数理解析研究所講究録
第 1855 巻 2013 年 222-228

Strong Solutions of Infinite-dimensional SDEs and Random Matrices

Hirofumi Osada (Kyushu University)
Hideki Tanemura (Chiba University)

This paper is an announcement of our recent results that will be published in [6] and [7] as full papers.

Let S be a connected open set in $\mathbb{R}^d$. We will take $S=\mathbb{R}^d$ and $(0,\infty)$ for example. If $S=\mathbb{R}^2$, then we naturally regard $\mathbb{R}^2$ as $\mathbb{C}$. S is a space where infinitely many particles move. If S has a boundary, we will suppose no particles hit the boundary for simplicity.

We present a new method to construct *unique, strong* solutions of infinite-dimensional stochastic differential equations (ISDEs) describing interacting Brownian motions (IBMs). Namely, we will solve ISDEs on $S^{\mathbb{N}}$. Our method can be applied to IBMs with Ruelle's class interaction potentials (with minimal smoothness just for the necessity to consider SDEs) such as Lennard-Jones 6-12 potentials in $\mathbb{R}^3$ and 2D Coulomb potentials in $\mathbb{R}$ and $\mathbb{R}^2$ related to random matrix theory such as Dyson's model (sine random point fields) and Bessel random point fields. As an application, we detect and solve the ISDEs whose unlabeled dynamics reversible with respect to Airy$_\beta$ random point fields with inverse temperatures $\beta=1,2,4$.

When $\beta=2$, infinite-dimensional stochastic dynamics has been constructed by a method of space-time correlation functions, which we will refer to the algebraic method, by Spohn and Johansson and others. We will prove that their dynamics is same as our stochastic dynamics given by the strong solution of the ISDE.

1 A new and equivalent notion of strong solutions of ISDEs

Since our method is flexible and seems to be applied in various situation, we state it in a general framework.

Let $W(S^{\mathbb{N}})=C([0,T];S^{\mathbb{N}})$ and let W_{sol} be a Borel subset of $W(S^{\mathbb{N}})$. Let $\sigma^i, b^i : W_{\mathrm{sol}}\to W(S^{\mathbb{N}})$. Let $\mathbf{S}_0$ be a Borel subset of $S^{\mathbb{N}}$.

We consider a quadruplet $(\{\sigma^i\},\{b^i\},W_{\mathrm{sol}},\mathbf{S}_0)$ and the ISDE on $S^{\mathbb{N}}$ of the form

$$dX_t^i=\sigma^i(\mathbf{X})_t dB_t^i+b^i(\mathbf{X})_t dt \quad (i\in\mathbb{N}) \tag{1.1}$$

$$\mathbf{X}_0=\mathbf{s}=(s_i)_{i\in\mathbb{N}}\in\mathbf{S}_0 \tag{1.2}$$

$$\mathbf{X}\in W_{\mathrm{sol}}. \tag{1.3}$$

Here $\mathbf{X}=\{\mathbf{X}_t\}_{t\in[0,T]}=\{(X_t^i)_{i\in\mathbb{N}}\}_{t\in[0,T]}\in W_{\mathrm{sol}}$, and $\mathbf{B}=\{B^i\}$ $(i\in\mathbb{N})$ is the $S^{\mathbb{N}}$-valued standard Brownian motion. By definition, $\mathbf{B}$ is a product of independent copy of S-valued standard Brownian motions.

Let $W^{\mathbf{0}}(S^{\mathbb{N}})=\{\mathbf{X}\in W(S^{\mathbb{N}})\,;\,\mathbf{X}_0=\mathbf{0}\}$. We assume:

(P1) The ISDE (1.1) has a solution $(\mathbf{X},\mathbf{B})\in W(S^{\mathbb{N}})\times W^{\mathbf{0}}(S^{\mathbb{N}})$ for each $\mathbf{s}\in\mathbf{S}_0$.

