sup_{0<t<t_0} \big|\int_0^t \fint_{B_R(x)} u(\tau)\, \mathrm{d}\tau\big|\big), \tag{2.3}$$

and if $\lambda = d$ then

$$|\xi_u(t_0; x, R)| \leq C\big(R\|u\|_{X_\lambda}(1 + \log\|u\|_{X_\lambda}) + \sup_{0<t<t_0} \big|\int_0^t \fint_{B_R(x)} u(\tau)\, \mathrm{d}\tau\big|\big). \tag{2.4}$$

Here C depends only on d, α, p. Moreover, the same estimate (2.3) *also holds for the case $\lambda = 2d/\alpha + d$ provided $\|u\|_{X_\lambda}$ is sufficiently small.*

2.3. **Estimates for the bilinear form.** We denote by $\mathcal{E}_K^{(t)}$ and $\mathcal{E}_{v(t)}$ the bilinear forms

$$\mathcal{E}_K^{(t)}(f, g) = \frac{1}{2}\int_{\mathbb{R}^{2d}} [f][g](x, y)K(t, x, y)\, \mathrm{d}x\, \mathrm{d}y, \qquad [f](x, y) = f(x) - f(y),$$

$$\mathcal{E}_{v(t)}(f, g) = - < f, v(t) \cdot \nabla g > := -\int_{\mathbb{R}^d} f(x)v(t, x) \cdot \nabla g(x)\, \mathrm{d}x,$$

Let $\mathrm{Lip}_0(\mathbb{R}^d)$ be the class of compactly-supported Lipschitz functions. For $\Psi \in \mathrm{Lip}([0, \infty) \times \mathbb{R}^d)$ with $\Psi(t, \cdot) \in \mathrm{Lip}_0(\mathbb{R}^d)$, we set

$$\Gamma(\Psi)(t, x) = e^{-2\Psi(t,x)}\Gamma(e^\Psi, e^\Psi)(t, x), \tag{2.5}$$

$$\Lambda(\Psi) = \max\{\|\Gamma(\Psi)\|_{L^\infty_{t,x}},\ \|\Gamma(-\Psi)\|_{L^\infty_{t,x}}\}, \tag{2.6}$$

where $\Gamma(f, g)$ is the function defined by

$$\Gamma(f, g)(t, x) = \int_{\mathbb{R}^d} [f][g](x, y)K(t, x, y)\, \mathrm{d}y. \tag{2.7}$$

The following coercive-type estimate, established by [9], represents the diffusion effect for the Dirichlet form $\mathcal{E}_K^{(t)}$.

Lemma 2.4 ([9, Theorem 3.9]). *Let $\Psi \in \mathrm{Lip}([0, \infty) \times \mathbb{R}^d)$ with $\Psi(t, \cdot) \in \mathrm{Lip}_0(\mathbb{R}^d)$. Let $r \in [1, \infty)$. Then for $f \in C_0^\infty(\mathbb{R}^d)$ with $f \geq 0$ it follows that*

$$\mathcal{E}_K^{(t)}(e^\Psi f^{r-1}, e^{-\Psi} f) \geq \frac{2}{r}\mathcal{E}_K^{(t)}(f^{\frac{r}{2}}, f^{\frac{r}{2}}) - Cr\Lambda(\Psi)\|f\|_{L^r}^r. \tag{2.8}$$

Here C is a numerical constant.

In fact, [9] considered the case when the kernel K and Ψ are independent of t. The dependence on t however does not change any arguments to obtain (2.8). On the other hand, the divergence free condition for v with the integral by parts immediately yields the following identity for the bilinear form $\mathcal{E}_{v(t)}$.

Lemma 2.5. *Let $\psi \in \mathrm{Lip}_0(\mathbb{R}^d)$. For $r \in [1, \infty)$ it follows that*

$$\mathcal{E}_{v(t)}(e^{\psi} f^{r-1}, e^{-\psi} f) = \int_{\mathbb{R}^d} f^r(x)\ v(t,x) \cdot \nabla\psi(x)\, \mathrm{d}x. \tag{2.9}$$

3. Pointwise upper bounds

In this section we will prove Theorem 1.3. In most parts of the proof we follow the argument in [25].

