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A system of fifth-order PDE's describing surfaces containing 2 families of circular arcs and the reduction to a system of fifth-order ODE's

By

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Abstract

In our previous papers we gave a system of partial differential equations describing a surface germ including two continuous families of circular arcs, and an estimate of the dimension of its solution space. In this paper we prove that this system of partial differential equations of fifth-order reduces to a finite system of ordinary differential equations of fifth-order.

Let $f(x,y)$ be a C^5-class real valued function defined in a neighborhood of $(0,0)\in\mathbb{R}^2$ satisfying

$$f(0,0)=f_x(0,0)=f_y(0,0)=f_{xy}(0,0)=0,\quad f_{xx}(0,0)-f_{yy}(0,0)\neq 0.$$

If a surface germ $z=f(x,y)$ at the origin of $\mathbb{R}^3$ generically includes ℓ continuous families of circular arcs for an integer $\ell\geq 1$, then by our previous results in [1], [2], [3] we obtain the following system of equations:

$$(0.1)\qquad \begin{cases} Z(T_k(x,y))=0,\quad T_k(0,0)=t_k,\\ \displaystyle\sum_{j=0}^{5}\binom{5}{j}T_k(x,y)^j\partial_x^{5-j}\partial_y^j f(x,y)=\frac{24N(T_k(x,y))}{R(T_k(x,y))K(T_k(x,y))^3},\\ (k=1,\ldots,\ell).\end{cases}$$

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Here $Z(T), N(T), R(T), K(T)$, which will be given later, are the polynomials of T whose coefficients are also polynomials of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$. Further $\{t_k; k = 1, \dots, \ell\}$ are non-zero, real, simple and distinct roots of $Z(t)|_{x=y=0} = 0$, and $\{T_k; k = 1, \dots, \ell\}$ are analytic functions of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$ defined by the equation $Z(T; x, y) = 0$. As the well-known solutions of these system; that is, the surfaces including two continuous families of circles, we have Darboux cyclides [5]:

$$\alpha\Big(\sum_{i=1}^{3} x_i^2\Big)^2 + \Big(\sum_{i=1}^{3} x_i^2\Big)\sum_{i=1}^{3}\beta_i x_i + \sum_{i,j=1}^{3}\gamma_{ij}x_i x_j + \sum_{i=1}^{3}\delta_i x_i + \epsilon = 0,$$

where $(x_1, x_2, x_3) = (x, y, z)$ and $\alpha, \beta_i, \gamma_{ij}, \delta_i, \epsilon$ are real constants for $i, j = 1, 2, 3$. According to N. Takeuchi [6], any Darboux cyclide with generic coefficients $\alpha, \beta_i, \gamma_{ij}, \delta_i, \epsilon$ includes just two or six continuous families of circles off the umbilical points, which constitute a discrete subset if the surface is neither a sphere nor a plane. Further, any Darboux cyclide includes just 1—6 continuous families of circles off the umbilical points. These results are a generalization of R. Blum's result [7]. On the other hand, in [8], [9], H. Pottmann, H. L. Shi, F. Nilov and M. Skopenkov found a surface which is not a cyclide, but includes 2 continuous families of circles:

$$(x^2 + y^2 + z^2)^2 - 4y^2z^2 - 4x^2 = 0.$$

Here y =Const., or z =Const. becomes circles. Indeed, we can rewrite this as follows:

$$x = \pm\sqrt{1 - y^2} \pm \sqrt{1 - z^2}\,.$$

Further they proved the following as Theorem 3.4 of [9]:

Theorem 0.1. *Let Φ be a smooth closed surface in $\mathbb{R}^3$ homeomorphic to either a sphere or a torus. If through each point of the surface one can draw at least* **4 *distinct circles fully contained in the surface*** (*and continuously depending on the point*) *then the surface is* ***a cyclide***.

They used Takeuchi's idea on intersection numbers of fundamental groups and a classical theorem on the relationship between cospherical circles and cyclides. So the proof relies on the global information of the surface. On the other hand their counter example is not a closed surface, but a surface with singularities. At the same time, they gave a conjecture:

3 distinct continuous families of circles $\Longrightarrow$ cyclides.

Our method is based on elementary analysis and differential equations, and so it is very different from their approach based on algebraic geometry.