For a probability measure $\bar{P}_{\mathbf{s}}$ on $W(S^{\mathbb{N}})\times W^{\mathbf{0}}(S^{\mathbb{N}})$ we denote by $\bar{P}_{\mathbf{s},\mathbf{B}}$ the regular conditional probability. We set

$$\bar{P}_{\mathbf{s},\mathbf{B}}=\bar{P}_{\mathbf{s}}(\mathbf{X}\in\cdot\,|\mathbf{B}),\quad \mathbf{P}_{\mathbf{s}}=\bar{P}_{\mathbf{s}}(\mathbf{X}\in\cdot),\quad P_{\mathrm{Br}}^{\infty}=\bar{P}_{\mathbf{s}}(\mathbf{B}\in\cdot) \tag{1.4}$$

We next introduce a system of finite-dimensional SDEs associated with the ISDE (1.1)–(1.3). For this we prepare a couple of notations.

For a path $\mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \in W(S^{\mathbb{N}})$ and $m \in \mathbb{N}$, we set

$$\mathbf{X}^{m*} = (0, \dots, 0, X_t^{m+1}, X_t^{m+2}, \dots) \in W(S^{\mathbb{N}}).$$

For $\mathbf{X} \in W_{\mathrm{sol}}$, $\mathbf{s} \in \mathbf{S}_0$, and $m \in \mathbb{N}$, we consider a system of finite-dimensional SDEs (1.5) on $\mathbf{Y}^m = (Y_t^{m,1}, \dots, Y_t^{m,m})$.

$$dY_t^i = \sigma^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dt \quad (i = 1, \dots, m) \tag{1.5}$$
$$\mathbf{Y}_0^m = (s_1, \dots, s_m) \in S^m, \quad \text{where } \mathbf{s} = (s_i)_{i=1}^{\infty}, \quad .$$

Here $\mathbf{X}^{m*}$ is interpreted as a part of the coefficients of the SDE (1.5), and we set

$$\mathbf{Y}^m + \mathbf{X}^{m*} = (Y_t^{m,1}, \dots, Y_t^{m,m}, X_t^{m+1}, X_t^{m+2}, \dots). \tag{1.6}$$

Let $W_{\mathrm{sol}}^{\mathbf{s}} = \{\mathbf{X} \in W_{\mathrm{sol}}\,;\, \mathbf{X}_0 = \mathbf{s}\}$. We assume:

(P2) For each $\mathbf{s} \in \mathbf{S}_0$ and $\mathbf{X} \in W_{\mathrm{sol}}^{\mathbf{s}}$, the SDE (1.5) has a unique, strong solution $\mathbf{Y}^m$ for each $m \in \mathbb{N}$. Moreover, $\mathbf{Y}^m$ satisfies

$$\mathbf{Y}^m + \mathbf{X}^{m*} \in W_{\mathrm{sol}} \tag{1.7}$$

We remark that the SDEs in (P2) are all in *finite*-dimensions. We will solve ISDEs by introducing the consistent family of finite-dimensional SDEs in (P2) and the tail σ-field concerning the path space, which we now define.

Let $\mathit{Tail}_{\mathrm{path}}(S^{\mathbb{N}})$ be the tail σ-field of $W(S^{\mathbb{N}})$ defined by

$$\mathit{Tail}_{\mathrm{path}}(S^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]. \tag{1.8}$$

For a probability measure $\mathbf{P}$ on $W(S^{\mathbb{N}})$, we set

$$\mathit{Tail}_{\mathrm{path}}^{[1]}(\mathbf{P}) = \{A \in \mathit{Tail}_{\mathrm{path}}(S^{\mathbb{N}})\,;\, \mathbf{P}(A) = 1\}.$$

(P3) For each $\mathbf{s} \in \mathbf{S}_0$, the tail σ-field $\mathit{Tail}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathbf{s}}$-trivial.

We state the main theorems in this section.

