数理解析研究所講究録
第 1856 巻 2013 年 56-74

Weak solvability for abstract nonlinear Schrödinger equations

東京理科大学大学院・理学研究科 D3 鈴木敏行 (Toshiyuki Suzuki)
Department of Mathematics, Tokyo University of Science

1. Introduction

In this paper we consider the following Cauchy problem for nonlinear Schrödinger equations with inverse-square potential:

$$\textbf{(CP)}_a \qquad \begin{cases} i\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + f(u) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases}$$

where $i=\sqrt{-1}$, $a>-(N-2)^2/4$. The feature for $\textbf{(CP)}_a$ is the presence of a strongly singular potential $a|x|^{-2}$; note that $-\Delta$ and $a|x|^{-2}$ are the same scale symmetry:

$$(-\Delta)[u(\lambda x)] = \lambda^2(-\Delta u)(\lambda x), \quad (|x|^{-2})[u(\lambda x)] = \lambda^2(|\cdot|^{-2}u)(\lambda x), \quad \forall\,\lambda>0.$$

This implies that the so-called scaling argument can not be applied to

$$P_a := -\Delta + \frac{a}{|x|^2}.$$

In other words, $\textbf{(CP)}_a$ can not be reduced to the case with $|a|$ and $\|u_0\|_{H^1}$ small enough.

On the other hand, from the viewpoint of operator theory, $P_a = -\Delta + a|x|^{-2}$ is nonnegative and selfadjoint in $L^2(\mathbb{R}^N)$ (as form-sum) if $N\ge 3$ and $a\ge -(N-2)^2/4$. Moreover, P_a is nonnegative and selfadjoint in $H^{-1}(\mathbb{R}^N)$ with domain $D(P_a)=H^1(\mathbb{R}^N)$ if $N\ge 3$ and $a>-(N-2)^2/4$. These are consequences of the **Hardy inequality**

$$\frac{N-2}{2}\Big\|\frac{u}{|x|}\Big\|_{L^2} \le \|\nabla u\|_{L^2} \quad \forall\,u\in H^1(\mathbb{R}^N),\ N\ge 3.$$

If $f(u)\equiv 0$ and $u_0\in H^1(\mathbb{R}^N)$, then $e^{-itP_a}u_0$ is a unique solution to $\textbf{(CP)}_a$ in $H^{-1}(\mathbb{R}^N)$.

Now we consider $f(u)\not\equiv 0$. If $V\in L^\infty(\mathbb{R}^N)+L^p(\mathbb{R}^N)$ for some $p\ge N/2$, then $V(x)u+f(u)$ can be regarded as a nonlinear term. For example, let $V(x):=|x|^{-\alpha}$ ($\alpha>0$). Then $V\in L^\infty(\mathbb{R}^N)+L^p(\mathbb{R}^N)$ for all $p<N/\alpha$. In paticular, if $0<\alpha<2$, then $|x|^{-\alpha}\in L^\infty(\mathbb{R}^N)+L^{N/2}(\mathbb{R}^N)$. However, $V(x)=|x|^{-2}\notin L^\infty(\mathbb{R}^N)+L^p(\mathbb{R}^N)$ ($\alpha=2$) for any $p\ge N/2$. Hence we can not regard the term $a|x|^{-2}u$ as a part of the nonlinear term.

The above consideration suggests that we may apply the preceding methods to $\textbf{(CP)}_a$ by replacing $-\Delta$ with P_a. However, there exist a lot of difficulties for solving $\textbf{(CP)}_a$ by the preceding methods: Ginibre-Velo's [**6**], Kato's [**8**], Cazenave-Weissler's [**5**] and Cazenave's [**3, 4**].

(i) There is no work for the dispersive estimates for e^{-itP_a}:

$$\|e^{-itP_a}\varphi\|_{L^q} \le C|t|^{(N/q)-(N/2)}\|\varphi\|_{L^{q'}} \quad \forall\,t\in\mathbb{R},\ 2\le p\le\infty,$$

where $P_a := -\Delta + a|x|^{-2}$. Hence we can not apply Ginibre-Velo's method [**6, 7**] because L^q-$L^{q'}$ type estimates is essentially used;

(ii) We can apply Kato's method [**8**] since the Strichartz estimates are available in [**1**]:

$$\|e^{-itP_a}u_0\|_{L^\tau(\mathbb{R};L^\rho(\mathbb{R}^N))} \le C\,\|u_0\|_{L^2(\mathbb{R}^N)}, \quad \frac{2}{\tau} + \frac{N}{\rho} = \frac{N}{2}, \ \tau,\ \rho \ge 2.$$

Applying Kato's method, Okazawa, Suzuki and Yokota [**10**] showed the following fact: define $f(u) := \lambda|u|^{p-1}u$, where λ and p satisfies $1 \le p < (N+2)/(N-2)$ ($\lambda > 0$) or $1 \le p < 1 + 4/N$ ($\lambda < 0$). Assume

$$a > \Big[\frac{N(p-1)}{2(p+1)}\Big]^2 - \frac{(N-2)^2}{4}. \tag{1.1}$$

Then for every $u_0 \in H^1(\mathbb{R}^N)$ there exists a unique global weak solution u to $\mathbf{(CP)}_a$. To establish this fact we evaluate $\|\nabla u\|_{L^\tau(I;L^\rho(\mathbb{R}^N))}$. Since ∇ and e^{-itP_a} are not commutative, we use the following Strichartz type estimates:

$$\|\nabla e^{-itP_a}u_0\|_{L^\tau(\mathbb{R};L^\rho(\mathbb{R}^N))} \le C\,\|\nabla u_0\|_{L^2(\mathbb{R}^N)} \quad [a + (N-2)^2/4]^{1/2} > 2/\tau.$$

To construct local weak solutions we choose

$$(\tau, \rho) = (\infty, 2) \text{ and } (\tau, \rho) = \Big(\frac{4(p+1)}{N(p-1)}, p+1\Big).$$

The latter pair applies to give the unsatisfactory restriction (1.1) on a.

(iii) Cazenave-Weissler [**5**] and Cazenave [**4**, Chapter 3] developed other methods. But those are not applicable to the critical case (for example, $f(u) := (W * |u|^2)u$ for $W \in L^{N/4}(\mathbb{R}^N)$); this critical case can be dealt with Ginibre-Velo [**7**] when $a = 0$.

(iv) Cazenave's method [**4**, Chapter 3] is useful because solvability of $\mathbf{(CP)}_a$ with $a = 0$ is verified without either the dispersive estimates or the Strichartz estimates. But his method uses the m-accretivity of $-\Delta$ in $L^q(\mathbb{R}^N)$. Here $P_a = -\Delta + a|x|^{-2}$ does not seem to be m-accretive in $L^q(\mathbb{R}^N)$ if a is near to $-(N-2)^2/4$. More precisely, Okazawa [**9**] proved the m-accretivity of P_a in $L^q(\mathbb{R}^N)$ with

$$a > \begin{cases} \dfrac{(q-1)(2q-N)N}{q^2}, & q \in \Big[\dfrac{2(N-1)}{N}, \infty\Big), \\[2ex] -\dfrac{(q-1)(N-2)^2}{q^2}, & q \in \Big[1, \dfrac{2(N-1)}{N}\Big]. \end{cases}$$

The lower bounds of a is greater than $-(N-2)^2/4$ if $q \ne 2$.

Thus we need another new approach to solve $\mathbf{(CP)}_a$. In Section 2 we introduce energy methods for abstract nonlinear Schrödinger equations. Application to $\mathbf{(CP)}_a$ with power type nonlinearity is stated in Section 3. Application to $\mathbf{(CP)}_a$ with nonlocal nonlinearity (Hartree type equations) is given in Section 4. Section 5 is devoted to the proof of the solvability of Hartree type equations in Section 4. Finally, some remarks are in order in Section 6.