§ 1. Our previous results

Definition 1.1 (The Key Polynomial $Z(T)$). Let $z = f(x, y)$ be a C^4-class function defined in a neighborhood of $(0, 0) \in \mathbb{R}^2$. Put the Taylor coefficients of f at (x, y) as follows:

$$\begin{cases} a := f_x(x,y), & b := f_y(x,y), & \\ c_0 := \frac{1}{2} f_{xx}(x,y), & c_1 := f_{xy}(x,y), & c_2 := \frac{1}{2} f_{yy}(x,y), \\ d_0 := \frac{1}{3!} f_{xxx}(x,y), & d_1 := \frac{1}{2!} f_{xxy}(x,y), & \\ d_2 := \frac{1}{2!} f_{xyy}(x,y), & d_3 := \frac{1}{3!} f_{yyy}(x,y), & \\ e_0 := \frac{1}{4!} f_{xxxx}(x,y), & e_1 := \frac{1}{3!} f_{xxxy}(x,y), & e_2 := \frac{1}{2!2} f_{xxyy}(x,y), \\ e_3 := \frac{1}{3!} f_{xyyy}(x,y), & e_4 := \frac{1}{4!} f_{yyyy}(x,y). & \end{cases}$$

Then we define some polynomials and the key polynomial $Z(T)$ in T as follows:

$$\begin{aligned} &C(T) = c_0 + c_1 T + c_2 T^2, \quad D(T) = d_0 + d_1 T + d_2 T^2 + d_3 T^3, \\ &E(T) = e_0 + e_1 T + e_2 T^2 + e_3 T^3 + e_4 T^4, \\ &R(T) = (b^2+1)T^2 + 2abT + a^2 + 1, \\ &S(T) = D(T)R(T) - 2(bT+a)C(T)^2, \\ &K(T) = R'(T)C(T) - R(T)C'(T), \quad W(T) = bS(T) + C(T)K(T), \\ &Z(T) \equiv Z(T; x, y) \\ &\quad := K(T)^2\big(R(T)E(T) - C(T)^3\big) + R(T)K(T)D(T)\big(D'(T)R(T) \\ &\qquad - 3(b^2+1)TD(T)\big) + D(T)^2R(T)[-ab\big(2K(T) + TK'(T)\big) \\ &\qquad - 2(a^2+1)(b^2+1)C(T) + \big((a^2+1)c_2 + (b^2+1)c_0\big)R(T)] \\ &\qquad + 2R(T)C(T)[(bT+a)\{D(T)K'(T)C(T) + D(T)K(T)C'(T) \\ &\qquad - D'(T)K(T)C(T)\} - bD(T)C(T)K(T)] \\ &\qquad + 4C(T)^4(bT+a) \times \{\big((a^2-1)c_2 + (b^2+1)c_0\big)(bT+a) \\ &\qquad - \frac{1}{2}ac_1R'(T) + 2a(c_2 - c_0) - bc_1\}, \end{aligned}$$

where $C'(T) = \partial_T C(T), R'(T) = \partial_T R(T)$ etc.

Then the following theorems were obtained in [1], [2], [3].

Theorem 1.2. *Let $z = f(x, y)$ be a C^5-function defined in a neighborhood $U_{\delta_0} = \{(x, y);\ x^2 + y^2 < \delta_0^2\}$ $(\delta_0 > 0)$ satisfying*

$$f(0,0) = 0, f_x(0,0) = 0, f_y(0,0) = 0, f_{xy}(0,0) = 0, \tag{1.1}$$

$$c_2(0,0) - c_0(0,0) = \frac{1}{2}(f_{yy}(0,0) - f_{xx}(0,0)) \neq 0. \tag{1.2}$$

Then we have the following (i), (ii).
(i) Let $t(x,y), s(x,y)$ be real-valued continuous functions defined in a neighborhood of $(0,0)$ such that, for some $\delta > 0$ and any $(x_0, y_0) \in U_\delta$, the set

$$M \cap \{y - y_0 = t(x_0,y_0)(x - x_0) + s(x_0,y_0)(z - f(x_0,y_0))\} \tag{1.3}$$

coincides with a circle in a neighborhood of $(x_0, y_0, f(x_0,y_0))$. Assume that $t(0,0) \neq 0$ and that $Z'(t(0,0);0,0) \neq 0$.

