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Maillet type theorem, convolution equations and multisummability of formal solutions

By

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Abstract

Let $P(\lambda)$ be a polynomial of degree m. In this note, we consider the following linear singular partial differential equation

$$\text{(E)} \qquad P(t\partial_t)u = \sum_{j+|\alpha|\leq L} a_{j,\alpha}(t)(t\partial_t)^j\partial_x^\alpha u + f(t,x)$$

with $(t,x)\in\mathbb{C}_t\times\mathbb{R}_x^N$ (or $(t,x)\in\mathbb{C}_t\times\mathbb{C}_x^N$) and with holomorphic coefficients $a_{j,\alpha}(t)$. First, we present a Maillet type theorem for formal solutions of this equation (E), then we give an analogue of Maillet type theorem in convolution partial differential equations, and finally we give an application to multisummability of formal solutions of (E). Only the results are written in this note: the details will be published elsewhere.

§ 1. Preliminaries

We denote by (t,x) the variables in $\mathbb{C}_t\times\mathbb{R}_x^N$. Let $D_r=\{t\in\mathbb{C}\,;\,|t|<r\}$ with $r>0$, and let V be an open subset of $\mathbb{R}_x^N$. For $\sigma>0$, we denote by $G^{\{\sigma\}}(V)$ the set of all functions $u(x)\in C^\infty(V)$ satisfying the estimates

$$\sup_{x\in V}|\partial_x^\alpha u(x)|\leq Ch^{|\alpha|}(|\alpha|!)^\sigma, \quad \forall\alpha\in\mathbb{N}^N$$

for some $C>0$ and $h>0$. A function $u(x)\in G^{\{\sigma\}}(V)$ is called a function of the Gevrey class of order σ. For $u(x)\in G^{\{\sigma\}}(V)$ we write

$$|||u|||_\rho=\sum_{|\alpha|\geq 0}\frac{\|\partial_x^\alpha u\|_V}{(|\alpha|!)^\sigma}\rho^{|\alpha|}$$

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where $\|\cdot\|_V$ denotes the supremum norm on V. We see: a function $u(x) \in C^\infty(V)$ belongs to the class $G^{\{\sigma\}}(V)$ if and only if $|||u|||_\rho$ is convergent in a neighborhood of $\rho = 0$.

Similarly, we denote by $G^{\{1,\sigma\}}(D_r \times V)$ the set of all functions $f(t,x) \in C^\infty(D_r \times V)$ holomorphic in $t \in D_r$ and satisfying the estimates

$$\sup_{(t,x)\in D_r\times V} |\partial_x^\alpha f(t,x)| \le Ch^{|\alpha|}(|\alpha|!)^\sigma, \quad \forall \alpha \in \mathbb{N}^N$$

for some $C > 0$ and $h > 0$. We write also

$$|||f(t)|||_\rho = \sum_{|\alpha|\ge 0} \frac{\|\partial_x^\alpha f(t)\|_V}{(|\alpha|!)^\sigma} \rho^{|\alpha|}.$$

§ 2. Maillet type theorem

Let m be a positive integer, let Λ be a finite subset of $\mathbb{N} \times \mathbb{N}^n$, and let us consider the following model equation:

$$P(t\partial_t)u = \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t)(t\partial_t)^j \partial_x^\alpha u + f(t,x), \tag{2.1}$$

where

$$P(\lambda) = \lambda^m + c_1\lambda^{m-1} + \cdots + c_{m-1}\lambda + c_m$$

is a polynomial of degree m, $a_{j,\alpha}(t)$ ($(j,\alpha) \in \Lambda$) are holomorphic functions on D_r and $f(t,x) \in G^{\{1,\sigma\}}(D_r \times V)$. It is easy to see that if the conditions

$$P(n) \ne 0 \ \text{ for any } n = 0, 1, 2, \dots, \ \text{ and} \tag{2.2}$$

$$a_{j,\alpha}(0) = 0 \ \text{ for any } (j,\alpha) \in \Lambda \tag{2.3}$$

are satisfied, the equation (2.1) has a unique formal solution

$$\hat{u}(t,x) = \sum_{n=0}^\infty u_n(x)t^n \in G^{\{\sigma\}}(V)[[t]]. \tag{2.4}$$

We denote by $\mathrm{ord}_t(a)$ the order of the zero of the function $a(t)$ at $t = 0$. We set $q_{j,\alpha} = \mathrm{ord}_t(a_{j,\alpha})$ ($(j,\alpha) \in \Lambda$): since (2.3) is supposed, we have $q_{j,\alpha} \ge 1$ for any $(j,\alpha) \in \Lambda$. We define the index $s \ge 1$ by

$$s = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda} \frac{j + \sigma|\alpha| - m}{q_{j,\alpha}}\right]. \tag{2.5}$$