2. Abstract theory for nonlinear Schrödinger equations

Let S be a nonnegative selfadjoint operator in a complex Hilbert space X. Put $X_S := D(S^{1/2})$. Then we have the usual triplet: $X_S \subset X = X^* \subset X_S^*$. Under this setting S can be extended to a nonnegative selfadjoint operator in X_S^* with domain X_S. Now we consider

(ACP)
$$\begin{cases} i\dfrac{du}{dt} = Su + g(u), \\ u(0) = u_0, \end{cases}$$

where $g : X_S \to X_S^*$ is a nonlinear operator satisfying

(G1) Existence of energy functional: there exists $G \in C^1(X_S;\mathbb{R})$ such that $G' = g$, that is, given $u \in X_S$, for every $\varepsilon > 0$ there exists $\delta = \delta(u,\varepsilon) > 0$ such that

$$|G(u+v) - G(u) - \mathrm{Re}\,\langle g(u), v\rangle_{X_S^*,X_S}| \le \varepsilon\,\|v\|_{X_S} \quad \forall\, v \in X_S \text{ with } \|v\|_{X_S} < \delta;$$

(G2) Local Lipschitz continuity: for all $M > 0$ there exists $C(M) > 0$ such that

$$\|g(u) - g(v)\|_{X_S^*} \le C(M)\|u - v\|_{X_S} \quad \forall\, u,\, v \in X_S \text{ with } \|u\|_{X_S},\, \|v\|_{X_S} \le M;$$

(G3) Hölder-like continuity of energy functional: given $M > 0$, for all $\delta > 0$ there exists a constant $C_\delta(M) > 0$ such that

$$|G(u) - G(v)| \le \delta + C_\delta(M)\|u - v\|_X \quad \forall\, u,\, v \in X_S \text{ with } \|u\|_{X_S},\, \|v\|_{X_S} \le M;$$

(G4) Gauge type condition for the conservation of charge:

$$\mathrm{Im}\,\langle g(u), u\rangle_{X_S^*,X_S} = 0 \quad \forall\, u \in X_S;$$

(G5) Closedness type condition: given a bounded open interval $I \subset \mathbb{R}$, let $\{w_n\}_n$ be any bounded sequence in $L^\infty(I;X_S)$ such that

$$\text{(2.1)} \qquad \begin{cases} w_n(t) \to w(t)\ (n \to \infty) & \text{weakly in } X_S \text{ a.a. } t \in I, \\ g(w_n) \to f\ (n \to \infty) & \text{weakly}^* \text{ in } L^\infty(I;X_S^*). \end{cases}$$

Then

$$\text{(2.2)} \qquad \mathrm{Im}\int_I \langle f(t), w(t)\rangle_{X_S^*,X_S}\,dt = \lim_{n\to\infty} \mathrm{Im}\int_I \langle g(w_n(t)), w_n(t)\rangle_{X_S^*,X_S}\,dt.$$

Here $f = g(w)$ is guaranteed if

$$\text{(2.3)} \qquad w_n(t) \to w(t)\ (n \to \infty) \text{ strongly in } X \text{ a.a. } t \in I;$$

(G6) Lower boundedness of the energy: there exist $\varepsilon \in (0,1]$ and $C_0(\cdot) \ge 0$ such that

$$G(u) \ge -\frac{1-\varepsilon}{2}\|u\|_{X_S}^2 - C_0(\|u\|_X) \quad \forall\, u \in X_S.$$

Here a function u is said to be a **local weak solution** on I to **(ACP)** if u belongs to $L^\infty(I;X_S) \cap W^{1,\infty}(I;X_S^*)$ and satisfies **(ACP)** in $L^\infty(I;X_S^*)$. If I coincides with $\mathbb{R}$, then the local weak solution is called a **global weak solution**.

Theorem 2.1 (Local existence, [**11**]). *Assume that* $g : X_S \to X_S^*$ *satisfies* **(G1)**–**(G5)**. *Then for every* $u_0 \in X_S$ *with* $\|u_0\|_{X_S} \le M$ *there exist* $T_M > 0$ *and a local weak solution on* $(-T_M, T_M)$. *Moreover*

$$\|u(t)\|_X = \|u_0\|_X, \quad E(u(t)) \le E(u_0) \quad \forall\, t \in [-T_M, T_M],$$

where $E(\cdot)$ *is the energy given by* $E(\varphi) := (1/2)\|S^{1/2}\varphi\|_X^2 + G(\varphi)$, $\varphi \in X_S$.

Proof of Theorem 2.1 is based on Cazenave's method. But we avoid to apply L^p theory to the nonnegative selfadjoint operator S. Now we give a sketch of the proof of Theorem 2.1.

Proof of Theorem 2.1 (Step 1). We consider the approximate problems in X:

$$(\mathbf{ACP})_\varepsilon \qquad \begin{cases} i\dfrac{du_\varepsilon}{dt} = Su_\varepsilon + g_\varepsilon(u_\varepsilon), \\ u_\varepsilon(0) = u_0, \end{cases}$$

where g_ε is the approximation of g defined as

$$g_\varepsilon(u) := (1+\varepsilon S)^{-1/2} g((1+\varepsilon S)^{-1/2} u)$$

Note that g_ε is locally Lipschitz continuous on X.

By virtue of **(G1)**, **(G2)** and **(G4)**, $(\mathbf{ACP})_\varepsilon$ admits a unique global weak solution u_ε belonging to $C(\mathbb{R}; X_S) \cap C^1(\mathbb{R}; X_S^*)$. More precisely, we apply the following theorem to $g_0(u) := g_\varepsilon(u)$ and $G_0(u) := G((1+\varepsilon S)^{-1/2}u)$.

Proposition 2.2 ([4, Theorem 3.3.1]). *Let* $S : X_S \subset X_S^* \to X_S^*$ *be a nonnegative selfadjoint operator. Assume that* $g_0 : X \to X$ *satisfies*

(i) *there exists* $G_0 \in C^1(X_S; \mathbb{R})$ *such that* $G_0' = g_0$, *that is, given* $u \in X_S$, *for every* $\varepsilon > 0$ *there exists* $\delta = \delta(u, \varepsilon) > 0$ *such that*

$$|G_0(u+v) - G_0(u) - \mathrm{Re}\,\langle g_0(u), v\rangle_X| \le \varepsilon\, \|v\|_{X_S} \quad \forall\, v \in X_S \text{ with } \|v\|_{X_S} < \delta;$$

(ii) g_0 *is locally Lipschitz continuous on* X:

$$\|g_0(x) - g_0(y)\|_X \le L(M)\|x-y\|_X \quad \forall\, x,\, y \in X \text{ with } \|x\|_X,\ \|y\|_X \le M;$$

(iii) $\mathrm{Re}\,\langle g_0(u), iu\rangle_X = 0\ \forall\, u \in X$.

Then for every $x \in X_S$ *there exists a unique solution* $u \in C(\mathbb{R}; X_S) \cap C^1(\mathbb{R}; X_S^*)$ *to*

$$\begin{cases} i\dfrac{du}{dt} = Su + g_0(u) \quad \text{in } \mathbb{R}, \\ u(0) = x, \end{cases}$$

Moreover conservation laws hold:

$$\|u(t)\|_X = \|x\|_X, \quad E_0(u(t)) = E_0(x) \quad \forall\, t \in \mathbb{R},$$

where E_0 *is the energy defined as* $E_0(y) := (1/2)\|S^{1/2}y\|_X^2 + G_0(y)$, $y \in X_S$.

Proof of Theorem 2.1 (Step 2). We prove the uniform boundedness with respect to $\varepsilon > 0$:

$$\|u_\varepsilon(t)\|_{X_S} \le M' \quad \forall\, t \in [-T_M, T_M].$$

This fact is proved in a way similar to Cazenave's method. See [**11**, Lemma 4.2] (or [**4**, Theorem 3.3.5]).

Proof of Theorem 2.1 (Step 3). Since $\|u_\varepsilon(t)\|_{X_S}$ and $\|u_\varepsilon'(t)\|_{X_S^*}$ are uniformly bounded in $[-T_M, T_M]$, the Ascoli type lemma (see [**4**, Proposition 1.1.2]) yields that there exist a sequence $\{u_{\varepsilon_k}\}_k \subset \{u_\varepsilon\}_\varepsilon$ and a function $u \in C_{\mathrm{w}}([-T_M, T_M]; X_S) \cap W^{1,\infty}(-T_M, T_M; X_S^*)$ such that

$$u_{\varepsilon_k}(t) \to u(t) \ (k \to \infty) \text{ weakly in } X_S \quad \forall\, t \in [-T_M, T_M]. \tag{2.4}$$

Hence it follows from $u_{\varepsilon_k}' \to u'$ $(k \to \infty)$ weakly* in $L^\infty(-T_M, T_M; X_S^*)$ that u satisfies

$$\textbf{(ACP)}' \qquad \begin{cases} i\dfrac{du}{dt} = Su + f & \text{in } L^\infty(-T_M, T_M; X_S^*), \\ u(0) = u_0. \end{cases}$$

It suffices to show that $f = g(u)$. To end this, we suppose weak closedness condition **(G5)**. In fact, since (2.4) and

$$g((1+\varepsilon_k S)^{-1/2} u_{\varepsilon_k}) \to f \ (k \to \infty) \quad \text{weakly* in } L^\infty(-T_M, T_M; X_S^*),$$

(G5) and **(G4)** imply that

$$\frac{1}{2}\|u(t)\|_X^2 - \frac{1}{2}\|u_0\|_X^2 = \mathrm{Im} \int_0^t \langle f(s), u(s)\rangle_{X_S^*, X_S}\, ds = 0$$

This is nothing but the conservation law of charge. (2.4) and $\|u_{\varepsilon_k}(t)\|_X = \|u_0\|_X = \|u(t)\|_X$ yield that $u_{\varepsilon_k}(t) \to u(t)$ $(k \to \infty)$ strongly in X. Hence we see from **(G5)** that $f = g(u)$.