Fix $L \geq 0$, $R > 0$, $t_0 > 0$, and $x_0,\ y_0 \in \mathbb{R}^d$. Let ψ be the function defined by

$$\psi(x) = L(R - |x - x_0|)_+. \tag{3.1}$$

Set $\xi(t; x_0, R) \in \mathbb{R}^d$ be the solution to (2.2) with $R > 0$ and $u(t,x) = v(t_0 - t, x)$, $0 \leq t \leq t_0$. If we put $\xi(t; x_0) = \xi(t_0 - t; x_0, R)$ then $\xi(t; x_0)$ solves the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\xi(t;x_0) = -\displaystyle⨍_{B_R(x_0+\xi(t;x_0))} v(t), & 0 \leq t \leq t_0, \\ \xi(t_0; x_0) = 0. \end{cases} \tag{3.2}$$

We also set

$$\Psi(t,x) = \psi(x - \xi(t;x_0)), \qquad 0 \leq t \leq t_0. \tag{3.3}$$

Then it is easy to see

$$\|\Psi\|_{L^\infty} \leq LR, \qquad \mathrm{Lip}(\Psi(t)) \leq L, \qquad \mathrm{supp}\ \Psi(t) = B_R(x_0 + \xi(t;x_0)). \tag{3.4}$$

The next lemma for $\Lambda(\Psi)$ corresponds with [25, Lemma 3.7].

Lemma 3.1. *Let Ψ be the function defined by (3.3). Then*

$$\Lambda(\Psi) \leq C e^{3LR} R^{-\alpha}. \tag{3.5}$$

Here C depends only on d, α, and C_0.

Proof. From $(e^t - 1)^2 \leq \min\{t^2 e^{2t}, 2e^{2t}\}$ for $t \geq 0$ and (3.4) we have

$$e^{-2\Psi(t,x)} \int_{\mathbb{R}^d} [e^{\Psi(t,\cdot)}]^2(x,y) K(t,x,y)\,\mathrm{d}y = \int_{\mathbb{R}^d} (e^{\Psi(t,x)-\Psi(t,y)} - 1)^2 K(t,x,y)\,\mathrm{d}y$$

$$\leq L^2 e^{2LR} \int_{|x-y|\leq R} |x-y|^2 K(t,x,y)\,\mathrm{d}y + 2e^{2LR} \int_{|x-y|\geq R} K(t,x,y)\,\mathrm{d}y.$$

It is straightforward from (1.4) to get

$$\int_{|x-y|\leq R} |x-y|^2 K(t,x,y)\,\mathrm{d}y \leq C R^{2-\alpha}$$

As for the second term, we have from (1.4)) and (1.5) that

$$\int_{|x-y|\geq R} K(t,x,y)\,\mathrm{d}y = \sum_{k=0}^{\infty} \int_{2^k R \leq |x-y| < 2^{k+1} R} K(t,x,y)\,\mathrm{d}y \leq C R^{-\alpha}.$$

This completes the proof. □

We next apply the weighted estimate for the fundamental solution $P_K(t,x;s,y)$ used in [25]. Without loss of generality we may take $s=0$.

Proposition 3.1. *Let Ψ be the function defined by (3.3).*
(i) *If $\lambda \in (2d/\alpha - d, d]$ then*
(3.6)

$$P_{K,v}(t,x;0,y) \le Ct^{-\frac{d}{\alpha}} \exp\left(-\Psi(t,x)+\Psi(0,y)+C\big(\Lambda(\Psi)t+\|v\|_{X_\lambda}^2 L^2 R^{\frac{\alpha\lambda}{d}} t^{\frac{2}{\alpha}-\frac{\lambda}{d}}\big)\right).$$

(ii) *If $\lambda \in (d, 2d/\alpha + d)$ then*
(3.7)

$$P_{K,v}(t,x;0,y) \le Ct^{-\frac{d}{\alpha}} \exp\left(-\Psi(t,x)+\Psi(0,y)+C\big(\Lambda(\Psi)t+\|v\|_{X_\lambda} L R^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}} t^{\frac{1}{\alpha}-\frac{\lambda}{2d}+\frac{1}{2}}\big)\right).$$

Here the positive constant C depends only on d, α, and C_0.