About the estimates of the coefficients $u_n(x)$ ($n = 0, 1, \dots$) of the formal solution (2.4), we have:

Theorem 2.1 (Maillet type theorem). *Suppose the conditions* (2.2) *and* (2.3). *Let* $\hat{u}(t,x)$ *be the unique formal solution of* (2.1). *Then, there are constants* $A>0$, $H>0$ *and* $\rho>0$ *such that*

$$\|u_n\|_\rho \le AH^n n!^{s-1}, \quad n=0,1,2,\dots \tag{2.6}$$

In [2], this kind of theorem is called *a Maillet type theorem.* Similar results are obtained in [8] for formal solutions in $G^\sigma(V)[[t]]$ of nonlinear partial differential equations in the case $\sigma \ge 1$. We note that in the above theorem our assumption is $\sigma > 0$.

§ 3. Convolution partial differential equations

Next, let us give an analogue of Maillet type theorem in the following convolution partial differential equation

$$P(k\xi^k)w = f(\xi,x) + \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(\xi) *_k (\mathscr{M}_{j,\alpha}[\partial_x^\alpha w]) \ \text{ on } S_I \times V,$$

where $P(\lambda)$ is a polynomial of degree m, and Λ is a finite subset of $\mathbb{N}\times\mathbb{N}^n$.

§ 3.1. An analogue of Maillet type theorem

For an open interval $I=(\theta_1,\theta_2)$ we write $S_I=\{\xi\in\mathcal{R}(\mathbb{C}\setminus\{0\})\,;\,\theta_1<\arg\xi<\theta_2\}$ (where $\mathcal{R}(\mathbb{C}\setminus\{0\})$ denotes the universal covering space of $\mathbb{C}\setminus\{0\}$), and $|I|=\theta_2-\theta_1$. For $k>0$ and two holomorphic functions $f(\xi)$ and $g(\xi)$ on S_I we define the k-convolution $(f *_k g)(\xi)$ of $f(\xi)$ and $g(\xi)$ by

$$(f *_k g)(\xi) = \int_0^\xi f(\tau) g((\xi^k-\tau^k)^{1/k})\,d\tau^k, \quad \xi\in S_I.$$

Let $k>0$ and $\sigma>0$ be fixed. For $(j,\alpha)\in\mathbb{N}\times\mathbb{N}^n$ we write

$$\mathscr{M}_{j,\alpha}[W] = \begin{cases} \dfrac{\xi^{k\sigma|\alpha|-k}}{\Gamma(\sigma|\alpha|)} *_k \left((k\xi^k)^j W\right), & \text{if } |\alpha|>0,\\ (k\xi^k)^j W, & \text{if } |\alpha|=0. \end{cases}$$

In this section, as a model we will consider the following convolution partial differential equation

$$P(k\xi^k)w = f(\xi,x) + \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(\xi) *_k (\mathscr{M}_{j,\alpha}[\partial_x^\alpha w]) \tag{3.1}$$

on $S_I \times V$, where

$$P(\lambda) = \lambda^m + c_1\lambda^{m-1} + \cdots + c_{m-1}\lambda + c_m$$

is a polynomial of degree m. We suppose: $k > 0$ is a real number, $0 < |I| < 2\pi/k$, $\sigma > 0$, $f(\xi, x) \in G^{\{1,\sigma\}}(S_I \times V)$, and $a_{j,\alpha}(\xi)$ $((j,\alpha) \in \Lambda)$ are holomorphic functions on the sector S_I. Moreover, we suppose that there are real numbers $\mu > 0$ and $q_{j,\alpha} > 0$ $((j,\alpha) \in \Lambda)$ such that the estimates

$$|||f(\xi)|||_\rho \leq F|\xi|^{\mu-k} \exp(c|\xi|^k) \quad \text{on } S_I,$$
$$|a_{j,\alpha}(\xi)| \leq A_{j,\alpha}|\xi|^{q_{j,\alpha}-k} \exp(c|\xi|^k) \quad \text{on } S_I \quad ((j,\alpha) \in \Lambda)$$

hold for some $\rho > 0$, $F \geq 0$, $c > 0$ and $A_{j,\alpha} \geq 0$ $((j,\alpha) \in \Lambda)$.

