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Quasi-Subdifferential Operators and Quasi-Subdifferential Evolution Equations

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1. INTRODUCTION

Based on our previous paper [13], we introduce some useful concepts for studying variational and quasi-variational problems associated with a general, i.e., not Euler–Lagrange, partial differential operator.

Consider the following elliptic variational inequality:

\[
\begin{aligned}
(VI) \quad \{u \in K, \\
\int_{\Omega} \{a(u, \nabla u) \cdot \nabla (u - z) + a_0(u)(u - z)\} \, dx \\
\leq (f, u - z) \quad \forall z \in K,
\end{aligned}
\]

where \( K \subset H^1(\Omega) \) is a closed convex set, \( \Omega \subset \mathbb{R}^N (N \geq 1) \) is a bounded domain, \( f \in L^2(\Omega) \) is a given function, \((\cdot, \cdot)\) denotes the inner product in \( L^2(\Omega) \), \( a(r, p) = \partial_p \hat{a}(r, p) \), \( \hat{a} \in C^1(\mathbb{R} \times \mathbb{R}^N) \), and \( a_0 \in C(\mathbb{R}) \) with appropriate growth conditions.

If it holds that

\( \hat{a}(r, p) \) is convex jointly in \((r, p) \in \mathbb{R} \times \mathbb{R}^N \) and \( a_0 = \partial_r \hat{a} \),

(1)

then we have

\[
(VI) \iff (f, z - u) \leq \psi(z) - \psi(u) \quad \forall z \in K,
\]

\[
\iff \partial \psi(u) \ni f,
\]

where \( \partial \psi \) is the subdifferential of a proper, lower-semicontinuous (l.s.c.), and convex function \( \psi : L^2(\Omega) \to \mathbb{R} \cup \{\infty\} \).

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\{+\infty\} \text{ defined by}
\[
\psi(z) := \begin{cases}
\int_{\Omega} \hat{a}(z, \nabla z) dx, & \text{if } z \in K, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

However, condition (1) is too restrictive for a general case. We have, in general:
\[
(VI) \iff (f, z - u) \leq \varphi(u; z) - \varphi(u; u) \quad \forall z \in K \\
\iff \partial \varphi(u; u) \ni f,
\]
where \(\partial \varphi\) is the subdifferential with respect to the second variable of a parameterized convex function \(\varphi : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}\) given by
\[
\varphi(v; z) := \begin{cases}
\int_{\Omega} \hat{a}(v, \nabla z) dx + \int_{\Omega} a_0(v) z dx, & \text{if } v \in H^1(\Omega) \text{ and } z \in K, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

Thus, we are led to the notion of a \textit{quasi-subdifferential operator}, which we define in the next section.

2. \textbf{QUASI-SUBDIFFERENTIAL OPERATORS (QSOs)}

In the following, \(H\) denotes a real Hilbert space with norm \(|\cdot|_H\) and inner product \((\cdot, \cdot)\).

**Definition 2.1.** ([13, Definition 2.1]) A (possibly multi-valued) map \(A : H \rightarrow H\) is called a \textit{quasi-subdifferential operator} (QSO) if
\[
Au = \partial \varphi(u; u) \quad \text{for } u \in D(A)
\]
where \(\varphi : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}\) satisfies:
\begin{itemize}
\item \(\varphi(v; \cdot) : H \rightarrow \mathbb{R} \cup \{+\infty\}\) is l.s.c. and convex \(\forall v \in H\).
\end{itemize}
\[ D(A) := \{ v \in H | \varphi(v; \cdot) \neq +\infty, v \in D(\partial \varphi(v; \cdot)) \} \neq \emptyset. \]

We call \( \varphi \) the defining convex function of \( A \), and write \( A^\varphi \) when this needs to be specified.

We have the following existence theorem for an equation with a quasi-subdifferential operator.

**Theorem 2.2.** ([13, Theorem 2.2]) Let \( A \) be a QSO defined by \( \varphi \). Let \( X \) be a reflexive Banach space with compact embedding \( X \subset H \), and \( K \) be a closed convex subset of \( X \). Assume that \( D(\varphi(v; \cdot)) \subset K \) for all \( v \in K \), and that there exist \( C_1, C_2, C_3 > 0 \), \( p > q \geq 1 \) satisfying the following conditions.

(A1) There exists \( z_0 \in H \) such that for all \( v \in K \)

\[ \varphi(v; z_0) \leq C_1 (|v|_X^q + 1). \]

(A2) For all \( v \in K \) and \( z \in X \)

\[ \varphi(v; z) \geq C_2 |z|_X^p - C_3 (|v|_X^q + 1). \]

(A3) For all \( v \in K \)

\[ D(\varphi(v; \cdot)) \ni z \mapsto \varphi(v; z) \text{ is strictly convex.} \]

(A4) If \( K \ni v_n \rightharpoonup v \) weakly in \( X \), then \( \varphi(v_n; \cdot) \rightharpoonup \varphi(v; \cdot) \) in the sense of Mosco.