Proof. The argument below is almost parallel to the one used in the proof of [25, Proposition 3.8]. Set
(3.8)

$$\theta(t,x) = e^{\Psi(t,x)} \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y) e^{-\Psi(0,y)} f(y)\,\mathrm{d}y, \qquad f \in C_0^\infty(\mathbb{R}^d),\ f \ge 0,$$

and let $r:\ [0,t_0) \to [1,\infty)$ be a continuously differentiable function to be specified later. By direct calculation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\theta(t)\|_{L^{r(t)}} = \frac{r'}{r^2}\|\theta\|_{L^r}^{-r}\int |\theta|^r \log\frac{|\theta|^r}{\|\theta\|_{L^r}^r}\,\mathrm{d}x + \|\theta\|_{L^r}^{-r}\int \theta^{r-1}\partial_t\theta\,\mathrm{d}x.$$

Then we have from Lemma 2.4, Lemma 2.5, and (1.5),

$$\begin{aligned}
&\int \theta^{r-1}\partial_t\theta\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \theta^r \partial_t\Psi\,\mathrm{d}x + < e^{\Psi}\theta^{r-1}, \partial_t(e^{-\Psi}\theta) > \\
&= -\,\mathcal{E}_K(e^{\Psi}\theta^{r-1}, e^{-\Psi}\theta) - \mathcal{E}_{v(t)}(e^{\Psi}\theta^{r-1}, e^{-\Psi}\theta) + \int_{\mathbb{R}^d} \theta^r \partial_t\Psi\,\mathrm{d}x \\
&\le -\,\frac{2}{r}\mathcal{E}_K(\theta^{\frac{r}{2}}, \theta^{\frac{r}{2}}) + Cr\Lambda(\Psi)\|\theta_M\|_{L^r}^r + \int_{\mathbb{R}^d} \theta^r(\partial_t\Psi - v\cdot\nabla\Psi)\,\mathrm{d}x \\
&\le -\,\frac{2c_0}{r}\|(-\Delta)^{\frac{\alpha}{4}}\theta^{\frac{r}{2}}\|_{L^2}^2 + Cr\Lambda(\Psi)\|\theta\|_{L^r}^r + \int_{\mathbb{R}^d} \theta^r(\partial_t\Psi - v\cdot\nabla\Psi)\,\mathrm{d}x. \qquad (3.9)
\end{aligned}$$

Here c_0 is a constant depending only on d, α, and C_0.

We now divide the proof by the value of λ in the assumption **(C)** and first consider the case $\lambda \le d$. By using (3.2)-(3.4) and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\theta^r(\partial_t\Psi - v\cdot\nabla\Psi)\,\mathrm{d}x &= \int_{B_R(x_0+\xi(t;x_0))}\theta^r\Big(\fint_{B_R(x_0+\xi(t;x_0))} v - v\Big)\cdot\nabla\Psi\,\mathrm{d}x\\
&\le L\|\theta^{\frac{r}{2}}\|^2_{L^{\frac{4d}{2d-\alpha}}}\Big(\int_{B_R(x_0+\xi(t;x_0))}|v-\fint_{B_R(x_0+\xi(t;x_0))} v|^{\frac{2d}{\alpha}}\,\mathrm{d}x\Big)^{\frac{\alpha}{2d}}\\
&\le CLR^{\frac{\alpha\lambda}{2d}}\|(-\Delta)^{\frac{\alpha}{4}}\theta^{\frac{r}{2}}\|_{L^2}\|\theta\|^{\frac{r}{2}}_{L^r}\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\\
&\le \frac{c_0}{r}\|(-\Delta)^{\frac{\alpha}{4}}\theta^{\frac{r}{2}}\|^2_{L^2} + CrL^2R^{\frac{\alpha\lambda}{d}}\|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\|\theta\|^r_{L^r}. \qquad (3.10)\end{aligned}$$

Plugging this in (3.9), we have

$$\int\theta^{r-1}\partial_t\theta\,\mathrm{d}x \le -\frac{c_0}{r}\|(-\Delta)^{\frac{\alpha}{4}}\theta^{\frac{r}{2}}\|^2_{L^2} + r\Lambda(\Psi)\|\theta\|^r_{L^r} + rL^2R^{\frac{\alpha\lambda}{d}}\|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\|\theta\|^r_{L^r}.$$

Then we apply Lemma 2.1 with $\beta = \dfrac{c_0\pi^{\frac{\alpha}{2}}r}{r'}$ to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\log\|\theta(t)\|_{L^{r(t)}}\\
&\le -\frac{r'}{r^2}\Big(d + \frac{\alpha\Gamma(\frac{d}{2})}{2\Gamma(\frac{d}{\alpha})} + \frac{d}{\alpha}\big(\log\frac{\pi^{\frac{\alpha}{2}}}{c_0} + \log\frac{r}{r'}\big)\Big) + Cr\big(\Lambda(\Psi) + L^2R^{\frac{\alpha\lambda}{d}}\|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\big).\end{aligned}$$

