数理解析研究所講究録
第 1854 巻 2013 年 1-9

Picone identities for half-linear elliptic equations with $p(x)$-Laplacians and applications

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1 Introduction

Since the pioneering work of M. Picone [4], efforts have been made to establish Picone identities (or Picone-type inequalities) for differential equations of various type. Picone identities play an important role in the study of Sturmian comparison theorems (cf. [6]) and oscillation results for ordinary or partial differential equations or systems. In 1909, Picone [4] derived the so-called Picone identity

$$\begin{aligned} & \frac{d}{dt}\left(\frac{u}{v}(a(t)u'v - A(t)v'u)\right) \\ = \; & (a(t)-A(t))(u')^2 + (C(t)-c(t))u^2 + A(t)\left[v\left(\frac{u}{v}\right)'\right]^2 \\ & + \frac{u}{v}(vq[u] - uQ[v]) \end{aligned}$$

to obtain Sturmian comparison theorems for ordinary differential operators q, Q defined by

$$\begin{aligned} q[u] &= (a(t)u')' + c(t)u, \\ Q[v] &= (A(t)v')' + C(t)v. \end{aligned}$$

Recently, much current interest has been focused on various mathematical problems with variable exponent growth condition (cf. [2, 3]). The study of such problems arise from nonlinear elasticity theory, electrorheological fluids (see [5, 12]).

The operator $\nabla \cdot (|\nabla u|^{p(x)-2}\nabla u)$ $(p(x) > 1)$ is said to be $p(x)$-Laplacian, and becomes p-Laplacian $\nabla \cdot (|\nabla u|^{p-2}\nabla u)$ if $p(x) = p$ (constant), where the dot $\cdot$ denotes the scalar product and $\nabla = (\partial/\partial x_1, ..., \partial/\partial x_n)$.

The paper [11] by Zhang seems to be the first paper dealing with oscillations of solutions of $p(t)$-Laplacian equations of the form

$$(|u'|^{p(t)-2}u')' + t^{-\theta(t)}g(t,u) = 0, \quad t > 0.$$

In this work we present Picone identity, Picone-type inequality and Riccati inequality (which is reduced from Picone identity) to establish Sturmian comparison theorems and oscillation theorems for quasilinear elliptic operators with $p(x)$-Laplacians (cf. [1, 7–10]).

2 Half-linear elliptic inequalities

We establish Picone identities for half-linear elliptic inequalities

$$uq[u] \geq 0, \tag{1}$$
$$vQ[v] \leq 0, \tag{2}$$

where q and Q are defined by

$$\begin{aligned} q[u] \ &:= \ \nabla \cdot (a(x)|\nabla u|^{\alpha(x)-1}\nabla u) - a(x)(\log|u|)|\nabla u|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla u \\ &\quad +|\nabla u|^{\alpha(x)-1}b(x)\cdot\nabla u + c(x)|u|^{\alpha(x)-1}u, & (3) \\ Q[v] \ &:= \ \nabla \cdot (A(x)|\nabla v|^{\alpha(x)-1}\nabla v) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad +|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v, & (4) \end{aligned}$$

to derive Sturmian comparison theorems for q and Q. Let G be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G. It is assumed that $a(x), A(x) \in C(\overline{G}; (0,\infty))$, $b(x), B(x) \in C(\overline{G}; \mathbb{R}^n)$, $c(x), C(x) \in C(\overline{G}; \mathbb{R})$, and that $\alpha(x) \in C^1(\overline{G}; (0,\infty))$. The domain $\mathcal{D}_q(G)$ of q is defined to be the set of all functions u of class $C^1(\overline{G}; \mathbb{R})$ such that $a(x)|\nabla u|^{\alpha(x)-1}\nabla u \in C^1(G; \mathbb{R}^n) \cap C(\overline{G}; \mathbb{R}^n)$. The domain $\mathcal{D}_Q(G)$ of Q is defined similarly. We note that $\log|u|$ in (3) has singularities at zeros x_0 of $u(x)$, but $u\log|u|$ in (1) is continuous at every zero x_0 if we define $u\log|u| = 0$ at $x = x_0$, in view of $\lim_{\varepsilon\to+0}\varepsilon\log\varepsilon = 0$. We make the similar remark in (4). By a *solution* u [resp. v] of (1) [resp. (2)] we mean a function $u \in \mathcal{D}_q(G)$ [resp. $v \in \mathcal{D}_Q(G)$] which satisfies (1) [resp. (2)] in G. We note that (1) and (2) are *half-linear* in the sense that a constant multiple of a solution u [resp. v] is also a solution of (1) [resp. (2)] in light of