Consider a continuous function

$$T(x,y) := \frac{t(x,y) + f_x(x,y)s(x,y)}{1 - f_y(x,y)s(x,y)} \tag{1.4}$$

defined in a neighborhood of $(0,0)$. Further assume that $t(x,y), s(x,y)$ are constant on each circular arc (1.3). Then, $T(x,y)$ is a C^1-function satisfying

$$Z(T(x,y);x,y) = 0, \tag{1.5}$$

$$(\partial_x + T(x,y)\partial_y)T(x,y) = \frac{2S(T)}{K(T)}. \tag{1.6}$$

Further, $s(x,y), t(x,y)$ are also C^1-functions written as

$$s(x,y) = \frac{S(T)}{W(T)}, \quad t(x,y) = \frac{TK(T)C(T) - aS(T)}{W(T)}. \tag{1.7}$$

(ii) Conversely, let $T(x,y)$ be a real-valued C^1-function defined in a neighborhood of $(0,0)$ satisfying $T(0,0) \neq 0$, (1.5) and (1.6). Then, $t(x,y)$, $s(x,y)$ defined by (1.7) belong to $C^1(U_\delta)$ for a $\delta > 0$, and satisfy that, for any $(x_0,y_0) \in U_\delta$, the set

$$M \cap \{y - y_0 = t(x_0,y_0)(x - x_0) + s(x_0,y_0)(z - f(x_0,y_0))\}$$

coincides with a circle in a neighborhood of $(x_0, y_0, f(x_0,y_0))$, and that $t(x,y)$, $s(x,y)$ are constant on this circular arc.

Though the equation (1.6) obtained in Theorem 1.2

$$(\partial_x + T(x,y)\partial_y)T(x,y) = \frac{2S(T)}{K(T)}$$

looks like a first order PDE, this is a fifth-order PDE for $f(x,y)$ because T is an analytic function of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$ through the equation $Z(T) = 0$. Indeed we have the following:

Theorem 1.3. *Under the conditions in Theorem 1.2, we have*

$$\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^j f(x,y)=\frac{24N(T)}{R(T)K(T)^3}, \tag{1.8}$$

where $N(T)$ is a polynomial in T of degree 14 given by

$$\begin{aligned}
N(T)=&-5R(T)K(T)^2E'(T)\big[R(T)D(T)-2(bT+a)C(T)^2\big]\\
&+D(T)^3R(T)B_1(T)+2D(T)^2D'(T)R(T)^2B_2(T)\\
&-D(T)^2R(T)^2K(T)\big[(3d_3T+d_2)(5R(T)-(b^2+1)T^2)+(d_1T+3d_0)(b^2+1)\big]\\
&+D(T)^2B_3(T)+2(bT+a)D(T)D'(T)R(T)C(T)B_4(T)\\
&+10(bT+a)D(T)D''(T)R(T)^2K(T)C(T)^2+D(T)B_5(T)\\
&-4(bT+a)D'(T)R(T)K(T)C(T)^3\big[5(bT+a)C'(T)+2bC(T)\big]\\
&+4(bT+a)C(T)^4K(T)\Big[3d_0B_6(T)+d_1B_7(T)+d_2B_8(T)-3d_3TB_9(T)\Big]\\
&+4C(T)^4B_{10}(T).
\end{aligned}$$

Further, $B_1(T),\dots,B_{10}(T)$ are some polynomials in T,a,b,c_*.

Remark. We omitted the explicit forms of $B_1(T),\dots,B_{10}(T)$ for saving the pages. The readers can get them in [2], [3], or a file "check-fifth" in our websites:

`http://www.u-gakugei.ac.jp/~nobuko/manycircles.html`
`http://agusta.ms.u-tokyo.ac.jp/microlocal/manycircles.html`

Let $M : z = f(x,y)$ be a C^5-class surface germ whose Taylor expansion at the origin satisfies conditions (1.1), (1.2) in Theorem 1.2. For an integer ℓ $(2\le \ell\le 10)$, suppose that there exist ℓ real numbers $\{t_k\}_{k=1}^{\ell}$ satisfying $t_k\neq 0$, $Z(t_k;0,0)=0$, $Z'(t_k;0,0)\neq 0$, and that M includes ℓ continuous families of circular arcs associated with $\{t_k\}_{k=1}^{\ell}$. Let $T_k(x,y)$ be the function T corresponding to the non-zero simple root t_k; that is, $T_k(0,0)=t_k$ $(k=1,\dots,\ell)$. Then f is a solution of the following system of PDE's:

$$\begin{cases}
Z(T_k(x,y))=0, \quad T_k(0,0)=t_k,\\
\displaystyle\sum_{j=0}^{5}\binom{5}{j}T_k(x,y)^j\partial_x^{5-j}\partial_y^j f(x,y)=\frac{24N(T_k(x,y))}{R(T_k(x,y))K(T_k(x,y))^3},\\
(1\le k\le \ell).
\end{cases} \tag{1.9}$$