Set $s(t) = 1/r(t)$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\theta(t)\|_{L^{\frac{1}{s}}} \le s'\Big(C_{d,\alpha} + \frac{d}{\alpha}\log\big(-\frac{s}{s'}\big)\Big) + \frac{C}{s}\big(\Lambda(\Psi) + L^2R^{\frac{\alpha\lambda}{d}}\|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\big).$$

Integrating from 0 to t_0, we get

$$\begin{aligned}\log\|\theta(t_0)\|_{L^{\frac{1}{s(t_0)}}} - \log\|\theta(0)\|_{L^{\frac{1}{s(0)}}} &\le \int_0^{t_0} s'\big(C_{d,\alpha} + \frac{d}{\alpha}\log s\big)\,\mathrm{d}t - \frac{d}{\alpha}\int_0^{t_0} s'\log(-s')\,\mathrm{d}t\\
&\quad + \int_0^{t_0}\frac{C}{s}\big(\Lambda(\Psi) + L^2R^{\frac{\alpha\lambda}{d}}\|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\big)\,\mathrm{d}t\end{aligned}$$

Choosing $s(t) = (1 - t/t_0)^q$ so that $s(t_0) = 0$, $s(0) = 1$ with $q \in (0, 2/\alpha - \lambda/d)$, we have

$$\int_0^{t_0} s'\big(C_{d,\alpha} + \frac{d}{\alpha}\log s\big)\,\mathrm{d}t = [C_{d,\alpha}s(t) - \frac{\alpha}{d}s(t)(\log s(t) - 1)]_{t=0}^{t_0} = -C_{d,\alpha}.$$

Moreover, the other integrals are estimated as follows:

$$-\int_0^{t_0} s'\log(-s')\,\mathrm{d}t \le -\log\, t_0 + C,\quad \int_0^{t_0}\frac{\mathrm{d}t}{s} = Ct_0,$$

$$\begin{aligned}\int_0^{t_0} \|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \frac{\mathrm{d}t}{s} &= \int_0^{t_0} \Big(-\int_t^{t_0} \|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\,\mathrm{d}\tau\Big)' \frac{\mathrm{d}t}{s} \\ &= \int_0^{t_0} \|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\,\mathrm{d}\tau - \int_0^{t_0} s's^{-2}\int_t^{t_0} \|v\|^2_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\,\mathrm{d}\tau\,\mathrm{d}t \\ &\leq C\|v\|^2_{X_\lambda} t_0^{\frac{2}{\alpha}-\frac{\lambda}{d}}.\end{aligned}$$

Summing up these estimates and replacing t_0 by t, we obtain

(3.11)

$$\log\|\theta(t)\|_{L^\infty} - \log\|\theta(0)\|_{L^1} \leq -\frac{d}{\alpha}\log t + C\big(1+\Lambda(\Psi)t + \|v\|^2_{X_\lambda} L^2 R^{\frac{\alpha\lambda}{d}} t^{\frac{2}{\alpha}-\frac{\lambda}{d}}\big),$$

which proves the desired estimate .

We next consider the case $d < \lambda \leq 2d/\alpha+d$. By using the characterization $\mathcal{L}^{2d/\alpha,\lambda} = \dot{C}^{\alpha\lambda/(2d)-\alpha/2}$, the last term in (3.9) can be estimated as the follows

$$\begin{aligned}&\int_{\mathbb{R}^d} \theta^r(\partial_t\Psi - v\cdot\nabla\Psi)\,\mathrm{d}x \\ &= \int_{B_R(x_0+\xi(t;x_0))} \theta^r\Big(⨍_{B_R(x_0+\xi(t;x_0))} v - v\Big)\cdot\nabla\Psi\,\mathrm{d}x \\ &\leq R^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}} \sup_{x,y\in\mathbb{R}^d} \frac{|v(x)-v(y)|}{|x-y|^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}}} L\|\theta\|^r_{L^r} \leq \|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} R^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}} L\|\theta\|^r_{L^r}.\end{aligned} \tag{3.12}$$