Therefore we have proved Theorem 2.1.

Here we compare Cazenave's method with ours (see Figure 1). First we can make more moderate approximation. Cazenave used the m-accretivity of $-\Delta$ in $L^p(\mathbb{R}^N)$ and that the resolvent $(1-\varepsilon\Delta)^{-1}$ maps from $L^2(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$ $(2 \le r < 2N/(N-2))$ and from $L^{\rho'}(\mathbb{R}^N)$ $(2 \le \rho < 2N/(N-2))$ to $L^2(\mathbb{R}^N)$. Here $r = 2N/(N-2) = \rho$ is excluded because he applied Rellich's compactness theorem to verifying $f = g(u)$ in Step 3. We do not need to apply Rellich's compactness theorem by virtue of **(G5)**.

Next we introduce the global existence of weak solutions to **(ACP)**. Note that we need the uniqueness of local weak solution to **(ACP)**.

Theorem 2.3 (Global existence, [**11**, Theorem 2.4]). *Assume that $g : X_S \to X_S^*$ satisfies* **(G1)**–**(G6)** *and the uniqueness of local weak solutions to* **(ACP)**. *Then for every $u_0 \in X_S$ there exists a global weak solution $u \in C(\mathbb{R}; X_S) \cap C^1(\mathbb{R}; X_S^*)$ to* **(ACP)** *and the conservation laws hold:*

$$\|u(t)\|_X = \|u_0\|_X, \quad E(u(t)) = E(u_0), \quad t \in \mathbb{R}.$$

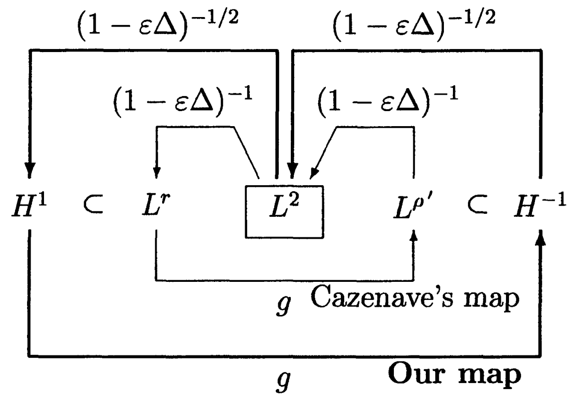

Figure 1: Comparison of Cazenave's composite mapping and ours

3. Solvability for nonlinear Schrödinger equations of power type

We can apply Theorem 2.3 to $(\mathbf{CP})_a$ with power type nonlinearity. Assume that $f : \mathbb{C} \to \mathbb{C}$ satisfies

(N1) $f(0) = 0$;

(N2) There exist $p \in [1, (N+2)/(N-2))$ and $K \geq 0$ such that

(3.1) $$|f(u) - f(v)| \leq K(1 + |u|^{p-1} + |v|^{p-1})|u - v| \quad \forall\, u,\, v \in \mathbb{C};$$

(N3) $f(x) \in \mathbb{R}$ $(x > 0)$ and $f(e^{i\theta}z) = e^{i\theta}f(z)$ $(z \in \mathbb{C},\ \theta \in \mathbb{R})$;

(N4) There exist $q \in [1, 1 + 4/N)$ and $L_1, L_2 \geq 0$ such that

(3.2) $$F(x) := \int_0^x f(s)\, ds \geq -L_1 x^2 - L_2 x^{q+1} \quad \forall\, x > 0.$$

The conditions **(N1)**–**(N4)** are nothing but what was imposed by Ginibre-Velo [**6**] and Kato [**8**]. Typical example of **(N1)**–**(N4)** is $f(u) := \lambda |u|^{p-1}u$ with

(i) $\lambda > 0$ and $1 \leq p < (N+2)/(N-2)$;

(ii) $\lambda < 0$ and $1 \leq p < 1 + 4/N$.

Applying Theorem 2.3, we obtain

Theorem 3.1 ([**11**, Theorem 5.1]). *Let $N \geq 3$, $a > -(N-2)^2/4$. Assume $f : \mathbb{C} \to \mathbb{C}$ satisfies* **(N1)**–**(N4)**. *Then for all $u_0 \in H^1(\mathbb{R}^N)$ there exists a unique global weak solution to* $(\mathbf{CP})_a$. *Moreover, u belongs to $C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ and satisfies conservation laws*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0) \quad \forall\, t \in \mathbb{R},$$

where the "energy" is defined as

$$E(\varphi) := \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{a}{2}\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 + \int_{\mathbb{R}^N}\int_0^{|\varphi(x)|} f(s)\, ds.$$

This is an improvement of [**10**, Theorem 1.2].

4. Solvability for Hartree type equations

Next we consider the following problem:

$$\textbf{(HE)}_a \qquad \begin{cases} i\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + K(|u|^2)u & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases}$$

where K is an integral operator:

$$K(f)(x) = Kf(x) := \int_{\mathbb{R}^N} k(x,y)f(y)\,dy. \tag{4.1}$$

The feature for $\textbf{(HE)}_a$ is the nonlocal nonlinearities $K(|u|^2)u$. Let $a=0$ and $k(x,y) = W(x-y)$. Then $\textbf{(HE)}_a$ is the usual Hartree equation (see [**7**]).

We consider the kernel k of the integral operator K [defined by (4.1)].

Definition 4.1. $L^\beta_x(L^\alpha_y) = L^\beta_x(\mathbb{R}^N; L^\alpha_y(\mathbb{R}^N))$ is the family of $k : \mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ such that

$$\|k\|_{L^\beta_x(L^\alpha_y)} := \Big(\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}|k(x,y)|^\alpha\,dy\Big)^{\beta/\alpha}dx\Big)^{1/\beta} < \infty. \tag{4.2}$$

Now we assume that the kernel k satisfies the following three conditions:

(K1) k is a symmetric real-valued function, that is, $k(x,y) = k(y,x) \in \mathbb{R}$ a.a. x, $y \in \mathbb{R}^N$;
(K2) $k \in L^\infty_y(L^\infty_x) + L^\beta_y(L^\alpha_x)$ and $k - k_R \to 0$ in $L^\beta_y(L^\alpha_x)$ for some α, $\beta \in [1,\infty]$ such that $\alpha \le \beta$, $\alpha^{-1}+\beta^{-1} \le 4/N$;
(K3) $k_- := -\min\{k, 0\} \in L^\infty_y(L^\infty_x) + L^{\tilde\beta}_y(L^{\tilde\alpha}_x)$ and $k_- - (k_-)_R \to 0$ in $L^{\tilde\beta}_y(L^{\tilde\alpha}_x)$ for some $\tilde\alpha$, $\tilde\beta \in [1,\infty]$ such that $\tilde\alpha \le \tilde\beta$, $\tilde\alpha^{-1}+\tilde\beta^{-1} \le 2/N$.

Here k_R is defined as

$$k_R(x,y) := \begin{cases} k(x,y) & |k(x,y)| \le R, \\ R & k(x,y) > R, \\ -R & k(x,y) < -R. \end{cases} \tag{4.3}$$

For example, let $W \in L^p(\mathbb{R}^N)$. Then $k(x,y) := W(x-y)$ belongs to $L^\infty_x(L^p_y)$ and satisfies $\|k\|_{L^\infty_x(L^p_y)} = \|W\|_{L^p}$.