For $\ell\ge 2$, this system is an analytic elliptic system of fifth-order equations for f. Indeed, the k-th equation is an analytic quasi-linear equation with fifth order principal symbol

$$(\xi+T_k(x,y)\eta)^5$$

(ξ, η are the symbols for ∂_x, ∂_y, respectively), and further each T_k is an analytic function of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f$ defined by the equation $Z(T_k; x, y) = 0$ for $k = 1, \dots, \ell$.

Theorem 1.4. *Let $M : z = f(x, y)$ be a $C^{5+\theta}$-class surface satisfying (1.1), (1.2) where θ $(0 < \theta < 1)$ is an exponent for Hölder continuity. Suppose that M contains two continuous families of circles in the sense of (i) of Theorem 1.2, where these families correspond to two distinct non-zero real simple roots t_1, t_2 of $Z(t; 0, 0) = 0$, respectively. Then, f is an analytic function which is uniquely determined only by the partial derivatives at $(0, 0)$ up to 8th-order. In particular, such surface-germs are classified by at most 21 real parameters A and B, where*

$$A := (\partial_x^p \partial_y^q f(0,0);\ 2 \le p + q \le 4, (p,q) \neq (1,1)),$$
$$B := \big((\partial_x + t_1\partial_y)^p(\partial_x + t_2\partial_y)^q f(0,0);\ p, q \le 4,\ 5 \le p + q \le 8\big).$$

§ 2. Our main results

Though the method of power series expansion is useful for getting the solvable condition of system (1.9) with $\ell = 2$ for an initial data (A, B), the calculation is too complicated even for one or two steps. Hence we need another method. We use one continuous family of circular arcs for an explicit expression of $M = \{z = f(x, y)\}$, and another continuous family of circular arcs for a partial differential equation for f. We prepare a lemma obtained as Lemma 2.3 in [4]:

Lemma 2.1. *Let $x_0, y_0, z_0, U_1, U_2, U_3, V_1, V_2, V_3$, and λ be real constants satisfying $U_3V_1 - U_1V_3 \neq 0$. Consider the following curve with parameter $t \in \mathbb{R} \cup \{\infty\}$ in $\mathbb{R}^3$ ($t = \infty \iff (x, y, z) = (x_0, y_0, z_0)$):*

$$x = x_0 + \frac{2\lambda(U_1 + V_1t)}{(U_1 + V_1t)^2 + (U_2 + V_2t)^2 + (U_3 + V_3t)^2},$$
$$y = y_0 + \frac{2\lambda(U_2 + V_2t)}{(U_1 + V_1t)^2 + (U_2 + V_2t)^2 + (U_3 + V_3t)^2},$$
$$z = z_0 + \frac{2\lambda(U_3 + V_3t)}{(U_1 + V_1t)^2 + (U_2 + V_2t)^2 + (U_3 + V_3t)^2}.$$

Then, this is a circle contained in a plane

$$y - y_0 = \frac{U_3V_2 - U_2V_3}{U_3V_1 - U_1V_3}(x - x_0) + \frac{U_2V_1 - U_1V_2}{U_3V_1 - U_1V_3}(z - z_0)$$

with center

$$x_C = x_0 + \frac{\lambda\big(U_1(V_2^2 + V_3^2) - V_1(U_2V_2 + U_3V_3)\big)}{(U_2V_3 - U_3V_2)^2 + (U_3V_1 - U_1V_3)^2 + (U_1V_2 - U_2V_1)^2},$$

$$y_C = y_0 + \frac{\lambda\big(U_2(V_3^2+V_1^2) - V_2(U_3V_3+U_1V_1)\big)}{(U_2V_3-U_3V_2)^2+(U_3V_1-U_1V_3)^2+(U_1V_2-U_2V_1)^2},$$
$$z_C = z_0 + \frac{\lambda\big(U_3(V_1^2+V_2^2) - V_3(U_1V_1+U_2V_2)\big)}{(U_2V_3-U_3V_2)^2+(U_3V_1-U_1V_3)^2+(U_1V_2-U_2V_1)^2},$$

and radius

$$R = \frac{|\lambda|\sqrt{V_1^2+V_2^2+V_3^2}}{\sqrt{(U_2V_3-U_3V_2)^2+(U_3V_1-U_1V_3)^2+(U_1V_2-U_2V_1)^2}}.$$