Thus, arguing as the preceding case, we get

(3.13)

$$\log\|\theta(t)\|_{L^\infty} - \log\|\theta(0)\|_{L^1} \leq -\frac{d}{\alpha}\log t + C\big(1+\Lambda(\Psi)t + \|v\|_{X_\lambda} L R^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}} t^{\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}\big).$$

This completes the proof. □

Since C in Proposition 3.1 does not depend on $\|v\|_{X_\lambda}$, by taking $L = 0$ and letting $M \to \infty$, we obtain

Corollary 3.1. *For all $t > 0$, x, $y \in \mathbb{R}^d$ it follows that*

$$P_{K,v}(t,x;0,y) \leq Ct^{-\frac{d}{\alpha}}. \tag{3.14}$$

Here C depends only on d, α, and C_0.

Proof of Theorem 1.3. We give the proof only for the case $\lambda \in (d, 2d/\alpha+d)$; the other case is shown similarly. Without loss of generality, we may assume $s = 0$. Fix x_0, $y_0 \in \mathbb{R}^d$, and $t_0 > 0$. Let us take $R = |x_0 - y_0|$ in Proposition 3.1. First we consider the case $R \leq C_* t_0^{\frac{1}{\alpha}}$, where $C_* \geq 1$ will be specified later. In this case we have from Corollary 3.1,

$$P_{K,v}(t_0,x_0;0,y_0) \le Ct_0^{-\frac{d}{\alpha}} \le Ct_0^{-\frac{d}{\alpha}}(1+C_*^{-1}t_0^{-\frac{1}{\alpha}}R)^{-d-\alpha}$$

$$\le CC_*^{d+\alpha}t_0^{-\frac{d}{\alpha}}(1+t_0^{-\frac{1}{\alpha}}R)^{-d-\alpha}. \tag{3.15}$$

Next we consider the case $R \ge C_* t_0^{\frac{1}{\alpha}}$. We may assume that $R \ge 2F[v](t_0, 0, x_0, y_0)$, otherwise the desired estimate always holds by Corollary 3.1. Take $L = \eta R^{-1}\log(R^\alpha/t_0)$ for some $\eta > 0$. Then Lemma 3.1 implies $\Lambda(\Psi)t_0 \le C$, where C depends only on d, α, γ, and C_0. Hence, applying Proposition 3.1 and $\Psi(0, y_0) = 0$, we have

$$P_{K,v}(t_0,x_0;0,y_0) \le Ct_0^{-\frac{d}{\alpha}}\exp(-\Psi(t_0,x_0) + C\|v\|_{X_\lambda} LR^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}}t_0^{\frac{1}{\alpha}-\frac{\lambda}{2d}+\frac{1}{2}}).$$

Taking C_* sufficient large depending on d, α and λ, we can estimate

$$LR^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}}t_0^{\frac{1}{\alpha}-\frac{\lambda}{2d}+\frac{1}{2}} = \eta(\frac{t_0}{R^\alpha})^{\frac{1}{\alpha}-\frac{\lambda}{2d}+\frac{1}{2}}\log(\frac{R^\alpha}{t_0}) \le C\eta.$$

for $R \ge C_* t_0^{1/\alpha}$. Thus, by the definition of Ψ, we get

$$P_{K,v}(t_0,x_0;0,y_0) \le Ct_0^{-\frac{d}{\alpha}}\exp\big(-L(R-|\xi(t_0;x_0)|)_+\big). \tag{3.16}$$

As in the proof of [25, Proposition 3.10], we have

$$-(R-|\xi(t_0;x_0)|)_+ \le -\frac{R}{4} \tag{3.17}$$

when $R \ge C_* t_0^{\frac{1}{\alpha}}$ and $R \ge 2F[v](t_0, 0, x_0, y_0)$. Hence, taking $\eta = \dfrac{4(d+\alpha)}{\alpha}$, we get

$$P_{K,v}(t_0,x_0;0,y_0) \le Ct_0^{-\frac{d}{\alpha}}\exp\big(-L(R-|\xi(t_0;x_0)|)_+\big) \le Ct_0^{-\frac{d}{\alpha}}\exp\big(-\frac{LR}{4}\big)$$

$$= Ct_0^{-\frac{d}{\alpha}}\exp\big(-\frac{\eta}{4}\log\frac{R^\alpha}{t_0}\big) = Ct_0R^{-d-\alpha}.$$

Here C depends only on d, α, C_0, and $\|v\|_{X_\lambda}$. The proof is complete. □

References

[1] H. Abels and M. Kassmann, The Cauchy problem and the martingale problem for integro-differential operators with non-smooth kernels. Osaka J. Math. **46** (2009) 661-683.