Theorem 4.1 ([**13**, Theorem 1.3]). *Let $N \ge 3$ and $a > -(N-2)^2/4$. Assume that k satisfies* **(K1)**–**(K3)**. *Then for every $u_0 \in H^1(\mathbb{R}^N)$ there exists a unique global weak solution u to* $\textbf{(HE)}_a$. *Moreover, u belongs to $C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ and satisfies conservation laws*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0) \quad \forall\, t \in \mathbb{R}, \tag{4.4}$$

where the "energy" is defined as

$$E(\varphi) := \frac{1}{2}\|\nabla\varphi\|^2_{L^2} + \frac{a}{2}\Big\|\frac{\varphi}{|x|}\Big\|^2_{L^2} + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)|\varphi(x)|^2|\varphi(y)|^2\,dxdy.$$

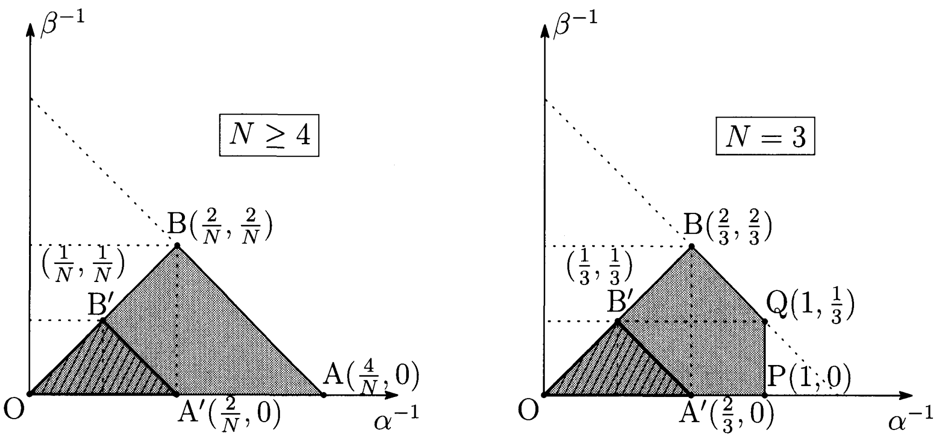

Figure 2: Admissible exponents for **(K2)** and **(K3)**

Now it is possible to take $k(x,y) = W(x-y)$ in the definition of integral operator (4.1) as in the usual Hartree equations. In this context let $W \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfy the following three conditions:

(W1) W is a real-valued even function, that is, $W(-x) = W(x) \in \mathbb{R}$ a.a. $x \in \mathbb{R}^N$;

(W2) There exists $p \geq \max\{1, N/4\}$ such that $W \in L^\infty(\mathbb{R}^N) + L^p(\mathbb{R}^N)$;

(W3) There exists $q \geq N/2$ such that $W_- := -\min\{W, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$.

We can show that $k(x,y) = W(x-y)$ belongs to $L^\infty_x(L^\infty_y) + L^\infty_x(L_y^{1\vee(N/4)})$. Thus the admissible pair in the convolution case is located on the edge OA ($N \geq 4$) or OP ($N = 3$) as in Figure 2. Hence we have the following:

Corollary 4.2. *Let $N \geq 3$ and $a > -(N-2)^2/4$. Assume that W satisfies* **(W1)–(W3)**. *Then for every $u_0 \in H^1(\mathbb{R}^N)$ there exists a unique global weak solution u to*

$$\text{(4.5)} \qquad \begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta u + \dfrac{a}{|x|^2}u + (W * |u|^2)u & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover, u satisfies conservation laws (4.4) *with*

$$\text{(4.6)} \qquad \begin{aligned} E(\varphi) :=& \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{a}{2}\left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 \\ &+ \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)|\varphi(x)|^2|\varphi(y)|^2\,dxdy, \quad \varphi \in H^1(\mathbb{R}^N). \end{aligned}$$

Proof of Corollary 4.2. By virtue of Theorem 4.1, it suffices to show that $k(x,y) := W(x-y)$ satisfies **(K1)–(K3)**. Here **(K1)** follows from **(W1)**. Next we show **(K2)**

and **(K3)**, that is, k belongs to $L^\infty_x(L^\infty_y) + L^\infty_x(L_y^{1\vee(N/4)})$ and k_- belongs to $L^\infty_x(L^\infty_y) + L^\infty_x(L_y^{N/2})$. For $R > 0$ set

$$W_R(x) := \begin{cases} W(x) & (|W(x)| \le R), \\ R & (W(x) > R), \\ -R & (W(x) < -R). \end{cases}$$

Then we have $|W_R(x)| \le R$ on $\mathbb{R}^N$ so that

$$|W(x)| \le |W(x) - W_R(x)| + |W_R(x)| \le |W(x) - W_R(x)| + R. \tag{4.7}$$

By **(W2)** we have $W - W_R \in L^{q(N)}(\mathbb{R}^N)$, where $q(N) := (N/4) \vee 1$. Hence

$$\begin{aligned} \int_{\mathbb{R}^N} |W(x-y) - W_R(x-y)|^{q(N)}\, dx &= \int_{\mathbb{R}^N} |W(x) - W_R(x)|^{q(N)}\, dx \\ &= \|W - W_R\|^{q(N)}_{L^{q(N)}} \quad \forall\, y \in \mathbb{R}^N. \end{aligned}$$

Thus we obtain

$$\|k - k_R\|_{L^\infty_x(L_y^{q(N)})} = \|W - W_R\|_{L^{q(N)}} \to 0 \quad (R \to \infty).$$

Therefore $k(x,y) = W(x-y)$ belongs to $L^\infty_x(L^\infty_y) + L^\infty_x(L_y^{q(N)})$ and satisfies **(K2)**.

Since $k_-(x,y) = W_-(x-y)$, we conclude that k_- belongs to $L^\infty_x(L^\infty_y) + L^\infty_x(L_y^{N/2})$ and satisfies **(K3)** in a way similar to **(K2)**. ∎

5. Proof of Theorem 4.1

In **[13]** the proof of Theorem 4.1 is mostly omitted. Thus we fully give the proof of Theorem 4.1 in this section. To show Theorem 4.1 we verify **(G1)**–**(G6)** with

$$g(u)(x) := (uK(|u|^2))(x) = u(x) \int_{\mathbb{R}^N} k(x,y)|u(y)|^2\, dy, \quad u \in H^1(\mathbb{R}^N), \tag{5.1}$$

$$G(u) := \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k(x,y)|u(x)|^2|u(y)|^2\, dxdy, \quad u \in H^1(\mathbb{R}^N) \tag{5.2}$$

and the uniqueness of local weak solutions to $\mathbf{(HE)}_a$.

First we show the uniqueness of local weak solutions to $\mathbf{(HE)}_a$. To prove it, we use the Strichartz estimates for $\{e^{-itP_a}\}$ established by Burq, Planchon, Stalker and Tahvildar-Zadeh **[1]** (see also **[10**, Theorems 2.3 and 2.5]):

Lemma 5.1. *Let $N \ge 3$ and (p,q) be a Schrödinger admissible pair, i.e.,*

$$\frac{2}{p} + \frac{N}{q} = \frac{N}{2}, \quad p, q \ge 2.$$

Then the following inequality holds:

$$\|e^{-itP_a}\varphi\|_{L^p(\mathbb{R};L^q)} \le C\,\|\varphi\|_{L^2} \quad \forall\, \varphi \in L^2(\mathbb{R}^N). \tag{5.3}$$

Moreover let (p_j, q_j) $(j = 1,\ 2)$ *be Schrödinger admissible pairs. Then*

$$\Big\| \int_0^t e^{-i(t-s)P_a}\Phi(s,x)\,ds \Big\|_{L^{p_2}(\mathbb{R};L^{q_2})} \le C'\,\|\Phi\|_{L^{p_1'}(\mathbb{R};L^{q_1'})} \quad \forall\, \Phi \in L^{p_1'}(\mathbb{R};L^{q_1'}(\mathbb{R}^N)). \tag{5.4}$$

In fact the endpoint case $p_1 = 2 = p_2$ in (5.4) has been restricted in [**10**, Theorems 2.5]. But we can remove the restriction by Pierfelice [**12**, Theorem 2 in Section 3] (see also [**2**, Theorem 3]).

Lemma 5.2. *Let* u_j $(j = 1,\ 2)$ *be local weak solutions to* $\mathbf{(HE)}_a$ *on* $(-T,T)$ *with initial values* $u_j(0) = u_{j,0}$. *Then for* $t \in (-T,T)$

$$\|u_1(t) - u_2(t)\|_{L^2} \le C\,\|u_{1,0} - u_{2,0}\|_{L^2}, \tag{5.5}$$

where C *is a constant depending on* $\|u_j\|_{L^\infty(-T,T;L^2)}$ *and* $\|u_j\|_{L^\infty(-T,T;L^{2\gamma})}$ $(j = 1,\ 2)$.