Then, for each \( f \in H \), there exists \( u \in K \) satisfying

\[ Au \ni f. \]

The idea of the proof of this theorem is as follows. For each \( v \in K \), assumptions (A2) and (A3) mean that there exists a unique \( z_v \in K \) minimizing \( \varphi(v; z) - (f, z) \) \((z \in H)\). By (A1) and (A2), the map \( v \mapsto z_v \), if restricted to an appropriate compact and convex set \( \bar{K} \subset K \), maps to itself. By (A4), this map is continuous with respect to the topology of \( H \). Therefore, from Schauder’s fixed point theorem, it follows that there is a fixed point \( u \) that is a
solution of the desired equation. We refer to [13] for the
detail.

We note that, under different assumptions, we can use
another type of fixed point theorem to obtain an existence
theorem of a different type. In the next section, we intro-
duce a concept based on such an argument.

This theorem can be applied to (VI) as well as to the
following quasi-variational inequality (cf. [13, Section 3]):

\[
\begin{aligned}
\left(\text{QVI}\right) \quad & \begin{cases}
    u \in K(u), \\
    \int_{\Omega} \{a(u, \nabla u) \cdot \nabla (u - z) + a_0(u)(u - z)\} \, dx \\
    \leq (f, u - z) \quad \forall z \in K(u)
\end{cases}
\end{aligned}
\]

Here, \( K(v) \subset H^1(\Omega) \) is a closed convex set depending on
\( v \). We have

\[
\text{(QVI)} \iff A u \ni f,
\]

where \( A \) is a QSO defined by

\[
\varphi(v; z) := \begin{cases}
    \int_{\Omega} \hat{a}(v, \nabla z) \, dx + \int_{\Omega} a_0(v) z \, dx, & \text{if } v \in H^1(\Omega) \text{ and } z \in K(v), \\
    +\infty, & \text{otherwise}.
\end{cases}
\]

For a pseudo-monotone operator approach to (VI) and
(QVI), we refer to Kenmochi et al. [10, 5]. For an earlier
study of elliptic quasi-variational inequalities, see Joly and
Mosco [4].
3. QUASI-VARIATIONAL PRINCIPLES

A variational principle is expressed using a proper, l.s.c., and convex function $\psi$ and its subdifferential as follows:

$$\partial\psi(u) \ni 0 \iff \psi(u) = \min_z \psi(z).$$

Here, the equation (or inclusion) $\partial\psi(u) \ni 0$ represents a variational inequality or a differential equation with a boundary condition according to the constraint posed by the function $\psi$. This principle has played an important role in mathematical physics and related fields. However, there is a simple limitation to the principle, since it can only be applied to problems associated with Euler–Lagrange-type differential operators. Problems associated with non-Euler–Lagrange-type differential operators, e.g., the Navier–Stokes equations, the diffusion equation with a convection term and so on, are not derived directly from the variational principle.

Let us consider the following idea:

$$\partial\varphi(u; u) \ni 0 \iff \begin{cases} u \text{ is a fixed point of } v \mapsto z_v : \varphi(v; z) = \min_z \varphi(v; z) \end{cases}.$$  \hspace{1cm} (2)

Here, we have a function $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ such that $\varphi(v; ::) : H \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and convex for each $v \in H$ and proper for some $v \in H$. In (2), $\partial\varphi$ denotes the subdifferential with respect to the second variable. Hence, we have

$$\partial\varphi(u; u) \ni 0 \iff A^\varphi \ni 0,$$

where $A^\varphi$ is the QSO defined by $\varphi$. We call the idea in (2) a quasi-variational principle (QVP). Thus, QVP is closely related to QSOs. A similar concept to this (2) was used by Joly and Mosco [4] to study quasi-variational inequalities, that is, variational inequalities with constraints depending on the unknown functions. However, the idea can be applied to various problems with non-Euler–Lagrange-type
differential operators. In fact, the proof of Theorem 2.2 is based on QVP and can be applied to variational and quasi-variational inequalities with non-Euler–Lagrange-type differential operators.

In addition to this, QVP plays an essential role in a standard proof of the existence theorem for the stationary Navier–Stokes equations. These are stated below in a slightly abstract form.