$$(ku)q[ku] = |k|^{\alpha(x)+1}uq[u] \ (k \in \mathbb{R}),$$
$$(kv)Q[kv] = |k|^{\alpha(x)+1}vQ[v] \ (k \in \mathbb{R}).$$

3 Picone identity

Lemma 1 (Picone identity for Q) *If $v \in \mathcal{D}_Q(G)$ and v has no zero in G, then we obtain the following Picone identity for any $u \in C^1(G;\mathbb{R})$ which has no zero in G:*

$$
\begin{aligned}
&-\nabla\cdot\left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\right)\\
&= -A(x)\left|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1}\\
&\quad +C(x)|u|^{\alpha(x)+1}\\
&\quad +A(x)\Bigg[\left|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1}\\
&\qquad +\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1}\\
&\qquad -(\alpha(x)+1)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}\Bigg(\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\\
&\qquad\qquad -\frac{u}{(\alpha(x)+1)A(x)}B(x)\Bigg)\cdot\left(\frac{u}{v}\nabla v\right)\Bigg]\\
&\quad -\frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}(vQ[v]) \quad \text{in } G, \qquad (5)
\end{aligned}
$$

where $\varphi(u)=|u|^{\alpha(x)-1}u=|u(x)|^{\alpha(x)-1}u(x)$.

Theorem 1 (Picone identity for q and Q) *Let $\alpha(x)\in C^2(G;(0,\infty))$ and $b(x)/a(x)\in C^1(G;\mathbb{R}^n)$. Assume that $u\in C^1(G;\mathbb{R})$, u has no zero in G, and that:*
(H_1) *there is a function $f\in C(\overline{G};\mathbb{R})\cap C^1(G;\mathbb{R})$ such that*

$$
\nabla f=\frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{b(x)}{(\alpha(x)+1)a(x)} \quad \text{in } G.
$$

If $e^f u\in\mathcal{D}_q(G)$, $v\in\mathcal{D}_Q(G)$ and v has no zero in G, then we obtain the following Picone identity:

$$
\begin{aligned}
&\nabla\cdot\left(e^{-(\alpha(x)+1)f}(e^f u)a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u)-\frac{u\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right)\\
&= a(x)\left|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1}\\
&\quad -A(x)\left|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1}
\end{aligned}
$$

$$
\begin{aligned}
&+(C(x)-c(x))|u|^{\alpha(x)+1} \\
&+A(x)\left[\left|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1}\right. \\
&\qquad +\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1} \\
&\qquad -(\alpha(x)+1)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}\left(\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\right. \\
&\qquad\qquad \left.\left.-\frac{u}{(\alpha(x)+1)A(x)}B(x)\right)\cdot\left(\frac{u}{v}\nabla v\right)\right] \\
&+e^{-(\alpha(x)+1)f}(e^f u)q[e^f u]-\frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}(vQ[v]) \quad in\ G.
\end{aligned}
$$

Theorem 2 (Sturmian comparison theorem) *Let $\alpha(x)\in C^2(G;(0,\infty))$ and $b(x)/a(x), B(x)/A(x)\in C^1(G;\mathbb{R}^n)$. Assume that there exists a function $u\in C^1(\overline{G};\mathbb{R})$ such that $u=0$ on ∂G, u has no zero in G, the hypothesis* (H_1) *of Theorem* 1 *holds and that:*
(H_2) *there is a function $F\in C(\overline{G};\mathbb{R})\cap C^1(G;\mathbb{R})$ such that*

$$
\nabla F=\frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{B(x)}{(\alpha(x)+1)A(x)} \quad in\ G.
$$