Theorem 1. (1) *Assume* (P1)–(P3). *Then ISDE* (1.1)–(1.3) *has a strong solution for each* $\mathbf{s} \in \mathbf{S}_0$.
(2) *Assume* (P2). *Let* $\mathbf{Y}_{\mathbf{s}}$ *and* $\mathbf{Y}'_{\mathbf{s}}$ *be strong solutions of ISDE* (1.1)–(1.3) *starting at* $\mathbf{s} \in \mathbf{S}_0$ *defined on the same space of Brownian motions* $\mathbf{B}$. *Then* $\mathbf{Y}_{\mathbf{s}} = \mathbf{Y}'_{\mathbf{s}}$ *a.s. if and only if*

$$\mathit{Tail}_{\mathrm{path}}^{[1]}(\mathrm{Law}(\mathbf{Y}_{\mathbf{s}})) = \mathit{Tail}_{\mathrm{path}}^{[1]}(\mathrm{Law}(\mathbf{Y}'_{\mathbf{s}})). \tag{1.9}$$

2 A Tail theorem

The first two assumptions (P1) and (P2) in Theorem 1 can be verified by the general theory developed in [2, 3, 4, 5] together with the classical theory of (finite-dimensional) SDEs. The third assumption (P3) requires the triviality of the tail σ-field of path space with respect to the label of the particles. This assumption is the most difficult one to be verified. We will deduce the tail triviality of path spaces from the tail triviality of the associated configuration spaces. We again give a general statement.

Let S be the configuration space over S. Set $S_r = \{s \in S\,;\, |s| < r\}$. Let $\mathit{Tail}(\mathsf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}]$ be the tail σ-field of S. Here $\pi_A : \mathsf{S} \to \mathsf{S}$ such that $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$. Let μ be a probability measure on S. We assume:

(Q1) $\mathit{Tail}(\mathsf{S})$ is μ-trivial, that is, $\mu(\mathsf{A}) \in \{0, 1\}$ for all $\mathsf{A} \in \mathit{Tail}(\mathsf{S})$.

Let $W(\mathsf{S}) = C([0,T]; \mathsf{S})$ and write $\mathsf{X} = \{\mathsf{X}_t\}_{0 \le t \le T} \in W(\mathsf{S})$. We lift the μ-triviality of $\mathit{Tail}(\mathsf{S})$ to the triviality of the labeled path spaces $W(S^{\mathbb{N}})$ with respect to a lift dynamics we now introduce. For this we equip S with a measurable subset S_0 and a family of probability measures $\{\mathsf{P_s}\}_{\mathsf{s} \in \mathsf{S}_0}$ on $W(\mathsf{S})$. We suppose that $\mathsf{P_s}(A)$ is measurable in $\mathsf{s} \in \mathsf{S}_0$ for each $A \in \mathcal{B}(W(\mathsf{S}_0))$, and $\int_{\mathsf{S}_0} \mathsf{P_s}\nu(d\mathsf{s})$ becomes a probability on $W(\mathsf{S}_0)$ for any probability measure ν on S_0.

For a given μ, $\{\mathsf{P_s}\}_{\mathsf{s} \in \mathsf{S}_0}$ are called **μ-lift dynamics** if $\{\mathsf{P_s}\}_{\mathsf{s} \in \mathsf{S}_0}$ satisfy (2.1)–(2.3).

$$\mu(\mathsf{S}_0) = 1, \quad \mathsf{P_s}(\mathsf{X}_0 = \mathsf{s}) = 1 \text{ for all } \mathsf{s} \in \mathsf{S}_0. \tag{2.1}$$

$$\mathsf{P}_{m\mu}^{\mathsf{X}_t} \prec \mu \text{ for all } t \in [0,T] \text{ and } m \in L^2(\mu). \tag{2.2}$$

$$\text{The density } p(t, \mathsf{s}, \mathsf{t}) \text{ is } \mathcal{B}([0,T]) \times \mathcal{B}(\mathsf{S}) \times \mathcal{B}(\mathsf{S})\text{-measurable.} \tag{2.3}$$

Here $\mathsf{P}_{m\mu}^{\mathsf{X}_t} = \mathsf{P}_{m\mu} \circ \mathsf{X}_t^{-1}$, $\mathsf{P}_{m\mu} = \int_{\mathsf{S}} \mathsf{P_s} m(\mathsf{s})\mu(d\mathsf{s})$, and $p(t, \mathsf{s}, \mathsf{t}) = \mathsf{P_s} \circ \mathsf{X}_t^{-1}(d\mathsf{t})/d\mu$. Moreover, for given Radon measures μ, ν, we denote by $\mu \prec \nu$ if μ is absolutely continuous with respect to ν.

