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Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians

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This note is a short presentation of recent results for semi-classical analysis of lowlying eigenvalues of spatially cut-off $P(\phi)_2$-Hamiltonians based on the author's recent research ([2, 3]). We refer the readers for semi-classical analysis in finite dimensions to [17, 21, 32, 33, 19, 20] and for $P(\phi)_2$-Hamiltonians to [12, 31, 34, 7].

First, we give a definition of spatially cut-off $P(\phi)_2$-Hamiltonians. Let $m>0$. Let μ be the Gaussian measure on the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ such that

$$\int_W {}_{\mathcal{S}(\mathbb{R})}\langle\varphi, w\rangle^2_{\mathcal{S}'(\mathbb{R})} d\mu(w)=\left((m^2-\Delta)^{-1/2}\varphi,\varphi\right)_{L^2}.$$

Let $\mathcal{E}$ be the Dirichlet form defined by

$$\mathcal{E}(f,f)=\int_W \|\nabla f(w)\|^2_{L^2(\mathbb{R},dx)}d\mu(w)\quad f\in \mathrm{D}(\mathcal{E}),$$

where $\nabla f(w)$ is the unique element in $L^2(\mathbb{R},dx)$ such that

$$\lim_{\varepsilon\to 0}\frac{f(w+\varepsilon\varphi)-f(w)}{\varepsilon}=(\nabla f(w),\varphi)_{L^2(\mathbb{R},dx)}.$$

The generator $-L(\geq 0)$ of $\mathcal{E}$ is one of expressions of a free Hamiltonian. Let $P(x)=\sum_{k=0}^{2M}a_kx^k$, where $a_{2M}>0$. Let $g\in C_0^\infty(\mathbb{R})$ with $g(x)\geq 0$ for all x and define for $h\in H^1(=H^1(\mathbb{R}))$,

$$V(h)=\int_{\mathbb{R}}P(h(x))g(x)dx$$
$$U(h)=\frac{1}{4}\int_{\mathbb{R}}\left(h'(x)^2+m^2h(x)^2\right)dx+V(h)$$

We want to consider an operator like

$$-L+\lambda V(w/\sqrt{\lambda})\quad \text{on}\quad L^2(\mathcal{S}'(\mathbb{R}),d\mu).$$

The difficulty is in the definition of $w(x)^k$ because w is an element of the Schwartz distribution. Instead of $w(x)^k$, we use Wick power $:w(x)^k:$ which requires renormalizations for which we refer the readers to [12, 31, 34, 7]. For $P=P(x)=\sum_{k=0}^{2M}a_kx^k$ with $a_{2M}>0$, define

$$\int_{\mathbb{R}}:P\left(\frac{w(x)}{\sqrt{\lambda}}\right):g(x)dx=\sum_{k=0}^{2M}a_k\int_{\mathbb{R}}:\left(\frac{w(x)}{\sqrt{\lambda}}\right)^k:g(x)dx.$$

We write

$$:V\left(\frac{w}{\sqrt{\lambda}}\right):=\int_{\mathbb{R}}:P\left(\frac{w(x)}{\sqrt{\lambda}}\right):g(x)dx,$$
$$V_\lambda(w)=\lambda:V\left(\frac{w}{\sqrt{\lambda}}\right):.$$

Definition 1. *The spatially cut-off $P(\phi)_2$-Hamiltonian $-L+V_\lambda$ is defined to be the unique self-adjoint extension operator of $(-L+V_\lambda, \mathfrak{F}C_b^\infty(\mathcal{S}'(\mathbb{R})))$.*

It is known that $-L+V_\lambda$ is bounded from below and the first eigenvalue $E_1(\lambda)$ is simple and the corresponding positive eigenfunction $\Omega_{1,\lambda}$ exists. See [12, 31, 34]. Formally, $-L+V_\lambda$ is unitarily equivalent to the infinite dimensional Schrödinger operator:

$$-\Delta_{L^2(\mathbb{R})}+\lambda U(w/\sqrt{\lambda})-\frac{1}{2}\mathrm{tr}(m^2-\Delta)^{1/2} \quad \text{on } L^2(L^2(\mathbb{R}),dw)$$