In this lemma, we put $t = 1/v$, $x_0 = 0$, $y_0 = u$, $z_0 = z_0(u)$, $\lambda = 1$, and

$$\begin{aligned} V_1 &= 1, & V_2 &= j(u)+k(u)V(u), & V_3 &= V(u),\\ U_1 &= 0, & U_2 &= k(u)U(u), & U_3 &= U(u). \end{aligned}$$

Then we obtain an expression of a surface M in $\mathbb{R}^3$ parametrized by (u, v) as

$$x = \frac{2v}{1+\big(k(u)U(u)v+j(u)+k(u)V(u)\big)^2+\big(U(u)v+V(u)\big)^2},$$

$$y = \frac{2\big(k(u)U(u)v+j(u)+k(u)V(u)\big)v}{1+\big(k(u)U(u)v+j(u)+k(u)V(u)\big)^2+\big(U(u)v+V(u)\big)^2}+u,$$

$$z = \frac{2\big(U(u)v+V(u)\big)v}{1+\big(k(u)U(u)v+j(u)+k(u)V(u)\big)^2+\big(U(u)v+V(u)\big)^2}+z_0(u),$$

by using 5 analytic functions

$$z_0(u),\ j(u),\ k(u),\ U(u),\ V(u)$$

of one-variable u. Here, u is the parameter concerning the family of circles, and v is the parameter for each circle. Since

$$\begin{aligned} M \cap \{x=0\} &= \{(0, u, z_0(u));\ u \in \mathbb{R}\},\\ y(u,v) - u &= j(u)x(u,v) + k(u)(z(u,v) - z_0(u)), \end{aligned}$$

we can choose an almost arbitrary initial curve $M \cap \{x = 0\}$ and an almost arbitrary normal vector $(j(u), -1, k(u))$ to the circle corresponding to u along $M \cap \{x = 0\}$ by suitable choices of functions $z_0(u), j(u), k(u)$. Further the center $(x_C(u,v), y_C(u,v), z_C(u,v))$ of the circle satisfies

$$(x_C, y_C - u, z_C - z_0(u)) = \frac{1}{U(u)}\vec{e_1}(u) + \frac{V(u)}{U(u)}\vec{e_2}(u)$$

with

$$\vec{e_1}(u) = \Big(\frac{-j(u)k(u)}{1+j(u)^2+k(u)^2}, \frac{k(u)}{1+j(u)^2+k(u)^2}, \frac{1+j(u)^2}{1+j(u)^2+k(u)^2}\Big),$$
$$\vec{e_2}(u) = \Big(\frac{-(1+k(u)^2)}{1+j(u)^2+k(u)^2}, \frac{-j(u)}{1+j(u)^2+k(u)^2}, \frac{j(u)k(u)}{1+j(u)^2+k(u)^2}\Big).$$

Since $\vec{e_1}(u) \times \vec{e_2}(u) = \dfrac{(j(u), -1, k(u))}{j(u)^2+k(u)^2+1}$, $\vec{e_1}(u), \vec{e_2}(u)$ span the plane

$$H := \{(x,y,z); y = j(u)x + k(u)z\}.$$

Hence our parametrization $(x(u,v), y(u,v), z(u,v))$ covers almost arbitrary circular surfaces M. Next, we can use u, v as the independent variables instead of x, y. Thus, we write

$$\partial_x = m_1\partial_u + m_2\partial_v, \qquad \partial_y = m_3\partial_u + m_4\partial_v,$$

where m_1, m_2, m_3, m_4 are given as follows:

$$m_1(u,v) := \frac{y_v}{x_u y_v - x_v y_u},$$
$$m_2(u,v) := -\frac{y_u}{x_u y_v - x_v y_u},$$
$$m_3(u,v) := -\frac{x_v}{x_u y_v - x_v y_u},$$
$$m_4(u,v) := \frac{x_u}{x_u y_v - x_v y_u}.$$

Therefore we can express the fifth-order PDE related to the second continuous family of circles by using v and the derivatives of $z_0(u), j(u), k(u), U(u), V(u)$ up to fifth order. Finally we get a fifth-order ordinary differential equation for $z_0(u), j(u), k(u), U(u), V(u)$ with analytic parameter v. However the calculations are too complicated to obtain any information about this ordinary differential equation.