If $e^f u\in\mathcal{D}_q(G)$, $(e^f u)q[e^f u]\geq 0$ in G, and

$$
\begin{aligned}
&\int_G\left[a(x)\left|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1}\right. \\
&-A(x)\left|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\
&\qquad\left.+(C(x)-c(x))|u|^{\alpha(x)+1}\right]dx\geq 0, \qquad (6)
\end{aligned}
$$

then every solution $v\in\mathcal{D}_Q(G)$ of (2) *must vanish at some point of $\overline{G}$.*

Corollary 1 (Sturmian comparison theorem) *Let $\alpha(x)\in C^2(G;(0,\infty))$, $b(x)/a(x), B(x)/A(x)\in C^1(G;\mathbb{R}^n)$. Assume that:*

(i) $\dfrac{b(x)}{a(x)}=\dfrac{B(x)}{A(x)}$ *in G;*

(ii) $a(x)\geq A(x)$, $C(x)\geq c(x)$ *in G.*

If there exists a function $u\in C^1(\overline{G};\mathbb{R})$ such that $u=0$ on ∂G, u has no zero in G, the hypotheses (H_1) *and* (H_2) *of Theorems* 1 *and* 2 *hold, $e^f u\in\mathcal{D}_q(G)$, $(e^f u)q[e^f u]\geq 0$ in G, then every solution $v\in\mathcal{D}_Q(G)$ of* (2) *must vanish at some point of $\overline{G}$.*

4 Picone-type inequality

We derive Picone-type inequality and Sturmian comparison theorem for the half-linear elliptic operator q defined by

$$\begin{aligned} q[u] \;:=\; & \nabla\cdot(a(x)|\nabla u|^{\alpha(x)-1}\nabla u) - a(x)(\log|u|)|\nabla u|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla u \\ & +|\nabla u|^{\alpha(x)-1}b(x)\cdot\nabla u + c(x)|u|^{\alpha(x)-1}u, \end{aligned}$$

and the quasilinear elliptic operator $\tilde{Q}$ defined by

$$\begin{aligned} \tilde{Q}[v] \;:=\; & \nabla\cdot(A(x)|\nabla v|^{\alpha(x)-1}\nabla v) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ & +|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v \\ & +D(x)|v|^{\beta(x)-1}v + E(x)|v|^{\gamma(x)-1}v, \end{aligned}$$

where $D(x), E(x) \in C(\overline{G};[0,\infty))$ and $\alpha(x), \beta(x), \gamma(x) \in C(\overline{G};(0,\infty))$ with $0 < \gamma(x) < \alpha(x) < \beta(x)$.

Theorem 3 (Picone-type inequality for q and $\tilde{Q}$) *Assume that $\alpha(x) \in C^2(G;(0,\infty))$, $b(x)/a(x) \in C^1(G;\mathbb{R}^n)$, and that $u \in C^1(G;\mathbb{R})$, u has no zero in G, and the hypothesis* (H$_1$) *of Theorem* 1 *holds. If $e^f u \in \mathcal{D}_q(G)$, $v \in \mathcal{D}_{\tilde{Q}}(G)$ and v has no zero in G, then we obtain the Picone-type inequality:*

$$\begin{aligned} & \nabla\cdot\Big(e^{-(\alpha(x)+1)f}(e^f u)a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) - \frac{u\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha(x)-1}\nabla v\Big) \\ \geq\; & a(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\right|^{\alpha(x)+1} \\ & -A(x)\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\ & +(C(x)+\tilde{C}(x)-c(x))|u|^{\alpha(x)+1} \\ & +A(x)\Bigg[\left|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\right|^{\alpha(x)+1} \\ & \qquad +\alpha(x)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)+1} \\ & \qquad -(\alpha(x)+1)\left|\frac{u}{v}\nabla v\right|^{\alpha(x)-1}\Big(\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) \\ & \qquad\qquad -\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\Big(\frac{u}{v}\nabla v\Big)\Bigg] \\ & +e^{-(\alpha(x)+1)f}(e^f u)q[e^f u] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}(v\tilde{Q}[v]) \quad \text{in } G, \end{aligned}$$

where

$$\tilde{C}(x) = \left(\frac{\beta(x)-\gamma(x)}{\alpha(x)-\gamma(x)}\right)\left(\frac{\beta(x)-\alpha(x)}{\alpha(x)-\gamma(x)}\right)^{\frac{\alpha(x)-\beta(x)}{\beta(x)-\gamma(x)}} D(x)^{\frac{\alpha(x)-\gamma(x)}{\beta(x)-\gamma(x)}} E(x)^{\frac{\beta(x)-\alpha(x)}{\beta(x)-\gamma(x)}}.$$