[2] D. G. Aronson, Non-negative solutions of linear parabolic equations. Ann. Scuola Norm. Sup. Pisa (3) **22** (1968) 607-694.

[3] M. Barlow, R. F. Bass, Z. Q. Chen, and M. Kassmann, Non-local Dirichlet forms and symmetric jump processes, Trans. Amer. Math. Soc. **361** (2009) 1963-1999.

[4] R. F. Bass, M. Kassmann, and T. Kumagai, Symmetric jump processes: localization, heat kernels and convergence, Ann. Inst. H. Poincaré Probab. Stat. **46** (2010) 59-71.

[5] R. F. Bass and D. A. Levin, Harnack inequalities for jump processes, Potential Anal. **17** (2002) 375-388.

[6] K. Bogdan and T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, Commun. Math. Phys. **271** (2007) 179-198.

[7] L. A. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. (2) **171** (2010) 1903-1930.

[8] E. A. Carlen and M. Loss, Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier-Stokes equation. A celebration of John F. Nash, Jr. Duke Math. J. **81** (1995) 135-157 (1996).

[9] E. A. Carlen, S. Kusuoka, and D. W. Stroock, Upper bounds for symmetric Markov transition functions, Ann. Inst. H. Poincaré Probab. Stat. **23** (1987) 245-287.

[10] D. Chae and J. Lee, Global well-posedness in the super-critical dissipative quasi-geostrophic equations, Commun. Math. Phys. **233** (2003) 297-311.

[11] P. Constantin, D. Córdoba, and J. Wu, On the critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J. **50** (2001) 97-107.

[12] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Commun. Math. Phys. **249** (2004) 511-528.

[13] P. Constantin, A. Majda, and E. Tabak, Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar, Nonlinearity **7** (1994) 1495-1533.

[14] A. Cotsiolis and N.K. Tavoularis, On logarithmic Sobolev inequalities for higher order fractional derivatives. C. R. Math. Acad. Sci. Paris **340** (2005) 205-208.

[15] E. B. Davies, Explicit constants for Gaussian upper bounds on heat kernels. Amer. J. Math. **109** (1987) 319-333.

[16] H. Dong and N. Pavlović, Regularity criteria for the dissipative quasi-geostrophic equations in Hölder spaces, Commun. Math. Phys. **290** (2009) 801-812.

[17] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton University Press, Princeton, NJ, 1983.

[18] T. Jakubowski and K. Szczypkowski, Time-dependent gradient perturbations of fractional Laplacian, J. Evol. Eq. **10** (2010) 319-339.

[19] A. Kiselev and F. Nazarov, A variation on a theme of Caffarelli and Vasseur, Zap. Nauchn. Sem. POMI **370** (2010) 58-72.

[20] A. Kiselev, F. Nazarov and A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math. **167** (2007) 445-453.

[21] T. Komatsu, Continuity estimates for solutions of parabolic equations associated with jump type Dirichlet forms, Osaka J. Math. **25** (1988) 697-728.

[22] T. Komatsu, Uniform estimates for fundamental solutions associated with non-local Dirichlet forms, Osaka J. Math. **32** (1995) 833-860.

[23] Y. Maekawa, A lower bound for fundamental solutions of the heat convection equations, Arch. Rational Mech. Anal., **189** (2008) 45-58.

[24] Y. Maekawa and H. Miura, On fundamental solutions for non-local diffusion equations with divergence free drift, Hokkaido university preprint series no.1014 (2012).

[25] Y. Maekawa and H. Miura, Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift, Hokkaido university preprint series no.1013 (2012).

[26] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. **80** (1958) 931-954.

[27] H. Osada, Diffusion processes with generators of generalized divergence form, J. Math. Kyoto Univ. **27** (1987) 597-619.

[28] J. Peetre, On the theory of $\mathcal{L}^{p,\lambda}$ spaces. J. Funct. Anal. **4** (1969) 71-87.

[29] G. Seregin, L. Silvestre, V. Sverak and A. Zlatos, On Divergence-free Drifts (preprint)

[30] Q. S. Zhang, A strong regularity result for parabolic equations. Commun. Math. Phys. **244** (2004) 245-260.