(Q2) There exist μ-lift dynamics $\{\mathsf{P_s}\}_{\mathsf{s} \in \mathsf{S}_0}$.

We set $\mathsf{S}_{\text{s.i.}} = \{\mathsf{s}\,;\, \mathsf{s}(S) = \infty, \mathsf{s}(\{x\}) \le 1 \text{ for all } x \in S\}$, and assume that

(Q3) $\mathsf{P_s}(W(\mathsf{S}_{\text{s.i.}})) = 1$ for all $\mathsf{s} \in \mathsf{S}_0$.

We call a measurable map $\mathfrak{l} : \mathsf{S} \to S^{\mathbb{N}}$ a label if $\mathfrak{u} \circ \mathfrak{l} = \text{id}$. Here $\mathfrak{u}$ is the unlabel map defined by $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$. Let $\mathfrak{l}(\mathsf{s}) = (\mathfrak{l}_n(\mathsf{s}))_{n \in \mathbb{N}}$ be a label. Let $\mathfrak{l}_{\text{path}}$ be the map $\mathfrak{l}_{\text{path}} : W(\mathsf{S}_{\text{s.i.}}) \to W(S^{\mathbb{N}})$ such that $\mathfrak{l}_{\text{path}}(\mathsf{X})_0 = \mathfrak{l}(\mathsf{X}_0)$. This map is well defined because the domain is restricted on $W(\mathsf{S}_{\text{s.i.}})$. We write $\mathbf{X} = \mathfrak{l}_{\text{path}}(\mathsf{X})$. If we write $\mathsf{X}_t = \sum_{n=1}^{\infty} \delta_{X_t^n}$, where $X_t^n \in C([0,T]; S)$, then by definition $\mathbf{X}_t = (X_t^n)_{n \in \mathbb{N}}$ for all t.

We set $W(S_r^c) = C([0,T]; S_r^c)$ and define $\mathsf{m}_r : W(\mathsf{S}_{\text{s.i.}}) \to \mathbb{N} \cup \{\infty\}$ by

$$\mathsf{m}_r(\mathsf{X}) = \inf\{m \in \mathbb{N}\,;\, X^n \in W(S_r^c) \text{ for all } m < n \in \mathbb{N}\}. \tag{2.4}$$

Here we set $\mathfrak{l}_{\text{path}}(\mathsf{X}) = (X^1, X^2, \ldots) \in W(S^{\mathbb{N}})$ and regard X^n as a map from $\mathsf{S}_{\text{s.i.}}$ to $W(S)$ by the correspondence $\mathsf{X} = \sum_{i=1}^{\infty} \delta_{X^i} \longmapsto X^n$. By construction, this map is the composition of $\mathfrak{l}_{\text{path}}$ and the path coordinate map $\mathbf{X} = (X^i)_{i \in \mathbb{N}} \mapsto X^n$.

(Q4) P_μ and the label $\mathfrak{l}$ satisfy the following.

$$\mathsf{P}_\mu(\mathsf{m}_r(\mathsf{X}) < \infty) = 1. \tag{2.5}$$

Theorem 2. *Assume* (Q1)–(Q4). *Let* $\mathbf{P}_{\mathsf{s}} = \mathsf{P}_{\mathsf{s}} \circ \mathfrak{l}_{\text{path}}^{-1}$, $\mathbf{s} = \mathfrak{l}(\mathsf{s})$, *and* $\mu^{\mathfrak{l}} = \mu \circ \mathfrak{l}^{-1}$. *Let* $\mathcal{G}$ *be a sub* σ*-field of* $\mathit{Tail}_{\text{path}}(S^{\mathbb{N}})$. *Assume that* $\mathcal{G}$ *is countably determined under* $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$. *Then*

(1) $\mathcal{G}$ *is* $\mathbf{P}_{\mathbf{s}}$*-trivial for* $\mu^{\mathfrak{l}}$*-a.s.* $\mathbf{s}$.