Theorem 4 (Sturmian comparison theorem) *Under the same assumptions of Theorem 2 with $C(x)$ in (6) replaced by $C(x)+\tilde{C}(x)$, every solution $v \in \mathcal{D}_{\tilde{Q}}(G)$ of $v\tilde{Q}[v] \le 0$ must vanish at some point of $\overline{G}$.*

Corollary 2 (Sturmian comparison theorem) *Let $\alpha(x) \in C^2(G;(0,\infty))$, $b(x)/a(x), B(x)/A(x) \in C^1(G;\mathbb{R}^n)$. Assume that:*

(i) $\dfrac{b(x)}{a(x)} = \dfrac{B(x)}{A(x)}$ *in G;*

(ii) $a(x) \ge A(x)$, $C(x)+\tilde{C}(x) \ge c(x)$ *in G.*

If there exists a function $u \in C^1(\overline{G};\mathbb{R})$ such that $u=0$ on ∂G, u has no zero in G, the hypotheses (H_1) *and* (H_2) *of Theorems* 1 *and* 2 *hold, $e^f u \in \mathcal{D}_q(G)$, $(e^f u)q[e^f u] \ge 0$ in G, then every solution $v \in \mathcal{D}_{\tilde{Q}}(G)$ of $v\tilde{Q}[v] \le 0$ must vanish at some point of $\overline{G}$.*

5 Riccati inequality

Let Ω be an exterior domain in $\mathbb{R}^n$, that is, Ω includes the domain $\{x \in \mathbb{R}^n; |x| \ge r_0\}$ for some $r_0 > 0$. It is assumed that $A(x) \in C(\Omega;(0,\infty))$, $B(x) \in C(\Omega;\mathbb{R}^n)$, $C(x) \in C(\Omega;\mathbb{R})$, and that $\alpha(x) \in C^1(\Omega;(0,\infty))$. The domain $\mathcal{D}_Q(\Omega)$ of Q is defined to be the set of all functions v of class $C^1(\Omega;\mathbb{R})$ such that $A(x)|\nabla v|^{\alpha(x)-1}\nabla v \in C^1(\Omega;\mathbb{R}^n)$.

A solution $v \in \mathcal{D}_Q(\Omega)$ of (2) is said to be *oscillatory* in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in \mathbb{R}^n;\ |x| > r\}.$$

We use the notation $A[r,\infty) = \{x \in \mathbb{R}^n;\ |x| \ge r\}$, and find that $\Omega_{r_1} = A(r_1,\infty)$ for some large $r_1 \ge r_0$. Noting Picone identity (5) holds in any domain of $\mathbb{R}$ and letting $u=1$ in (5), we obtain the following lemma.

Lemma 2 *If $v \in \mathcal{D}_Q(\Omega)$ and v has no zero in $A[r_2, \infty)$ for some $r_2 > r_1$, then we obtain the following:*

$$\begin{aligned} & -\nabla \cdot \left(\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v} \right) \\ = \;& C(x) + \alpha(x)A(x)\left|\frac{\nabla v}{v}\right|^{\alpha(x)+1} + B(x) \cdot \left(\frac{|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v} \right) \\ & -\frac{vQ[v]}{|v|^{\alpha(x)+1}} \quad \text{in } A[r_2, \infty). \end{aligned}$$

Based on Lemma 2 we obtain the following.