Proof. Let $u_j \in L^\infty(I;H^1(\mathbb{R}^N))$ $(j = 1,\ 2)$ be local weak solutions to $\mathbf{(HE)}_a$ on $(-T,T)$ with initial values $u_j(0) = u_{j,0}$. Then u_j $(j = 1,\ 2)$ satisfy the following integral equations:

$$u_j(t) = e^{-itP_a}u_{j,0} - i\int_0^t e^{-i(t-s)P_a}g(u_j(s))\,ds.$$

Therefore we see that $v(t) := u_1(t) - u_2(t)$ satisfies

$$v(t) = e^{-itP_a}[u_{1,0} - u_{2,0}] - i\int_0^t e^{-i(t-s)P_a}[g(u_1(s)) - g(u_2(s))]\,ds.$$

Now let $(r(\gamma), 2\gamma)$ be a Schrödinger admissible pair:

$$\frac{2}{r(\gamma)} + \frac{N}{2\gamma} = \frac{N}{2}, \quad \text{i.e.,}\ \ r(\gamma) := \frac{4\gamma}{N(\gamma-1)}.$$

Applying (5.23), (5.24) and the Strichartz estimates (5.3), (5.4), we see that for every Schrödinger admissible pair (τ, ρ),

$$\|e^{-itP_a}[u_{1,0} - u_{2,0}]\|_{L^\tau(-T,T;L^\rho)} \le C_\tau\|u_{1,0} - u_{2,0}\|_{L^2}, \tag{5.6}$$

$$\begin{aligned} &\Big\| \int_0^t e^{-i(t-s)P_a}[g_1(u_1(s)) - g_1(u_2(s))]\,ds \Big\|_{L^\tau(-T,T;L^\rho)} \\ &\le C_{\infty,\tau}\|g_1(u_1) - g_1(u_2)\|_{L^1(-T,T;L^2)} \\ &\le 2C_{\infty,\tau}RT\,[\|u_1\|^2_{L^\infty_t L^2} + \|u_1\|_{L^\infty_t L^2}\|u_2\|_{L^\infty_t L^2} + \|u_2\|^2_{L^\infty_t L^2}]\|v\|_{L^\infty(-T,T;L^2)}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} &\Big\| \int_0^t e^{-i(t-s)P_a}[g_2(u_1(s)) - g_2(u_2(s))]\,ds \Big\|_{L^\tau(-T,T;L^\rho)} \\ &\le C_{r(\gamma),\tau}\|g_2(u_1) - g_2(u_2)\|_{L^{r(\gamma)'}(-T,T;L^{(2\gamma)'})} \\ &\le C_{r(\gamma),\tau}(2T)^{1-2/r(\gamma)}\,\|\ell_R\|_{B(\alpha,\beta)} \\ &\quad \times [\|u_1\|^2_{L^\infty_t L^{2\gamma}} + \|u_1\|_{L^\infty_t L^{2\gamma}}\|u_2\|_{L^\infty_t L^{2\gamma}} + \|u_2\|^2_{L^\infty_t L^{2\gamma}}]\|v\|_{L^{r(\gamma)}(-T,T;L^{2\gamma})}, \end{aligned} \tag{5.8}$$

where $\|\cdot\|_{L_t^\infty L^p} := \|\cdot\|_{L^\infty(-T,T;L^p)}$.

Putting $(\tau,\rho) := (\infty, 2)$ and $(\tau,\rho) := (r(\gamma), 2\gamma)$ in (5.6), (5.7) and (5.8), we see that

$$
\begin{aligned}
(5.9)\qquad & \|v\|_{L^{r(\gamma)}(-T,T;L^{2\gamma})} + \|v\|_{L^\infty(-T,T;L^2)} \\
&\le (C_{r(\gamma)} + C_\infty)\|u_{1,0} - u_{2,0}\|_{L^2} + 6(C_{\infty,\infty} + C_{\infty,r(\gamma)})RM^2T\,\|v\|_{L^\infty(-T,T;L^2)} \\
&+ 3(C_{r(\gamma),\infty} + C_{r(\gamma),r(\gamma)})\|\ell_R\|_{L_x^\beta(L_y^\alpha)}M^2(2T)^{1-2/r(\gamma)}\,\|v\|_{L^{r(\gamma)}(-T,T;L^{2\gamma})},
\end{aligned}
$$

where

$$M := \max_{j=1,2}\{\|u_j\|_{L^\infty(-T,T;L^2)} \vee \|u_j\|_{L^\infty(-T,T;L^{2\gamma})}\}.$$

Case 1 ($\alpha^{-1}+\beta^{-1} < 4/N$). Take $T_0 \in (0,T)$ such that $6(C_{\infty,\infty}+C_{\infty,r(\gamma)})RM^2T_0 \le 1/2$ and $3(C_{r(\gamma),\infty} + C_{r(\gamma),r(\gamma)})\|\ell_R\|_{L_x^\beta(L_y^\alpha)}M^2(2T_0)^{1-2/r(\gamma)} \le 1/2$. Then by (5.9) we obtain

$$(5.10)\qquad \|v\|_{L^{r(\gamma)}(-T_0,T_0;L^{2\gamma})} + \|v\|_{L^\infty(-T_0,T_0;L^2)} \le 2(C_{r(\gamma)} + C_\infty)\|u_{1,0} - u_{2,0}\|_{L^2}.$$

Case 2 ($\alpha^{-1}+\beta^{-1} = 4/N$). This is the critical case because of $2\gamma = 2N/(N-2)$. Then we see from (5.9) that

$$
\begin{aligned}
& \|v\|_{L^2(-T,T;L^{2N/(N-2)})} + \|v\|_{L^\infty(-T,T;L^2)} \\
&\le (C_2 + C_\infty)\|u_{1,0} - u_{2,0}\|_{L^2} + 6(C_{\infty,\infty} + C_{\infty,2})RM^2T\,\|v\|_{L^\infty(-T,T;L^2)} \\
&+ 3(C_{2,\infty} + C_{2,2})\|\ell_R\|_{L_x^\beta(L_y^\alpha)}M^2\,\|v\|_{L^2(-T,T;L^{2N/(N-2)})}.
\end{aligned}
$$

Fix $R > 0$ so that $3(C_{2,\infty} + C_{2,2})\|\ell_R\|_{L_x^\beta(L_y^\alpha)}M^2 \le 1/2$. Next take $T_0 \in (0,T)$ such that $6(C_{\infty,\infty} + C_{\infty,r(\gamma)})RM^2T_0 \le 1/2$. Then we have (5.10).

Extending the interval step by step, we conclude (5.5). ∎

Selecting $u_{0,1} = u_{0,2}$, we see that $u_1 = u_2$ in $L^\infty(-T,T;H^1(\mathbb{R}^N))$. Hence we conclude the uniqueness of local weak solutions to $(\mathbf{HE})_a$ on $(-T,T)$.

Next we verify **(G1)**–**(G6)**. To end this, we apply the following two lemmas.

Lemma 5.3 ([**13**, Lemma 2.4]). *Let* α, β, γ, $\rho \in [1,\infty]$. *Assume that* $k \in L_x^\beta(L_y^\alpha) \cap L_y^\beta(L_x^\alpha)$ *and*

$$\alpha \le \rho \le \beta, \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1 + \frac{1}{\rho}.$$

Then the operator defined by (4.1) *is linear and bounded from* $L^\gamma(\mathbb{R}^N)$ *to* $L^\rho(\mathbb{R}^N)$. *Moreover*

$$(5.11)\qquad \|Kf\|_{L^\rho(\mathbb{R}^N)} \le (\|k\|_{L_x^\beta(L_y^\alpha)} \vee \|k\|_{L_y^\beta(L_x^\alpha)})\|f\|_{L^\gamma(\mathbb{R}^N)} \quad \forall\, f \in L^\gamma(\mathbb{R}^N).$$

Lemma 5.4 ([**13**, Lemma 2.5]). *Let* α, $\beta \in [1,\infty]$ *be two exponents such that* $\alpha \le \beta$ *and* $\alpha^{-1}+\beta^{-1} \le 4/N$. *Put* $\gamma^{-1} := 1-(\alpha^{-1}+\beta^{-1})/2$. *Assume that* $k \in L_x^\beta(L_y^\alpha)$ *is symmetric. Then for all* $u_j \in H^1(\mathbb{R}^N)$ ($j = 1, 2, 3, 4$)

$$(5.12)\qquad \|u_1K(u_2\bar{u}_3)\|_{L^{(2\gamma)'}} \le \|k\|_{L_x^\beta(L_y^\alpha)}\|u_1\|_{L^{2\gamma}}\|u_2\|_{L^{2\gamma}}\|u_3\|_{L^{2\gamma}},$$

$$(5.13)\qquad \left|\int_{\mathbb{R}^N} u_1\bar{u}_2K(u_3\bar{u}_4)\,dx\right| \le \|k\|_{L_x^\beta(L_y^\alpha)}\|u_1\|_{L^{2\gamma}}\|u_2\|_{L^{2\gamma}}\|u_3\|_{L^{2\gamma}}\|u_4\|_{L^{2\gamma}}.$$

Now we start to verify **(G1)**–**(G6)**.