(2) *For* $\mu^{\mathfrak{l}}$*-a.s.* $\mathbf{s}$, *the set* $\mathit{Tail}_{\text{path}}^{[1]}(S^{\mathbb{N}}, \mathcal{G}; \mathbf{P}_{\mathbf{s}}) = \{\mathbf{A} \in \mathcal{G}\,;\, \mathbf{P}_{\mathbf{s}}(\mathbf{A}) = 1\}$ *is independent of* $\mathbf{s}$ *and the particular choice of* $\{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{S}_0}$ *in* (Q2).

3 Strong solutions of interacting Brownian motions

In this section, we apply the results to interacting Brownian motions. We will prove the uniqueness and existence of strong solutions of interacting Brownian motions in infinite-dimensions.

We begin by introducing the ISDE. Let H be a measurable subset in S. Let $\mathfrak{u}$ be the unlabeled map, and set $\mathbf{H} = \mathfrak{u}^{-1}(\mathsf{H})$. Let $\sigma^i : S \times \mathsf{H} \to (\mathbb{R}^d)^{\mathbb{N}}$ and $\mathsf{b}^i : S \times \mathsf{H} \to (\mathbb{R}^{d^2})^{\mathbb{N}}$ be measurable functions. Let $\mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \in W(S^{\mathbb{N}})$ and set $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$, $\mathsf{X}_t^{i*} = \sum_{j\in\mathbb{N},\ j\neq i} \delta_{X_t^j}$. Consider the ISDE of Markovian type.

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i*})dB_t^i + \mathsf{b}(X_t^i, \mathsf{X}_t^{i*})dt \tag{3.1}$$

$$\mathbf{X}_0 = \mathbf{s} \in \mathbf{H} \tag{3.2}$$

$$\mathbf{X} \in \mathfrak{l}_{\text{path}}(\mathsf{H}). \tag{3.3}$$

We set $\mathsf{a}(x, \mathsf{y}) = \sigma(x, \mathsf{y})^t \sigma(x, \mathsf{y})$ and assume (R1)–(R6) below.

(R1) μ has a log derivative $\mathsf{d}_\mu(x, \mathsf{y})$ satisfying the identity

$$\mathsf{b}(x, \mathsf{y}) = \frac{1}{2}\{\nabla_x \mathsf{a}(x, \mathsf{y}) + \mathsf{a}(x, \mathsf{y})\mathsf{d}_\mu(x, \mathsf{y})\}.$$

(R2) μ is a (Φ, Ψ)-quasi Gibbs measure, and (Φ, Ψ) is upper semi continuous.

(R3) $\rho^1 \in L^1_{\text{loc}}(S, dx)$ and $\sigma_r^k \in L^2(S_r^k, d\mathbf{x}_k)$ for all $k, r \in \mathbb{N}$.
Here ρ^1 is the 1-correlation function of μ and σ_r^k are k-density functions on $S_r = \{|x| \le r\}$.

(R4) H is a subset of $\mathsf{S}_{\text{s.i.}}$ satisfying $\text{Cap}^\mu(\mathsf{H}^c) = 0$. Here $\mathsf{H} = \mathfrak{u}(\mathbf{H})$.

(R5) Each tagged particles are non-explosive. Namely,

$$P(\sup_{0\le t\le T} |X_t^i| < \infty, \quad \text{for all } T, i \in \mathbb{N}) = 1.$$

(R6) The assumption (P2) is satisfied by taking $\mathbf{S}_0 = \mathbf{H}$ and $W_{\text{sol}} = C([0,T]; \mathbf{H})$.

• "quasi-Gibbs meaures" and "log derivative" are most prime notions in our argument. We refer to [4, 3, 5] for the definition of quasi-Gibbs measures, and [3] for log derivatives.
• From (R2) and (R3), we deduce that $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is a quasi-regular Dirichlet form on $L^2(\mathsf{S}, \mu)$. Cap^μ in (R4) is the capacity associated with this Dirichlet space.
• The assumption (R6) is satisfied if the coefficients a and b satisfy "local Lipschitz conditions".