Lemma 3 *If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \le 0$ in Ω and v has no zero in $A[r_2, \infty)$ for some $r_2 > r_1$, then we derive the Riccati inequality:*

$$\nabla \cdot (\psi(x)W(x)) + d(x) + \frac{\alpha(x)}{\alpha(x)+1} e(x)|W(x)|^{1+(1/\alpha(x))} \le 0$$

in $A[r_2, \infty)$ for any $\psi(x) \in C^1(A[r_2, \infty); (0, \infty))$, where

$$\begin{aligned} e(x) &= \frac{\alpha(x)+1}{2}\psi(x)A(x)^{-1/\alpha(x)}, \\ d(x) &= \psi(x)C(x) - \frac{1}{\alpha(x)+1} e(x)^{-\alpha(x)}\psi(x)^{\alpha(x)+1}\left|\frac{B(x)}{A(x)} - \frac{\nabla\psi(x)}{\psi(x)}\right|^{\alpha(x)+1}. \end{aligned}$$

Lemma 4 *Assume that the following hypothesis holds:*

(H) $\alpha(x) \equiv \alpha(|x|)$ *in $A[r_0, \infty)$.*

If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \le 0$ in Ω and v has no zero in $A[r_2, \infty)$ for some $r_2 > r_1$, then we have the Riccati inequality:

$$Y'(r) + \int_{S_r} d(x)\, dS + \frac{\alpha(r)}{\alpha(r)+1}\Psi(r)^{-1/\alpha(r)}|Y(r)|^{1+(1/\alpha(r))} \le 0 \qquad (7)$$

for $r \ge r_2$, where

$$\begin{aligned} S_r &= \{x \in \mathbb{R}^n;\ |x| = r\}, \\ \Psi(r) &= \int_{S_r} e(x)^{-\alpha(r)}\psi(x)^{\alpha(r)+1}\, dS, \\ Y(r) &= \int_{S_r} \psi(x)\langle W(x), \nu(x)\rangle\, dS, \end{aligned}$$

$\nu(x)$ being the unit exterior normal vector x/r on S_r.

Theorem 5 *Assume that the hypothesis* (H) *of Lemma* 4 *holds. If there exists a function* $\psi(x) \in C^1(A[r_1,\infty);(0,\infty))$ *such that the Riccati inequality* (7) *has no solution on* $[r,\infty)$ *for all large* r*, then every solution* $v \in \mathcal{D}_Q(\Omega)$ *of* $vQ[v] \le 0$ *is oscillatory in* Ω.

We can obtain oscillation results for $vQ[v] \le 0$ by analyzing one-dimensional Riccati inequalities with variable exponent of the form

$$y'(r) + \frac{1}{\beta(r)}\frac{1}{p(r)}|y(r)|^{\beta(r)} \le -q(r),$$

where $\beta(r) > 1$, $p(r) \in C([r_1,\infty);(0,\infty))$ and $q(r) \in C([r_1,\infty);\mathbb{R})$.

For example, we obtain the following.

Corollary 3 *Assume that the hypothesis* (H) *of Lemma* 4 *holds. Let* $\mu > 1$ *and* ν *be a real number. If there exists a function* $\psi(x) \in C^1(A[r_1,\infty);(0,\infty))$ *such that*

$$\limsup_{r\to\infty}\frac{1}{r^\mu}\int_{r_1}^{r}\Big[\omega_n s^{\nu+n-1}(r-s)^\mu \overline{d}(s) \\ -\frac{1}{\alpha(s)+1}s^{\nu-\alpha(s)+1}|\nu r-(\mu+\nu)s|^{\alpha(s)+1}(r-s)^{\mu-\alpha(s)-1}\Psi(s)\Big]ds = \infty,$$

then every solution $v \in \mathcal{D}_Q(\Omega)$ *of* $vQ[v] \le 0$ *is oscillatory in* Ω*, where* ω_n *denotes the surface area of the unit sphere* S_1 *and* $\overline{d}(r)$ *denotes the spherical mean of* $d(x)$ *over the sphere* S_r.

6 Forced oscillations

We study oscillation criteria for $v(\tilde{Q}[v]-f(x)) \le 0$ with a forcing term $f(x)$. Under some hypotheses we can establish Riccati inequality which is similar to that obtained in Lemma 3. Utilizing the Riccati method as were used for $vQ[v] \le 0$, we can obtain oscillation results for $v(\tilde{Q}[v]-f(x)) \le 0$ (see [9]).

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