Verification of (G1). Let u, $v \in H^1(\mathbb{R}^N)$. Then we see from **(K1)** that

$$
\begin{aligned}
(5.14)\qquad & G(u+v) - G(u) - \mathrm{Re}\,\langle g(u), v\rangle_{H^{-1},H^1} \\
&= \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)[|(u+v)(x)|^2|(u+v)(y)|^2 - |u(x)|^2|u(y)|^2]\,dxdy \\
&- \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)[2\mathrm{Re}\,(u(x)\bar{v}(x))|u(y)|^2 + 2\mathrm{Re}\,(u(y)\bar{v}(y))|u(x)|^2]\,dxdy.
\end{aligned}
$$

Now let A, B, ξ, $\eta \in \mathbb{C}$. Then we see that

$$
\begin{aligned}
(5.15)\qquad & |A+\xi|^2|B+\eta|^2 - |A|^2|B|^2 - 2|B|^2\mathrm{Re}\,(A\bar{\xi}) - 2|A|^2\mathrm{Re}\,(B\bar{\eta}) \\
&= 4\mathrm{Re}\,(A\bar{\xi})\mathrm{Re}\,(B\bar{\eta}) + |\xi|^2(|B|^2 + 2\mathrm{Re}\,(B\bar{\eta})) + |\eta|^2(|A|^2 + 2\mathrm{Re}\,(A\bar{\xi})) + |\xi|^2|\eta|^2.
\end{aligned}
$$

Put $A := u(x)$, $B := u(y)$, $\xi = v(x)$, $\eta = v(y)$ in (5.15). It follows from (5.14) that

$$
(5.16)\qquad G(u+v) - G(u) - \mathrm{Re}\,\langle g(u), v\rangle_{H^{-1},H^1} = I_1 + I_2 + I_3,
$$

where

$$
\begin{aligned}
I_1 &:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)\mathrm{Re}\,(u(x)\bar{v}(x))\mathrm{Re}\,(u(y)\bar{v}(y))\,dxdy, \\
I_2 &:= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)|v(x)|^2(|u(y)|^2 + 2\mathrm{Re}\,(u(y)\bar{v}(y)))\,dxdy, \\
I_3 &:= \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)|v(x)|^2|v(y)|^2\,dxdy.
\end{aligned}
$$

Now let $R > 0$ so that $\ell_R \in L^\beta_x(L^\alpha_y)$. First we see for I_1 that

$$
\begin{aligned}
|I_1| \le & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |k_R(x,y)||u(x)||v(x)||u(y)||v(y)|\,dxdy \\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\ell_R(x,y)||u(x)||v(x)||u(y)||v(y)|\,dxdy.
\end{aligned}
$$

Applying Lemma 5.4, we have

$$
\begin{aligned}
(5.17)\qquad |I_1| &\le R\|u\|^2_{L^2}\|v\|^2_{L^2} + \|\ell_R\|_{L^\beta_x(L^\alpha_y)}\|u\|^2_{L^{2\gamma}}\|v\|^2_{L^{2\gamma}}. \\
&\le R\|u\|^2_{H^1}\|v\|^2_{H^1} + c^4\,\|\ell_R\|_{L^\beta_x(L^\alpha_y)}\,\|u\|^2_{H^1}\|v\|^2_{H^1}.
\end{aligned}
$$

In a similar way of the estimates for I_1, we see that

$$
\begin{aligned}
(5.18)\qquad |I_2| &\le \frac{R}{2}\|v\|^2_{H^1}(\|u\|^2_{H^1} + 2\|u\|_{H^1}\|v\|_{H^1}) \\
&+ \frac{c^4}{2}\|\ell_R\|_{L^\beta_x(L^\alpha_y)}\|v\|^2_{H^1}(\|u\|^2_{H^1} + 2\|u\|_{H^1}\|v\|_{H^1}), \\
(5.19)\qquad |I_3| &\le \frac{R}{4}\|v\|^4_{H^1} + \frac{c^4}{4}\|\ell_R\|_{L^\beta_x(L^\alpha_y)}\,\|v\|^4_{H^1}.
\end{aligned}
$$

Since $L^{2\gamma}(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$ we have from (5.16), (5.17), (5.18) and (5.19) that

$$\begin{aligned} &|G(u+v) - G(u) - \operatorname{Re}\langle g(u), v\rangle_{H^{-1},H^1}| \\ &\le \frac{R + c^4\|\ell_R\|_{L_x^\beta(L_y^\alpha)}}{4}\|v\|_{H^1}^2(6\|u\|_{H^1}^2 + 4\|u\|_{H^1}\|v\|_{H^1} + \|v\|_{H^1}^2). \end{aligned} \tag{5.20}$$

Let $M > 0$ and $\varepsilon > 0$. Then we see that

$$|G(u+v) - G(u) - \operatorname{Re}\langle g(u), v\rangle_{H^{-1},H^1}| \le \frac{R + c^4\|\ell_R\|_{L_x^\beta(L_y^\alpha)}}{4}(6M^2 + 4M + 1)\|v\|_{H^1}^2$$
$$\forall\, u,\ v \in H^1(\mathbb{R}^N) \text{ with } \|u\|_{H^1} \le M,\ \|v\|_{H^1} \le 1.$$

Hence by setting $\delta > 0$ as

$$\delta = \delta(u, \varepsilon) = 1 \wedge \frac{4\varepsilon}{(R + c^4\|\ell_R\|_{L_x^\beta(L_y^\alpha)})(6M^2 + 4M + 1)},$$

we conclude that

$$|G(u+v) - G(u) - \operatorname{Re}\langle g(u), v\rangle_{H^{-1},H^1}| \le \varepsilon\|v\|_{H^1} \quad \forall\, v \in H^1(\mathbb{R}^N) \text{ with } \|v\|_{H^1} \le \delta.$$

This is nothing but **(G1)**.

Verification of (G2). Let $u,\ v \in H^1(\mathbb{R}^N)$. Then we see that

$$g(u) - g(v) = K(|u|^2)u - K(|v|^2)v = K(|u|^2 - |v|^2)u + K(|v|^2)(u - v).$$

Now we divide K into K_R and L_R as

$$K_R(f)(x) := \int_{\mathbb{R}^N} k_R(x, y) f(y)\, dy, \tag{5.21}$$

$$L_R(f)(x) := \int_{\mathbb{R}^N} \ell_R(x, y) f(y)\, dy. \tag{5.22}$$

Note that $K = K_R + L_R$. Applying 5.4 with $L_x^\infty(L_y^\infty)$ and $\|k_R\|_{L_x^\infty(L_y^\infty)} \le R$ we have

$$\begin{aligned} &\|K_R(|u|^2)u - K_R(|v|^2)v\|_{H^{-1}} \\ &\le \|K_R(|u|^2 - |v|^2)u\|_{L^2} + \|K_R(|v|^2)(u - v)\|_{L^2} \\ &\le R\,(\|u\|_{L^2} + \|v\|_{L^2})\|u - v\|_{L^2}\|u\|_{L^2} + R\,\|v\|_{L^2}^2\|u - v\|_{L^2} \\ &\le R\,(\|u\|_{L^2}^2 + \|u\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2)\|u - v\|_{L^2}. \end{aligned} \tag{5.23}$$

On the other hand, applying Lemma 5.4 with $L_x^\beta(L_y^\alpha)$ we have

$$\begin{aligned} &c^{-1}\,\|L_R(|u|^2)u - L_R(|v|^2)v\|_{H^{-1}} \\ &\le \|L_R(|u|^2)u - L_R(|v|^2)v\|_{L^{(2\gamma)'}} \\ &\le \|\ell_R\|_{L_x^\beta(L_y^\alpha)}(\|u\|_{L^{2\gamma}}^2 + \|u\|_{L^{2\gamma}}\|v\|_{L^{2\gamma}} + \|v\|_{L^{2\gamma}}^2)\|u - v\|_{L^{2\gamma}} \\ &\le c^3\,\|\ell_R\|_{L_x^\beta(L_y^\alpha)}(\|u\|_{H^1}^2 + \|u\|_{H^1}\|v\|_{H^1} + \|v\|_{H^1}^2)\|u - v\|_{H^1}. \end{aligned} \tag{5.24}$$

Combining (5.23) and (5.24), we obtain **(G2)**:

$$(5.25)\quad \begin{aligned} \|g(u)-g(v)\|_{H^{-1}} &\le R\,(\|u\|_{L^2}^2+\|u\|_{L^2}\|v\|_{L^2}+\|v\|_{L^2}^2)\|u-v\|_{L^2}. \\ &\quad + c^4\,\|\ell_R\|_{L_x^\beta(L_y^\alpha)}(\|u\|_{H^1}^2+\|u\|_{H^1}\|v\|_{H^1}+\|v\|_{H^1}^2)\|u-v\|_{H^1} \\ &\le 3M^2(R+c^4\,\|\ell_R\|_{L_x^\beta(L_y^\alpha)})\|u-v\|_{H^1} \\ &\quad \forall\, u,\, v\in H^1(\mathbb{R}^N) \text{ with } \|u\|_{H^1},\, \|v\|_{H^1}\le M. \end{aligned}$$

Verification of (G3). Let $u,\ v\in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1},\ \|v\|_{H^1}\le M$. Then we see from **(K1)** that

$$G(u)-G(v)=\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}k(x,y)(|u(y)|^2-|v(y)|^2)(|u(x)|^2+|v(x)|^2)\,dxdy.$$