Let μ_{t} be the regular conditional probability defined by $\mu_{\mathsf{t}} = \mu(\cdot|\mathit{Tail}(\mathsf{S}))(\mathsf{t})$. Here $\mathsf{t} \in \mathsf{S}$. By construction we see that

$$\mu(\mathsf{A}) = \int_{\mathsf{S}} \mu_{\mathsf{t}}(\mathsf{A})\mu(d\mathsf{t}), \text{ and } \mu_{\mathsf{t}}(\mathsf{A}) \text{ is } \mathit{Tail}(\mathsf{S})\text{-measurable for each } \mathsf{A} \in \mathcal{B}(\mathsf{S}). \tag{3.4}$$

Lemma 3. *Assume that μ is a quasi-Gibbs measure. Then Tail* (S) *is μ_{t}-trivial for μ-a.s.* t. *Moreover, for μ-a.s.* t,

$$\mu_{\mathsf{t}}(\mathsf{A}) = 1_{\mathsf{A}}(\mathsf{t}) \quad \textit{for all } \mathsf{A} \in \textit{Tail}\,(\mathsf{S}). \tag{3.5}$$

Taking (3.5) into account, we introduce the equivalent relation $\mathsf{S}/\textit{Tail}\,(\mathsf{S})$ such that

$$\mathsf{t} \sim \mathsf{t}' \Leftrightarrow \mathsf{t}, \mathsf{t}' \in \mathsf{A} \text{ for all } \mathsf{A} \in \textit{Tail}\,(\mathsf{S}). \tag{3.6}$$

Theorem 4. *Assume* (R1)–(R6). *Then there exists* S_0 *such that* $\mu(\mathsf{S}_0) = 1$ *satisfying the following:*
(1) *The ISDE* (3.1)–(3.3) *has a strong solution* $(\mathbf{X}, \mathbf{P_s})$ *for each* $\mathbf{s} \in \mathbf{S}_0$.
(2) $\mathbf{S}_0$ *can be decomposed as a disjoint sum* $\mathbf{S}_0 = \sum_{\mathsf{S}/\textit{Tail}\,(\mathsf{S})} \mathbf{S}_{0,\mathsf{t}}$ *such that* $\mu_{\mathsf{t}}(\mathsf{S}_{0,\mathsf{t}}) = 1$, *where* $\mathsf{S}_{0,\mathsf{t}} = \mathbf{u}(\mathbf{S}_{0,\mathsf{t}})$, *and that the sub collection* $\{(\mathsf{X}, \mathsf{P}_{\mathsf{s}})\}_{\mathsf{s} \in \mathsf{S}_{0,\mathsf{t}}}$ *are* $\mathsf{S}_{0,\mathsf{t}}$*-valued,* μ_{t}*-reversible diffusion satisfying*

$$\mathsf{P}_{\mu_{\mathsf{t}}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathsf{t}} \quad \textit{for all } t \quad \textit{for } \mu\textit{-a.s. } \mathsf{t}. \tag{3.7}$$

(3) *A family of strong solutions* $\{(\mathbf{X}, \mathbf{P_s})\}_{\mathbf{s} \in \mathbf{S}_0}$ *of* (3.1)–(3.3) *satisfying* (3.7) *is unique for* μ^{l}*-a.s.* $\mathbf{s}$.

We next give examples that Theorem 4 can be applied to. Let $\beta > 0$ be an inverse temperature.