where dw is an infinite dimensional Lebesgue measure. The function U is a potential function such that

$$U(w)=\frac{1}{4}\int_{\mathbb{R}}w'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{4}w(x)^2+:P(w(x)):g(x)\right)dx$$

and $\Delta_{L^2(\mathbb{R})}$ denotes the "Laplacian"on $L^2(\mathbb{R},dx)$. Hence, by the analogy of Schrödinger operators in finite dimensions, it is natural to expect that asymptotic behavior of lowlying eigenvalues of $-L+V_\lambda$ in the semiclassical limit $\lambda\to\infty$ is related with the properties of global minimum points of U. In view of this, we consider the following assumptions.

Assumption 2. *Let U be the function on H^1 such that*

$$U(h)=\frac{1}{4}\int_{\mathbb{R}}h'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{4}h(x)^2+P(h(x))g(x)\right)dx \qquad \text{for } h\in H^1.$$

(A1) *The function U is non-negative and the zero point set*

$$\mathcal{Z}:=\{h\in H^1 \mid U(h)=0\}=\{h_1,\dots,h_n\}$$

is a finite set.
(A2) *For all $1\le i\le n$, the Hessian $\nabla^2U(h_i)$ is non-degenerate. That is, there exists $\delta_i>0$ for each i such that*

$$\begin{aligned}\nabla^2U(h_i)(h,h)&:=\frac{1}{2}\int_{\mathbb{R}}h'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{2}h(x)^2+P''(h_i(x))g(x)h(x)^2\right)dx\\&\ge\delta_i\|h\|^2_{L^2(\mathbb{R})} \qquad \text{for all } h\in H^1(\mathbb{R}).\end{aligned}$$

(A3) *For all x, $P(x)=P(-x)$ and $\mathcal{Z}=\{h_0,-h_0\}$, where $h_0\neq 0$.*

Let $E_1(\lambda)$ be the lowest eigenvalue of $-L+V_\lambda$. The first main result is as follows.

Theorem 3 ([3]). *Assume that* (A1) *and* (A2) *hold. Let* $E_1(\lambda) = \inf \sigma(-L + V_\lambda)$. *Then*

$$\lim_{\lambda \to \infty} E_1(\lambda) = \min_{1 \le i \le n} E_i,$$

where

$$E_i = \inf \sigma(-L + Q_i)$$

and Q_i *is given by*

$$Q_i(w) = \frac{1}{2} \int_{\mathbb{R}} : w(x)^2 : P''(h_i(x)) g(x) dx.$$

Remark 4. In the case of finite dimensional Schrödinger operators, there exist eigenvalues near the approximate eigenvalues E_i when λ is large. In Theorem 3, if $E_i < m + \min_{1 \le i \le n} E_i$, then the same results hold by the result of Hoegh-Krohn and Simon [34]. However, if it is not the case, it is not clear and they may be embedded eigenvalues in the essential spectrum. Under the assumptions in Theorem 6, $E_2(\lambda)$ is an eigenvalue for large λ. Simon [30] gave an example of $P(\phi)_2$-Hamiltonian for which an embedded eigenvalue exists.

Let

$$E_2(\lambda) = \inf \left\{ \sigma(-L + V_\lambda) \setminus \{E_1(\lambda)\} \right\}.$$

We can prove that $E_2(\lambda) - E_1(\lambda)$ is exponentially small when U is a symmetric double well potential function. The exponential decay rate is given by the Agmon distance which is defined below.

Definition 5. *Let* $0 < T < \infty$ *and* $h, k \in H^1(\mathbb{R})$. *Let* $AC_{T,h,k}(H^1(\mathbb{R}))$ *be the set of all absolutely continuous paths* $c : [0, T] \to H^1(\mathbb{R})$ *satisfying* $c(0) = h, c(T) = k$. *Let* U *be the potential function in* (2). *Assume* U *is non-negative. We define the Agmon distance between* h, k *by*

$$d_U^{Ag}(h, k) = \inf \left\{ \ell_U(c) \mid c \in AC_{T,h,k}(H^1(\mathbb{R})) \right\},$$

where

$$\ell_U(c) = \int_0^T \sqrt{U(c(t))} \|c'(t)\|_{L^2} dt.$$

The following estimate is the second main result.

