<table>
<thead>
<tr>
<th>Title</th>
<th>Quasi-Subdifferential Operators and Quasi-Subdifferential Evolution Equations (New developments of the theory of evolution equations in the analysis of non-equilibria)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kubo, Masahiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1856: 24-33</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195250">http://hdl.handle.net/2433/195250</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Quasi-Subdifferential Operators and
Quasi-Subdifferential Evolution Equations

Masahiro Kubo
(Nagoya Institute of Technology, Japan)

1. INTRODUCTION

Based on our previous paper [13], we introduce some useful concepts for studying variational and quasi-variational problems associated with a general, i.e., not Euler–Lagrange, partial differential operator.

Consider the following elliptic variational inequality:

\[
(VI) \begin{cases}
    u \in K, \\
    \int_{\Omega} \left\{ a(u, \nabla u) \cdot \nabla (u - z) + a_0(u)(u - z) \right\} \, dx \\
    \leq (f, u - z) \quad \forall z \in K,
\end{cases}
\]

where \( K \subset H^1(\Omega) \) is a closed convex set, \( \Omega \subset \mathbb{R}^N (N \geq 1) \) is a bounded domain, \( f \in L^2(\Omega) \) is a given function, \( (\cdot, \cdot) \) denotes the inner product in \( L^2(\Omega) \), \( a(r, p) = \partial_p \hat{a}(r, p) \), \( \hat{a} \in C^1(\mathbb{R} \times \mathbb{R}^N) \), and \( a_0 \in C(\mathbb{R}) \) with appropriate growth conditions.

If it holds that

\[ \hat{a}(r, p) \text{ is convex jointly in } (r, p) \in \mathbb{R} \times \mathbb{R}^N \text{ and } a_0 = \partial_r \hat{a}, \]

then we have

\[ (VI) \iff (f, z - u) \leq \psi(z) - \psi(u) \quad \forall z \in K, \]

\[ \iff \partial \psi(u) \ni f, \]

where \( \partial \psi \) is the subdifferential of a proper, lower-semicontinuous (l.s.c.), and convex function \( \psi : L^2(\Omega) \to \mathbb{R} \cup \]

\[ ^{1}\text{This work was supported by JSPS KAKENHI Grant Number 21540173} \]
\{+\infty\} defined by
\[
\psi(z) := \begin{cases} 
\int_{\Omega} \hat{a}(z, \nabla z)dx, & \text{if } z \in K, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

However, condition (1) is too restrictive for a general case. We have, in general:
\[
(VI) \iff (f, z-u) \leq \varphi(u; z) - \varphi(u; u) \quad \forall z \in K
\]
\[
\iff \partial\varphi(u; u) \ni f,
\]
where \(\partial\varphi\) is the subdifferential with respect to the second variable of a parameterized convex function \(\varphi : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}\) given by
\[
\varphi(v; z) := \begin{cases} 
\int_{\Omega} \hat{a}(v, \nabla z)dx + \int_{\Omega} a_0(v)zdx, & \text{if } v \in H^1(\Omega) \text{ and } z \in K, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

Thus, we are led to the notion of a quasi-subdifferential operator, which we define in the next section.

2. Quasi-subdifferential Operators (QSOs)

In the following, \(H\) denotes a real Hilbert space with norm \(\|\cdot\|_H\) and inner product \((\cdot, \cdot)\).

**Definition 2.1.** ([13, Definition 2.1]) A (possibly multi-valued) map \(A : H \to H\) is called a quasi-subdifferential operator (QSO) if
\[
Au = \partial\varphi(u; u) \quad \text{for } u \in D(A)
\]
where \(\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}\) satisfies:

- \(\varphi(v; \cdot) : H \to \mathbb{R} \cup \{+\infty\}\) is l.s.c. and convex \(\forall v \in H\).
• $D(A) := \{v \in H | \varphi(v; \cdot) \neq +\infty, v \in D(\partial \varphi(v; \cdot))\} \neq \emptyset$.

We call $\varphi$ the defining convex function of $A$, and write $A^\varphi$ when this needs to be specified.

We have the following existence theorem for an equation with a quasi-subdifferential operator.

**Theorem 2.2.** ([13, Theorem 2.2]) Let $A$ be a QSO defined by $\varphi$. Let $X$ be a reflexive Banach space with compact embedding $X \subset H$, and $K$ be a closed convex subset of $X$. Assume that $D(\varphi(v; \cdot)) \subset K$ for all $v \in K$, and that there exist $C_1, C_2, C_3 > 0$, $p > q \geq 1$ satisfying the following conditions.

(A1) There exists $z_0 \in H$ such that for all $v \in K$
\[ \varphi(v; z_0) \leq C_1 (|v|_X^q + 1). \]

(A2) For all $v \in K$ and $z \in X$
\[ \varphi(v; z) \geq C_2 |z|_X^p - C_3 (|v|_X^q + 1). \]

(A3) For all $v \in K$
\[ D(\varphi(v; \cdot)) \ni z \mapsto \varphi(v; z) \text{ is strictly convex}. \]

(A4) If $K \ni v_n \rightharpoonup v$ weakly in $X$, then $\varphi(v_n; \cdot) \rightharpoonup \varphi(v; \cdot)$ in the sense of Mosco.