Example 1. μ are canonical Gibbs measures with free potential Φ and Ruelle's class interacting potentials Ψ.

$$dX_t^i = dB_t^i + \frac{\beta}{2}\nabla\Phi(X_t^i)dt + \frac{\beta}{2}\sum_{i=1,\, j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt \quad (i \in \mathbb{N}).$$

Let $S = \mathbb{R}^3$, $\Phi = 0$ and $\Psi = \{|x|^{-12} - |x|^{-6}\}$ be Lennard-Jone's 6-12 potentials. Then

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1, j\neq i}^{\infty} \{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N}).$$

Example 2. μ are sine_β random point field with $\beta = 1, 2, 4$. Then $S = \mathbb{R}$ and

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r,\, j\neq i} \frac{1}{X_t^i - X_t^j}dt$$

Example 3. μ is Bessel_β random point field with $\beta = 2$. Then $S = (0, \infty)$ and $a > 1$,

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i}dt + \lim_{r\to\infty}\frac{\beta}{2} \sum_{|X_t^i - X_t^j| < r,\, j\neq i} \frac{1}{X_t^i - X_t^j}dt$$

The case $\beta = 1, 4$ is in progress.

Example 4. μ is Airy$_\beta$ random point field with $\beta = 1, 2, 4$. Then $S = \mathbb{R}$ and

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{|X_t^i - X_t^j| < r, j \neq i} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Here ϱ is the rescaled semi-circle function centered at 2, defined by

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x).$$

Example 5. μ is the Ginibre random point field. Then $S = \mathbb{R}^2$ and

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r, j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt.$$

A surprising fact is that $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ is a strong solution of another ISDE.

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j| < r, j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt.$$

Remark 1. (1) From the uniqueness of strong solutions, we deduce the uniqueness of quasi-regular local, Dirichlet forms on the configuration space (unlabeled dynamics) when the tail σ-field *Tail* (S) of the configuration space is μ-trivial, where μ is the reference measure of the Dirichlet space. In particular, Dirichlet spaces related to Lang's approximation and the first author's approximation are the same (if the tail σ-field *Tail* (S) is μ-trivial). The condition (R5) is essential for this.
(2) It is plausible that our method can be applied to Airy$_\beta$, Sine$_\beta$, Bessel$_\beta$ ensembles for general $0 < \beta < \infty$, and random point fields given by the zero points of Gaussian analytic functional.

4 Identification of Airy$_2$ stochastic dynamics in infinite-dimensions

As an application of the uniqueness of strong solutions, we see that the solution of the ISDE related to Airy$_2$ random ponit field is same as the Airy$_2$ stochastic dynamics given by the algebraic method in [1, 8].

Lemma 5. *The Airy$_2$ random point field has a trivial tail.*

Theorem 6. *The Airy$_2$ stochastic dynamics given by the space-time correlation functions in* [1, 8] *is the unique strong solution of the ISDE*

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \{ (\sum_{|X_t^i - X_t^j| < r,\ j \neq i} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \quad (i \in \mathbb{N}).$$

Here $\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0)}(x)$ *as before.*

References

[1] Johansson, K. *Discrete polynuclear growth and determinantal processes*, Commun. Math. Phys. **242**, 277-329 (2003)

[2] Osada, H., *Tagged particle processes and their non-explosion criteria*, J. Math. Soc. Japan, **62**, No. **3**, 867-894 (2010)

[3] Osada, H., *Infinite-dimensional stochastic differential equations related to random matrices*, Probability Theory and Related Fields, **153**, 471-509 (2012)

[4] Osada, H., *Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials*, Ann. of Probab. **41**, 1-49 (2013)

[5] Osada, H., *Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II : Airy random point field*, Stochastic Processes and their applications **123**, 813-838 (2013)

[6] Osada, H., Tanemura, H. *Infinite-dimensional stochastic differential equations related to Airy random point fields*, (in preparation).

[7] Osada, H., Tanemura, H. *Uniqueness and existence of strong solutions of infinite-dimensional stochastic differential equations describing interacting Brownian motions*, (in preparation).

[8] Prähofer, M., Spohn, H. *Scale invariance of the PNG droplet and the Airy process*, J. Stat. Phys. **108**, 1071-1106 (2002)

Hirofumi Osada
Faculty of Mathematics Kyushu University
Fukuoka 819-0395, Japan.
osada@math.kyushu-u.ac.jp

Hideki Tanemura
Department of Mathematics and Informatics
Faculty of Science, Chiba University
1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan.
tanemura@math.s.chiba-u.ac.jp