Theorem 2.2. *Let $M : z = f(x,y)$ be a $C^{5+\theta}$-class surface satisfying* (1.1), (1.2) *where θ $(0 < \theta < 1)$ is an exponent for Hölder continuity. Suppose that M contains two continuous families of circles in the sense of (ii) of Theorem* 1.2*, where these families correspond to two distinct non-zero real simple roots t_1, t_2 of $Z(t;0,0) = 0$, respectively. Then, f is described by* 5 *analytic functions $z_0(u), j(u), k(u), U(u), V(u)$ of one-variable u satisfying*

$$z_0(0) = z_0'(0) = V(0) = V'(0) - t_1 z_0''(0) = 0, \ j(0) = t_1, \ U(0) \neq 0,$$

and an ordinary differential equation of fifth-order with analytic parameter v:

$$\begin{cases} Z(T_2(x,y)) = 0, \quad T_2(0,0) = t_2, \\ \displaystyle\sum_{j=0}^{5} \binom{5}{j} T_2(x,y)^j \partial_x^{5-j}\partial_y^j f(x,y) = \frac{24N(T_2(x,y))}{R(T_2(x,y))K(T_2(x,y))^3}, \end{cases}$$

where $x = x(u,v), y = y(u,v), f(x,y) = z(u,v)$.

Theorem 2.3. *Under the same conditions as Theorem* 2.2, *we have the following equivalent conditions for* 5 *analytic functions* $z_0(u), j(u), k(u), U(u), V(u)$: *There exist some real polynomials*

$$P_\ell(z_0^{(i_1)}(u), j^{(i_2)}(u), k^{(i_3)}(u), U^{(i_4)}(u), V^{(i_5)}(u); i_1, \dots, i_5 = 0, 1, \dots, 5)$$

$(\ell = 1, \dots, L)$ *of* 30 *variables such that*

$$\begin{cases} z_0(0) = z_0'(0) = V(0) = V'(0) - t_1 z_0''(0) = 0, \ j(0) = t_1, \\ P_\ell\big(z_0^{(i_1)}(u), j^{(i_2)}(u), k^{(i_3)}(u), U^{(i_4)}(u), V^{(i_5)}(u)\big) = 0 \ (\ell = 1, \dots, L) \end{cases}$$

with $U(0) \neq 0$.

A sketch of the proof of Theorem 2.3. We can rewrite the equations in Theorem 2.2 for $z_0(u), j(u), k(u), U(u), V(u)$ as follows:

$$\begin{cases} Z(T_2) = 0, \\ 24N(T_2) - R(T_2)K(T_2)^3 \sum_{j=0}^{5} \binom{5}{j} (\partial_x^{5-j}\partial_y^j f)\, T_2^j = 0, \\ T_2(0,0) = t_2, \quad j(0) = t_1, \\ z_0(0) = z_0'(0) = V(0) = V'(0) - t_1 z_0''(0) = 0, \end{cases}$$

with $U(0) \neq 0$, where $x = x(u,v)$, $y = y(u,v)$, $f(x,y) = z(u,v)$. Then we can consider the first and the second equations as algebraic equations for T_2 with coefficients in $\mathcal{K}(v)$; $\mathcal{K}(v)$ is the field of rational functions of v with coefficients in the quotient field $\mathcal{K} :=$ Frac$(\mathcal{O}_{u,0})$ ($\mathcal{O}_{u,0}$ is the integral domain of all the germs at $u = 0$ of analytic functions in u). In fact, the left sides of the first and the second equations are polynomials in T_2 of degree 10 and 14, respectively. Put

$$X(T) := 24N(T) - R(T)K(T)^3 \sum_{j=0}^{5} \binom{5}{j} (\partial_x^{5-j}\partial_y^j f)\, T^j.$$