Thus we evaluate the following two integrals:

$$I(k_R):=\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}k_R(x,y)(|u(y)|^2-|v(y)|^2)(|u(x)|^2+|v(x)|^2)\,dxdy,$$
$$I(\ell_R):=\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\ell_R(x,y)(|u(y)|^2-|v(y)|^2)(|u(x)|^2+|v(x)|^2)\,dxdy.$$

Note that $G(u)-G(v)=I(k_R)+I(\ell_R)$. For $I(k_R)$ we calculate

$$|I(k_R)|\le\frac{R}{4}\|u-v\|_{L^2}(\|u\|_{L^2}+\|v\|_{L^2})(\|u\|_{L^2}^2+\|v\|_{L^2}^2)\le RM^3\|u-v\|_{L^2}.$$

On the other hand, for $I(\ell_R)$ we evaluate

$$|I(\ell_R)|\le\frac{\|\ell_R\|_{L_x^\beta(L_y^\alpha)}}{4}(\|u\|_{L^{2\gamma}}^2+\|v\|_{L^{2\gamma}}^2)^2\le c^4\|\ell_R\|_{L_x^\beta(L_y^\alpha)}M^4.$$

By virtue of **(K2)**, we see that

$$(5.26)\qquad \|\ell_R\|_{L_x^\beta(L_y^\alpha)}=\|k-k_R\|_{L_x^\beta(L_y^\alpha)}\to 0\quad\text{as } R\to\infty.$$

Hence for every $\delta>0$ there exists $R(\delta)>0$ such that

$$\|\ell_{R(\delta)}\|_{L_x^\beta(L_y^\alpha)}<\frac{\delta}{c^4M^4}.$$

Thus for all $u,\ v\in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1},\ \|v\|_{H^1}\le M$ we have

$$|G(u)-G(v)|\le\delta+R(\delta)M^3\|u-v\|_{L^2}\quad\forall\,\delta>0.$$

This is nothing but **(G3)**.

Verification of (G4). Let $u\in H^1(\mathbb{R}^N)$. Then **(K1)** implies **(G4)**:

$$\mathrm{Im}\,\langle g(u),u\rangle_{H^{-1},H^1}=\mathrm{Im}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}k(x,y)|u(y)|^2|u(x)|^2\,dydx=0.$$

Verification of (G5). Let $\{w_n\}_n$ be a sequence in $L^\infty(I;H^1(\mathbb{R}^N))$ satisfying

$$\begin{cases} w_n(t) \to w(t)\ (n\to\infty) & \text{weakly in } H^1(\mathbb{R}^N) \quad \text{a.a. } t\in I, \\ g(w_n) \to f\ (n\to\infty) & \text{weakly}^* \text{ in } L^\infty(I;H^{-1}(\mathbb{R}^N)). \end{cases} \tag{5.27}$$

Define $\sigma_1 := 2$, $\sigma_2 := 2\gamma$ and

$$g_1(u) := u(x)\int_{\mathbb{R}^N} k_R(x,y)|u(y)|^2\,dy, \quad g_2(u) := u(x)\int_{\mathbb{R}^N} \ell_R(x,y)|u(y)|^2\,dy.$$

Since $\{g_1(w_n)\}_n$ and $\{g_2(w_n)\}_n$ are bounded in $L^\infty(I;H^{-1}(\mathbb{R}^N))$ and the Sobolev embeddings, there exist a subsequence $\{w_{n(j)}\}_j$ of $\{w_n\}_n$ and f_1, $f_2 \in L^\infty(I;H^{-1}(\mathbb{R}^N))$ such that

$$g_l(w_{n(j)}) \to f_l\ (j\to\infty) \quad \text{weakly}^* \text{ in } L^\infty(I;L^{\sigma_l'}(\mathbb{R}^N))\ (l=1,\,2). \tag{5.28}$$

To confirm (2.2) let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open subset with C^1 boundary. Then

$$\begin{aligned} \langle f_l(t), w(t)\rangle_{L^{\sigma_l'}(\Omega),L^{\sigma_l}(\Omega)} &= \langle f_l(t) - g_l(w_{n(j)}(t)), w(t)\rangle_{L^{\sigma_l'}(\Omega),L^{\sigma_l}(\Omega)} \\ &\quad + \langle g_l(w_{n(j)}(t)), w(t) - w_{n(j)}(t)\rangle_{L^{\sigma_l'}(\Omega),L^{\sigma_l}(\Omega)} \\ &\quad + \langle g_l(w_{n(j)}(t)), w_{n(j)}(t)\rangle_{L^{\sigma_l'}(\Omega),L^{\sigma_l}(\Omega)} \\ &=: I_{l1}(t) + I_{l2}(t) + I_{l3}(t) \quad (l=1,\,2). \end{aligned} \tag{5.29}$$

The weak convergence (5.28) asserts that

$$\int_I I_{l1}(t)\,dt \to 0\ (j\to\infty), \quad l=1,\,2. \tag{5.30}$$

Next we consider I_{l2} ($l=1$, 2). Rellich's compactness theorem implies that $w_{n(j)}(t) \to w(t)$ ($j\to\infty$) strongly in $L^2(\Omega)$ a.a. $t\in I$. Hence it follows from the boundedness of $\{g_1(w_{n(j)}(t))\}_j$ in $L^2(\Omega)$ a.a. $t\in I$ that $I_{12}(t)\to 0$ ($j\to\infty$) for a.a. $t\in I$. Moreover, the boundedness of $\{w_{n(j)}\}_j$ and $\{g_1(w_{n(j)})\}_j$ in $L^\infty(I;L^2(\Omega))$ implies that

$$\int_I I_{12}(t)\,dt \to 0\ (j\to\infty). \tag{5.31}$$

On the other hand, for I_{22} we evaluate $|I_{22}(t)| \le 2M^4c^4\,\|\ell_R\|_{L^\beta_x(L^\alpha_y)}$. Note that the constant $2M^4c^4\|\ell_R\|_{L^\beta_x(L^\alpha_y)}$ does not depend on Ω and j. Hence we have

$$\left|\int_I I_{22}(t)\,dt\right| \le 2|I|M^4c^4\|\ell_R\|_{L^\beta_x(L^\alpha_y)}. \tag{5.32}$$

Since k_R and ℓ_R are real-valued, we see that $\operatorname{Im} I_{l3}(t) = 0$ a.a. $t \in I$ ($l = 1, 2$). Integrating (5.29) over I and using (5.30), (5.31) and (5.32), we obtain

$$\operatorname{Im} \int_I \langle f_1(t), w(t) \rangle_{L^2(\Omega)}\, dt = 0,$$
$$\left| \operatorname{Im} \int_I \langle f_2(t), w(t) \rangle_{L^{(2\gamma)'}(\Omega), L^{2\gamma}(\Omega)}\, dt \right| \le 2|I| M^4 c^4 \|\ell_R\|_{L_x^\beta(L_y^\alpha)}.$$

Since Ω is arbitrary and $f = f_1 + f_2$, we obtain (2.2) by letting $R \to \infty$ and using **(G4)**:

$$\operatorname{Im} \int_I \langle f(t), w(t) \rangle_{H^{-1}, H^1}\, dt = 0 = \lim_{n\to\infty} \operatorname{Im} \int_I \langle g(w_n(t)), w_n(t) \rangle_{H^{-1}, H^1}\, dt.$$

Next we show that $f = g(w)$ by assuming further that $w_n(t) \to w(t)$ ($n \to \infty$) in $L^2(\mathbb{R}^N)$ a.a. $t \in I$. Let $M := \sup_n \|w_n\|_{L^\infty(I;H^1)}$. It follows from (5.25) that

$$\|g(w_n(t)) - g(w(t))\|_{H^{-1}} \le 3M^2 R\, \|w_n(t) - w(t)\|_{L^2} + 6c^4 M^4 \|\ell_R\|_{L_x^\beta(L_y^\alpha)}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \|g(w_n(t)) - g(w(t))\|_{H^{-1}} \le 6c^4 M^4 \|\ell_R\|_{L_x^\beta(L_y^\alpha)} \quad \text{a.a. } t \in I.$$

Since R is arbitrary, we see that $g(w_n(t)) \to g(w(t))$ ($n \to \infty$) in $H^{-1}(\mathbb{R}^N)$ a.a. $t \in I$. Therefore we conclude that $f = g(w)$ and **(G5)** is verified.