Then, for each $f \in H$, there exists $u \in K$ satisfying
\[ Au \ni f. \]

The idea of the proof of this theorem is as follows. For each $v \in K$, assumptions (A2) and (A3) mean that there exists a unique $z_v \in K$ minimizing $\varphi(v; z) - (f, z)$ ($z \in H$). By (A1) and (A2), the map $v \mapsto z_v$, if restricted to an appropriate compact and convex set $\tilde{K} \subset K$, maps to itself. By (A4), this map is continuous with respect to the topology of $H$. Therefore, from Schauder's fixed point theorem, it follows that there is a fixed point $u$ that is a
solution of the desired equation. We refer to [13] for the detail.

We note that, under different assumptions, we can use another type of fixed point theorem to obtain an existence theorem of a different type. In the next section, we introduce a concept based on such an argument.

This theorem can be applied to (VI) as well as to the following quasi-variational inequality (cf. [13, Section 3]):

\[
\text{(QVI)} \begin{cases}
\quad u \in K(u), \\
\quad \int_{\Omega} \{a(u, \nabla u) \cdot \nabla(u - z) + a_0(u)(u - z)\} \, dx \\
\quad \quad \quad \leq (f, u - z) \quad \forall z \in K(u)
\end{cases}
\]

Here, \( K(v) \subset H^1(\Omega) \) is a closed convex set depending on \( v \). We have

\[
\text{(QVI)} \iff Au \ni f,
\]

where \( A \) is a QSO defined by

\[
\varphi(v; z) := \begin{cases}
\quad \int_{\Omega} a(v, \nabla z) \, dx + \int_{\Omega} a_0(v) z \, dx, \\
\quad \quad \quad \text{if } v \in H^1(\Omega) \text{ and } z \in K(v), \\
\quad +\infty, \quad \text{otherwise.}
\end{cases}
\]

For a pseudo-monotone operator approach to (VI) and (QVI), we refer to Kenmochi et al. [10, 5]. For an earlier study of elliptic quasi-variational inequalities, see Joly and Mosco [4].
3. QUASI-VARIATIONAL PRINCIPLES

A variational principle is expressed using a proper, l.s.c., and convex function $\psi$ and its subdifferential as follows:

$$\partial\psi(u) \ni 0 \iff \psi(u) = \min_z \psi(z).$$

Here, the equation (or inclusion) $\partial\psi(u) \ni 0$ represents a variational inequality or a differential equation with a boundary condition according to the constraint posed by the function $\psi$. This principle has played an important role in mathematical physics and related fields. However, there is a simple limitation to the principle, since it can only be applied to problems associated with Euler–Lagrange-type differential operators. Problems associated with non-Euler–Lagrange-type differential operators, e.g., the Navier–Stokes equations, the diffusion equation with a convection term and so on, are not derived directly from the variational principle.

Let us consider the following idea:

$$\partial\varphi(u; u) \ni 0 \iff \begin{cases} u \text{ is a fixed point of } v \mapsto z_v : \\ \varphi(v; z_v) = \min_z \varphi(v; z). \end{cases}$$

Here, we have a function $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ such that $\varphi(v; \cdot) : H \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and convex for each $v \in H$ and proper for some $v \in H$. In (2), $\partial\varphi$ denotes the subdifferential with respect to the second variable. Hence, we have

$$\partial\varphi(u; u) \ni 0 \iff A^\varphi \ni 0,$$

where $A^\varphi$ is the QSO defined by $\varphi$. We call the idea in (2) a quasi-variational principle (QVP). Thus, QVP is closely related to QSOs. A similar concept to this (2) was used by Joly and Mosco [4] to study quasi-variational inequalities, that is, variational inequalities with constraints depending on the unknown functions. However, the idea can be applied to various problems with non-Euler–Lagrange-type
differential operators. In fact, the proof of Theorem 2.2 is based on QVP and can be applied to variational and quasi-variational inequalities with non-Euler–Lagrange-type differential operators.

In addition to this, QVP plays an essential role in a standard proof of the existence theorem for the stationary Navier–Stokes equations. These are stated below in a slightly abstract form.