**Theorem 3.1. (abstract Navier–Stokes equations)** Let $V \subset H \subset V^*$ be a Hilbert triplet with compact embeddings, $\langle \cdot, \cdot \rangle$ be the duality pairing, and $F : V \to V^*$ be the duality map. Let $B : V \to V^*$ be a compact map satisfying $\langle B(z), z \rangle = 0$ for all $z \in V$. Let $A : H \to H$ be a QSO defined by

$$
\varphi(v; z) := \begin{cases} 
\frac{1}{2}|z|_V^2 + \langle B(v), z \rangle, & \text{if } v, z \in V, \\
+\infty, & \text{otherwise}.
\end{cases}
$$

Then, for each $f \in H$, there exists a $u \in H$ such that

$$Au = f.$$

This theorem can be proved as follows. For each $v \in V$, there exists a unique $z_v \in V$ such that

$$\Phi_{\lambda,f}(v; z_v) = \min_z \Phi_{\lambda,f}(v; z),$$

where, for $\lambda \in [0, 1]$, we define

$$\Phi_{\lambda,f}(v; z) := \begin{cases} 
\frac{1}{2}|z|_V^2 + \lambda(\langle B(v), z \rangle - (f, z)), & \text{if } v, z \in V, \\
+\infty, & \text{otherwise}.
\end{cases}$$

That is, we have

$$z_v + \lambda F^{-1}(B(v) - f) = 0.$$

By Leray–Schauder’s fixed point theorem, we can show that there exists a fixed point $u$ of the map $v \mapsto z_v$ that is a desired solution to the equation.
4. QUASI-SUBDIFFERENTIAL EVOLUTION EQUATIONS

In this section, we study quasi-subdifferential evolution equations (QSEs), which are evolution equations related to QSOs. We consider two types of QSE. The first is given as follows:

\[(QSE1) \quad u'(t) + A(t)u(t) \ni 0 \quad \text{a.e. } t \in (0, T).\]

Here, \(A(t) (0 \leq t \leq T)\) is a QSO defined by \(\varphi^t : H \times H \to \mathbb{R} \cup \{+\infty\}\). Consider the following conditions:

\((\Phi 1)\) \(\varphi^t(v; z) \geq G(|z|_X) \quad \forall (v, z) \in H \times H\), where \(X\) is a Banach space with compact embedding \(X \subset H\) and \(\lim_{r \to +\infty}G(r) = +\infty\).

\((\Phi 2)\) There are two functions \(\alpha \in W^{1,2}(0, T)\) and \(\beta \in W^{1,1}(0, T)\) such that, for all \(v, w \in H, 0 \leq s \leq t \leq T\) and \(z \in D(\varphi^s(v; \cdot))\), there exists \(\tilde{z} \in D(\varphi^t(v; \cdot))\) satisfying the following inequalities:

\[|\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)| (\varphi^s(v; z))^{1/2},\]

\[\varphi^t(w; \tilde{z}) - \varphi^s(v; z) \leq |\beta(t) - \beta(s)| \varphi^s(v; z) + |w - v|_H (\varphi^s(v; z))^{1/2}.\]

Put \(K(t) := \{z \in H | \varphi^t(z; z) < +\infty\}\).

**Theorem 4.1.** ([13, Theorem 4.1]) Assume (\(\Phi 1\)) and (\(\Phi 2\)). Then, for each \(u_0 \in K(0)\), there exists a solution \(u \in W^{1,2}(0, T; H)\) of (QSE1) satisfying \(u(0) = u_0\).

The idea of this theorem has been developed by Kenmochi, Kubo, Yamazaki, Shirakawa and Fukao [12, 16, 20, 17, 18, 2, 15, 3] and is based on the theory of time-dependent subdifferential evolution equations (TSEs). In fact, by assumption (\(\Phi 2\)), for each \(v \in W^{1,2}(0, T; H)\) the function

\[t \mapsto \Phi(t) := \varphi^t(v(t); \cdot)\]
satisfies the condition of the standard theory of TSEs developed by Kenmochi [8, 9] and Yamada [19]. Hence, there exists a unique solution of the problem:

\[
\begin{align*}
&u'(t) + \varphi^t(v(t); u(t)) \ni 0 \quad \text{a.e. } t \in (0, T), \\
&u(0) = u_0.
\end{align*}
\]

Using assumption (Φ1) and the energy inequality derived by TSE theory, we can show that there is a fixed point of the map \( v \mapsto u \) that gives a desired solution of (QSE1).

The second type of QSE is given as follows:

\[
(\text{QSE2}) \quad \mathcal{L}_{u_0}u + Au \ni 0 \quad \text{in } \mathcal{H}.
\]

Here, \( \mathcal{H} := L^2(0, T; H) \), \( A : \mathcal{H} \to \mathcal{H} \) is a QSO, \( \mathcal{L}_{u_0}u := u' \), and \( D(\mathcal{L}_{u_0}) := \{ w \in W^{1,2}(0, T; H) | w(0) = u_0 \} \).

This type of problem arises in hysteresis models, nonlocal obstacle problems, and so on (cf. [11, 1, 14, 6]). In particular, Kano, Murase and Kenmochi [7] studied this type of abstract problem by employing the theory of TSEs.

REFERENCES