Theorem 6 ([3]). *Assume that* U *satisfies* (A1),(A2),(A3). *Then it holds that*

$$\limsup_{\lambda \to \infty} \frac{\log (E_2(\lambda) - E_1(\lambda))}{\lambda} \le -d_U^{Ag}(h_0, -h_0).$$

Remark 7. (1) Agmon distance can be extended to a continuous distance function on $H^{1/2}(\mathbb{R})$. Moreover the topology defined by the Agmon distance coincides with the one defined by the Sobolev norm of $H^{1/2}(\mathbb{R})$.
(2) We can prove the existence of minimal geodesic between h_0 and $-h_0$ with respect to the Agmon metric. The uniqueness of the geodesics is not clear at the moment.
(3) The Agmon distance $d_U^{Ag}(h_0, -h_0)$ is equal to an Euclidean action integral of an instanton solution. This is an infinite dimensional example corresponding to the result of instanton in the case of Schrödinger operator which is due to Carmona and Simon [6].

The following is an example for which our main theorem is applicable.

Example 8. *Fix $g \in C_0^\infty(\mathbb{R})$. Let $n \in \mathbb{N}$. For sufficiently large $a > 0$, the polynomial*

$$P(x) = a(x^2 - 1)^{2n} - C$$

satisfies (A1), (A2), (A3). *Here C is a positive constant which depends on a, g.*

The same theorems are valid in the case where the space is a finite interval $I = [-l/2, l/2]$ as in the setting in [2]. In that framework, we show a simple example for which the Agmon distance and instanton can be calculated. Let a and x_0 be positive numbers. We consider the case where

$$U(h) = \frac{1}{4}\int_I h'(x)^2 dx + a\int_I \left(h(x)^2 - x_0^2\right)^2 dx.$$

For example, setting $b^2 = x_0^2 + \frac{m^2}{8a}$ and

$$P(x) = a(x^2 - b^2)^2 - a\left\{b^4 - \left(b^2 - \frac{m^2}{8a}\right)^2\right\},$$

we obtain the potential function above. Note $\mathcal{Z} = \{h_0, -h_0\}$, where $h_0(x) \equiv x_0$ is a constant function. $\pm x_0$ are the zero points also of the potential function

$$Q(x) = a(x^2 - x_0^2)^2 \quad x \in \mathbb{R}.$$

Let

$$d_{1dim}^{Ag}(-x_0, x_0) = \inf\left\{\int_{-T}^{T} \sqrt{Q(x(t))}|x'(t)|dt \;\middle|\; x(-T) = -x_0,\ x(T) = x_0\right\}.$$

This is the Agmon distance which corresponds to 1-dimensional Schrödinger operator $-\frac{d^2}{dx^2} + Q(x)$ defined in $L^2(\mathbb{R}, dx)$ and

$$d_{1dim}^{Ag}(-x_0, x_0) = \int_{-x_0}^{x_0} \sqrt{Q(x)}dx = \frac{4\sqrt{a}x_0^3}{3}.$$

We can prove the following.

Proposition 9. *Assume $2ax_0^2 l^2 \le \pi^2$. Let $u_0(t) = x_0 \tanh\left(2\sqrt{a}x_0 t\right)$. Then $u_0(t)$ is a solution to*

$$u''(t) = 2Q'(u(t)) \qquad -\infty < t < \infty,$$
$$\lim_{t\to-\infty} u(t) = -x_0, \qquad \lim_{t\to\infty} u(t) = x_0$$

and

$$\begin{aligned} I_{\infty,P}(u_0) &= \left(\frac{1}{4}\int_{-\infty}^{\infty} u_0'(t)^2 dt + \int_{-\infty}^{\infty} Q(u_0(t))dt\right) l, \\ &= d_{1dim}^{Ag}(-x_0, x_0) l \\ &= d_U^{Ag}(-h_0, h_0). \end{aligned}$$

The Proposition above claims that u_0 is the instanton for both operators: 1-dimensional Schrödinger operator $-\dfrac{d^2}{dx^2} + \lambda Q(\cdot/\sqrt{\lambda})$ and $-L + V_\lambda$.

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