Verification of (G6). Let $k_-(x, y) := (-k(x, y)) \vee 0$ and

$$k_R^-(x,y) := \begin{cases} k_-(x,y) & k^-(x,y) \le R, \\ R & k^-(x,y) > R, \end{cases} \tag{5.33}$$

$$\ell_R^-(x,y) := k_-(x,y) - k_R^-(x,y). \tag{5.34}$$

Then we see from (5.2) and **(K3)** that

$$\begin{aligned} G(u) \ge & -\frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k_R^-(x,y) |u(x)|^2 |u(y)|^2\, dxdy \\ & -\frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \ell_R^-(x,y) |u(x)|^2 |u(y)|^2\, dxdy \quad \forall\, u \in H^1(\mathbb{R}^N). \end{aligned}$$

Applying Lemma 5.4, we have for $u \in H^1(\mathbb{R}^N)$,

$$-\frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k_R^-(x,y) |u(x)|^2 |u(y)|^2\, dxdy \ge -\frac{1}{4} R\, \|u\|_{L^2}^4,$$
$$-\frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \ell_R^-(x,y) |u(x)|^2 |u(y)|^2\, dxdy \ge -\frac{1}{4} \|\ell_R^-\|_{L_x^{\tilde\beta}(L_y^{\tilde\alpha})} \|u\|_{L^{2\tilde\gamma}}^4,$$

where $\tilde{\gamma}^{-1} = 1 - (\tilde{\alpha}^{-1} + \tilde{\beta}^{-1})/2$. It follows from the Gagliardo-Nirenberg inequality that

$$(5.35) \qquad \|u\|_{L^{2\tilde{\gamma}}} \le c_0 \|u\|_{L^2}^{1-\theta} \|\nabla u\|_{L^2}^{\theta} \quad \forall\, u \in H^1(\mathbb{R}^N),$$

where

$$\theta = N\Big(\frac{1}{2} - \frac{1}{2\gamma}\Big) = \frac{N}{4}\Big(\frac{1}{\tilde{\alpha}} + \frac{1}{\tilde{\beta}}\Big).$$

Case 1 ($\tilde{\alpha}^{-1} + \tilde{\beta}^{-1} < 2/N$). Note that $N(\tilde{\alpha}^{-1} + \tilde{\beta}^{-1}) < 2$. Hence (5.35) and the Young inequality imply that

$$\begin{aligned}-\frac{1}{4}\|\ell_R^-\|_{L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})}\|u\|_{L^{2\tilde{\gamma}}}^4 &\ge -c_0^4\,\|\ell_R^-\|_{L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})}\|u\|_{L^2}^{4-N(\tilde{\alpha}^{-1}+\tilde{\beta}^{-1})}\|\nabla u\|_{L^2}^{N(\tilde{\alpha}^{-1}+\tilde{\beta}^{-1})}\\ &\ge -\delta\,\|\nabla u\|_{L^2}^2 - C_\delta(\|u\|_{L^2}) \quad \forall\, u \in H^1(\mathbb{R}^N).\end{aligned}$$

Putting $\delta := (1-\varepsilon)/2$ for some $\varepsilon \in (0,1)$, we see that **(G6)** is satisfied.
Case 2 ($\tilde{\alpha}^{-1} + \tilde{\beta}^{-1} = 2/N$). This is the critical case. In view of (5.35) we see that

$$-\frac{1}{4}\|\ell_R^-\|_{L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})}\|u\|_{L^{2\tilde{\gamma}}}^4 \ge -c_0^4\,\|\ell_R^-\|_{L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2.$$

Since $\|\ell_R^-\|_{L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})} \to 0$ as $R \to \infty$ by **(K3)**, there exists $R_1 = R_1(\|u\|_{L^2}, \varepsilon) > 0$ such that

$$c_0^4\,\|\ell_R^-\|_{L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})}\|u\|_{L^2}^2 < \frac{1-\varepsilon}{2}, \quad R > R_1.$$

Then we have

$$G(u) \ge -\frac{1-\varepsilon}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{4}R_1(\|u\|_{L^2}, \varepsilon)\,\|u\|_{L^2}^4 \quad \forall\, u \in H^1(\mathbb{R}^N).$$

This is nothing but **(G6)**.

Since **(G1)**–**(G6)** are verified and the uniqueness of local weak solutions for **(HE)**$_a$ is proved, Theorem 2.3 yields the global existence of weak solutions to **(HE)**$_a$.

6. Concluding remarks

Remark 6.1. In general, nonlocal nonlinearity does not satisfy the condition

$$(6.1) \qquad \begin{cases} u_n \to u\ (n \to \infty) & \text{weakly in } X_S, \\ g(u_n) \to f\ (n \to \infty) & \text{weakly in } X_S^* \end{cases} \Rightarrow\ f = g(u)$$

for any sequence $\{u_n\}_n$ in X_S (see Section 2.3 for notations). Let $X_S = H^1(\mathbb{R})$, $X = L^2(\mathbb{R})$, $X_S^* = H^{-1}(\mathbb{R})$ and consider $g(u) := \|u\|_{L^2}^2 u$ $(k(x,y) = 1)$. Then g satisfies **(G1)**–**(G6)**. Now we show that g does not verify (6.1). Let $\varphi \in H^1(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [-1,1]$. Put $w_n(x) := \varphi(x) + \varphi(x - 2n)$. Then $\{w_n\}_n$ is a bounded sequence in $H^1(\mathbb{R})$. It is easy to see that

$$w_n \to \varphi \quad \text{weakly in } L^2(\mathbb{R})$$

and hence weakly in $H^1(\mathbb{R})$. Since $\|w_n\|_{L^2}^2 = 2\|\varphi\|_{L^2}^2$ for all $n \in \mathbb{N}$, we have

$$g(w_n) \to f := 2\|\varphi\|_{L^2}^2\varphi \quad \text{weakly in } L^2(\mathbb{R})$$

and hence weakly in $H^{-1}(\mathbb{R})$. But $g(\varphi) = \|\varphi\|_{L^2}^2\varphi$ and so $f \neq g(\varphi)$.

On the other hand, local nonlinearity satisfies the condition (6.1). See [**11**] for details.

Remark 6.2. Assume that k satisfies **(K1)**–**(K3)**. By applying Lemma 5.2 we obtain the Lipschitz type dependence

$$\|u_1(t) - u_2(t)\|_{L^2} \leq L(M)e^{\omega(M)|t|}\|u_{1,0} - u_{2,0}\|_{L^2} \quad \forall\, t \in \mathbb{R},$$

where u_j $(j = 1, 2)$ are the global weak solutions to $\mathbf{(HE)}_a$ with initial values $u_j(0) = u_{j,0} \in H^1(\mathbb{R}^N)$, $\|u_{j,0}\|_{H^1} \leq M$. See also [**10**, Proposition 3.7] for details. Hence we conclude that $\mathbf{(HE)}_a$ is wellposed.

Remark 6.3. Another example of the kernel which belongs to $L_x^\beta(L_y^\alpha)$ is the following:

$$k(x,y) := U(x)W(x-y)U(y), \tag{6.2}$$

where U, W are real-valued functions and W is even such that $U \in L^p(\mathbb{R}^N)$ and $W \in L^q(\mathbb{R}^N)$. Then k belongs to $L_x^p(L_y^{pq/(p+q)})$. By virtue of the Hölder inequality we have

$$\|U(x)W(x-\cdot)U(\cdot)\|_{L^{pq/(p+q)}} \leq |U(x)|\, \|W\|_{L^q}\|U\|_{L^p} \quad \text{a.a.} x \in \mathbb{R}^N$$

and hence we obtain

$$\|k\|_{L_x^p(L_y^{pq/(p+q)})} \leq \|U\|_{L^p}\, \|W\|_{L^q}\|U\|_{L^p}.$$

References

[1] N. Burq, F. Planchon, J. Stalker, A. S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203** (2003), 519–549.

[2] N. Burq, F. Planchon, J. Stalker, A. S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay*, Indiana Univ. Math. J. **53** (2004), 1665–1680.

[3] T. Cazenave, "An introduction to nonlinear Schrödinger equation," Textos de Métodos Matemáticos, 22. Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 1989.

[4] T. Cazenave, "Semilinear Schrödinger Equations," Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, AMS, New York, 2003.

[5] T. Cazenave, F. B. Weissler, *The Cauchy problem for the nonlinear Schrödinger equation in* H^1, Manuscripta Math. **61** (1988), 477–494.

[6] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations.* I. *The Cauchy problem, general case*, J. Funct. Anal. **32** (1979), 1–32.

[7] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations with nonlocal interaction*, Math. Z. **170** (1980), 109–136.

[8] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. **46** (1987), 113–129.

[9] N. Okazawa, L^p*-theory of Schrödinger operators with strongly singular potentials*, Japan. J. Math. **22** (1996), 199–239.

[10] N. Okazawa, T. Suzuki, T. Yokota, *Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials*, Appl. Anal. **91** (2012), 1605–1629.

[11] N. Okazawa, T. Suzuki, T. Yokota, *Energy methods for abstract nonlinear Schrödinger equations*, Evolution Equations Control Theory **1** (2012), 337–354.

[12] V. Pierfelice, *Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces*, Math. Z. **260** (2008), 377–392.

[13] T. Suzuki, *Energy methods for Hartree type equations with inverse-square potentials*, Evolution Equations Control Theory, to appear.