Then we have $Z(T_1) = X(T_1) = 0$, where

$$T_1 = \frac{j(u) + k(u) f_x}{1 - k(u) f_y} \in \mathcal{K}(v).$$

This is because $M : z = f(x,y)$ includes the first continuous family of circular arcs. Therefore we have $\tilde{Z}(T), \tilde{X}(T) \in \mathcal{K}(v)[T]$ satisfying $Z(T) = (T - T_1)\tilde{Z}(T)$, $X(T) = (T - T_1)\tilde{X}(T)$. Consequently we get the following equations for T_2:

$$\tilde{Z}(T_2) = 0, \quad \tilde{X}(T_2) = 0.$$

Since $T_2(x(u,v),y(u,v)) \in \mathcal{O}_{(u,v)=(0,0)}$, there exists an irreducible polynomial $Y(T) \in \mathcal{K}(v)[T]$ such that $Y(T)|\tilde{Z}(T)$, $Y(T)|\tilde{X}(T)$, and $Y(T_2)=0$. Put $q := \text{degree}(Y(T)) \in \{1,2,\ldots,9\}$. Then we have the nine-cases corresponding to q: Applying the Euclidean algorithm for finding the common divisor to the system $\tilde{Z}(T),\tilde{X}(T)$ in $\mathcal{K}(v)[T]$ up to $(10-q)$-times, we get the vanishing conditions for the remainder term, which are equations in $\mathcal{K}(v)$. Hence, finally we obtain some polynomials

$$P_\ell(z_0^{(i_1)}(u), j^{(i_2)}(u), k^{(i_3)}(u), U^{(i_4)}(u), V^{(i_5)}(u); i_1,\ldots,i_5 = 0,1,\ldots,5)$$

$(\ell = 1,\ldots,L)$ such that $P_\ell = 0$ $(\forall \ell = 1,\ldots,L)$.

An example of successful calculations. Let $M = \{(x(u,v),y(u,v),z(u,v))\}$ be a C^1-surface for C^1-functions $z_0(u), j(u), k(u), U(u)(>0), V(u)$. Suppose that at any point $(x(u_0,v_0),y(u_0,v_0),z(u_0,v_0)) \in M$ the tangent plane to M is orthogonal to the plane including the circle $M \cap \{u = u_0\}$. Indeed, any sphere and any torus are the examples of such special surfaces. Then we have the following equivalent equations for $z_0(u), j(u), k(u)$:

$$\begin{aligned}
z_0'(u) &= \frac{-k(u)+j(u)V(u)}{1+j(u)^2+j(u)k(u)V(u)},\\
j'(u) &= \frac{1}{U(u)(k(u)-j(u)V(u))}\Big(j(u)U(u)^2 + j(u)^3U(u)^2 + j(u)k(u)^2U(u)^2\\
&\quad + V(u)U'(u) + j(u)^2V(u)U'(u) + k(u)^2V(u)U'(u)\\
&\quad - U(u)V'(u) - j(u)^2U(u)V'(u) - k(u)^2U(u)V'(u)\Big),\\
k'(u) &= \frac{-1}{U(u)(k(u)-j(u)V(u))(1+j(u)^2+j(u)k(u)V(u))}\\
&\quad \times \Big(-j(u)^2k(u)U(u)^2 - j(u)^4k(u)U(u)^2 - j(u)^2k(u)^3U(u)^2\\
&\quad - j(u)U(u)^2V(u) - j(u)^3U(u)^2V(u) - 2j(u)k(u)^2U(u)^2V(u)\\
&\quad - j(u)^3k(u)^2U(u)^2V(u) - j(u)k(u)^4U(u)^2V(u) + U'(u) + 2j(u)^2U'(u)\\
&\quad + j(u)^4U'(u) + k(u)^2U'(u) + j(u)^2k(u)^2U'(u) + j(u)k(u)V(u)U'(u)\\
&\quad + j(u)^3k(u)V(u)U'(u) + j(u)k(u)^3V(u)U'(u)\Big).
\end{aligned}$$

Hence we can take arbitrary C^1-functions as $U(u)(>0), V(u)$. On the other hand, N. Takeuchi proved in "A sphere as a surface which contains many circles. II, *J. Geom.* **34** (1989), 195–200" that under some global conditions such M is a sphere; for example, M is a simply connected complete smooth surface, or a compact smooth surface. The precise versions are as follows:

(1) Let M be a simply connected complete smooth surface in E^3. For each point $p \in M$, suppose that there exists a circle through p on M which is contained in a normal plane of M at p. Then M is a sphere.

(2) Let M be a compact smooth surface in E^3. For each point $p \in M$, suppose that there exists a normal plane $N(p)$ of M at p such that $N(p) \cap M$ is a circle (hence, any torus is excluded). Then M is a sphere.

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