**Theorem 3.1. (abstract Navier–Stokes equations)** Let $V \subset H \subset V^*$ be a Hilbert triplet with compact embeddings, $\langle \cdot ; \cdot \rangle$ be the duality pairing, and $F : V \to V^*$ be the duality map. Let $B : V \to V^*$ be a compact map satisfying $\langle B(z), z \rangle = 0$ for all $z \in V$. Let $A : H \to H$ be a QSO defined by

$$\varphi(v; z) := \begin{cases} \frac{1}{2} |z|_V^2 + \langle B(v), z \rangle, & \text{if } v, z \in V, \\ +\infty, & \text{otherwise}. \end{cases}$$

Then, for each $f \in H$, there exists a $u \in H$ such that

$$Au = f.$$  

This theorem can be proved as follows. For each $v \in V$, there exists a unique $z_v \in V$ such that

$$\Phi_{\lambda, f}(v; z_v) = \min_z \Phi_{\lambda, f}(v; z),$$

where, for $\lambda \in [0, 1]$, we define

$$\Phi_{\lambda, f}(v; z) := \begin{cases} \frac{1}{2} |z|_V^2 + \lambda(\langle B(v), z \rangle - (f, z)), & \text{if } v, z \in V, \\ +\infty, & \text{otherwise}. \end{cases}$$

That is, we have

$$z_v + \lambda F^{-1}(B(v) - f) = 0.$$  

By Leray–Schauder’s fixed point theorem, we can show that there exists a fixed point $u$ of the map $v \mapsto z_v$ that is a desired solution to the equation.
4. QUASI-SUBDIFFERENTIAL EVOLUTION EQUATIONS

In this section, we study quasi-subdifferential evolution equations (QSEs), which are evolution equations related to QSOs. We consider two types of QSE. The first is given as follows:

\[(QSE1) \quad u'(t) + A(t)u(t) \ni 0 \quad \text{a.e. } t \in (0, T).\]

Here, $A(t) \, (0 \leq t \leq T)$ is a QSO defined by $\varphi^t : H \times H \to \mathbb{R} \cup \{+\infty\}$. Consider the following conditions:

(\Phi 1) \quad \varphi^t(v; z) \geq G(|z|_X) \quad \forall (v, z) \in H \times H, \text{ where } X \text{ is a Banach space with compact embedding } X \subset H \text{ and } \lim_{r \to +\infty} G(r) = +\infty.

(\Phi 2) \quad \text{There are two functions } \alpha \in W^{1,2}(0, T) \text{ and } \beta \in W^{1,1}(0, T) \text{ such that, for all } v, w \in H, 0 \leq s \leq t \leq T \text{ and } z \in D(\varphi^s(v; \cdot)), \text{ there exists } \tilde{z} \in D(\varphi^t(v; \cdot)) \text{ satisfying the following inequalities:}

\[
|\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)| (\varphi^s(v; z))^{1/2},
\]

\[
\varphi^t(w; \tilde{z}) - \varphi^s(v; z)
\]

\[
\leq |\beta(t) - \beta(s)| \varphi^s(v; z) + |w - v|_H (\varphi^s(v; z))^{1/2}.
\]

Put $K(t) := \{z \in H | \varphi^t(z; z) < +\infty\}$.

**Theorem 4.1.** ([13, Theorem 4.1]) Assume (\Phi 1) and (\Phi 2). Then, for each $u_0 \in K(0)$, there exists a solution $u \in W^{1,2}(0, T; H)$ of (QSE1) satisfying $u(0) = u_0$.

The idea of this theorem has been developed by Kenmochi, Kubo, Yamazaki, Shirakawa and Fukao [12, 16, 20, 17, 18, 2, 15, 3] and is based on the theory of time-dependent subdifferential evolution equations (TSEs). In fact, by assumption (\Phi 2), for each $v \in W^{1,2}(0, T; H)$ the function

\[ t \mapsto \Phi(t) := \varphi^t(v(t); \cdot) \]
satisfies the condition of the standard theory of TSEs developed by Kenmochi [8, 9] and Yamada [19]. Hence, there exists a unique solution of the problem:

\[
\begin{align*}
&u'(t) + \varphi^t(v(t); u(t)) \ni 0 \quad \text{a.e. } t \in (0, T), \\
&u(0) = u_0.
\end{align*}
\]

Using assumption (Φ1) and the energy inequality derived by TSE theory, we can show that there is a fixed point of the map \( v \mapsto u \) that gives a desired solution of (QSE1).

The second type of QSE is given as follows:

\[
\begin{equation}
\text{(QSE2)} \quad \mathcal{L}_{u_0} u + \mathcal{A} u \ni 0 \quad \text{in } \mathcal{H}.
\end{equation}
\]

Here, \( \mathcal{H} := L^2(0, T; H) \), \( \mathcal{A} : \mathcal{H} \rightarrow \mathcal{H} \) is a QSO, \( \mathcal{L}_{u_0} u := u' \), and \( D(\mathcal{L}_{u_0}) := \{ w \in W^{1,2}(0, T; H) | w(0) = u_0 \} \).

This type of problem arises in hysteresis models, non-local obstacle problems, and so on (cf. [11, 1, 14, 6]). In particular, Kano, Murase and Kenmochi [7] studied this type of abstract problem by employing the theory of TSEs.

REFERENCES


tion equations generated by subdifferentials with nonlo-

[8] N. Kenmochi, *Some nonlinear parabolic variational in-

[9] N. Kenmochi, *Solvability of nonlinear evolution equa-