Under these assumptions, we set

$$s = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda}\left(\frac{j + \sigma|\alpha| - m}{q_{j,\alpha} + k[j + \sigma|\alpha| - m]_+}\right)\right]. \tag{3.2}$$

For a real number x we write $[x]_+ = \max\{x, 0\}$. We set

$$\mathcal{K} = \{q_{j,\alpha} + k[j + \sigma|\alpha| - m]_+ \,;\, (j,\alpha) \in \Lambda\} :$$

since this is a finite set, we can write $\mathcal{K} = \{\kappa_1, \ldots, \kappa_\ell\}$ where $\kappa_1, \ldots, \kappa_\ell$ are distinct positive real numbers. We set

$$\mathcal{N} = \mu + \sum_{i=1}^{\ell} \mathbb{N}\kappa_i,$$

that is, a real number n belongs to $\mathcal{N}$ if and only if n is expressed in the form $n = \mu + \kappa_1 q_1 + \cdots + \kappa_\ell q_\ell$ for some $q_i \in \mathbb{N}$ $(i = 1, 2, \ldots, \ell)$. Since $\mathcal{N}$ is a discreat set of positive real numbers, we can write it in the form $\mathcal{N} = \{n_0, n_1, n_2, \ldots\}$ with $n_0 = \mu$, $0 < n_0 < n_1 < n_2 < \ldots$, and $n_p \longrightarrow \infty$ (as $p \longrightarrow \infty$).

We let $\lambda_1, \ldots, \lambda_m$ be the roots of $P(\lambda) = 0$. We denote by $p : \mathcal{R}(\mathbb{C} \setminus \{0\}) \longrightarrow \mathbb{C}$ the natural projection. We have

Theorem 3.1 (Analog of Maillet type theorem). *Suppose the condition*

$$\lambda_i \in \mathbb{C} \setminus \overline{p(S_{kI})} \quad \textit{for } i = 1, 2, \ldots, m.$$

Then, the equation (3.1) *has a formal solution*

$$w(t, x) = \sum_{n\in\mathcal{N}} w_n(\xi, x), \quad w_n(\xi, x) \in G^{\{1,\sigma\}}(S_I \times V) \ (n \in \mathcal{N})$$

which satisfies the estimates

$$|||w_n(\xi)|||_\rho \leq \frac{AH^n n!^{s-1}}{\Gamma(n/k)} \frac{|\xi|^{n-k}}{(|\xi|^k + 1)^m} \exp(c_1|\xi|^k) \quad \textit{on } S_I, \ \forall n \in \mathcal{N} \tag{3.3}$$

for some $\rho > 0$, $A > 0$, $H > 0$ and $c_1 > 0$.

This is an analogue of Maillet type theorem. We note that the formula (3.2) is very similar to the formula (2.5) in Maillet type theorem: this indicates that we can prove Theorem 3.1 by a similar argument to the proof of Theorem 2.1.

§ 3.2. Analytic continuation in ξ

Let us show the possibility of analytic continuation of the solution of (3.1). First we define $k_1 > 0$ by the following:

Lemma 3.2. *Let s be the one in* (3.2). *Then we have:*

(1) $s = 1$ *holds, if and only if* $j + \sigma|\alpha| \leq m$ *holds for any* $(j, \alpha) \in \Lambda$. *In this case, we set* $k_1 = k$.

(2) $s > 1$ *holds, if and only if* $j + \sigma|\alpha| > m$ *holds for some* $(j, \alpha) \in \Lambda$. *In this case, we have* $s - 1 < 1/k$ *and so we can define a real number* $k_1 > 0$ *by the relation* $1/k_1 = 1/k - (s - 1)$.

For $\varepsilon > 0$ we write $S_I(\varepsilon) = \{\xi \in S_I\,;\, 0 < |\xi| < \varepsilon\}$. By combining the estimate (3.3) in Theorem 3.1 with the argument in [7] we have

Theorem 3.3 (Analytic continuation). *Suppose the condition*

$$\lambda_i = 0 \ \text{ or } \ \lambda_i \in \mathbb{C} \setminus \overline{p(S_{kI})} \quad \text{for } i = 1, 2, \ldots, m.$$

If a function $w(\xi, x) \in G^{\{1,\sigma\}}(S_I(\varepsilon) \times V)$ *(where* $\varepsilon > 0$*) satisfies* (3.1) *and*

$$|||w(\xi)|||_\rho \leq C|\xi|^{\mu-k} \quad \text{on } S_I(\varepsilon)$$

for some $C > 0$, *then* $w(\xi, x)$ *has an analytic continuation* $w^*(\xi, x) \in G^{\{1,\sigma\}}(S_I \times V)$ *as a solution of* (3.1) *that satisfies the following: for any* $I_1 \Subset I$ *there are* $\rho_1 > 0$, $M > 0$ *and* $c_1 > 0$ *such that*

$$|||w^*(\xi)|||_{\rho_1} \leq \frac{M|\xi|^{\mu-k}}{(|\xi|^k + 1)^m} \exp(c_1|\xi|^{k_1}) \quad \text{on } S_{I_1}. \tag{3.4}$$

§ 4. Entire functions of finite order

For $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$ we write $|x| = |x_1| + \cdots + |x_N|$. We say that $f(x)$ is an *entire function* if it is a holomorphic function on $\mathbb{C}^N$: for $\gamma > 0$ we say that $f(x)$ is an *entire function of order* γ if it is a holomorphic function on $\mathbb{C}^N$ satisfying

$$|f(x)| \leq A \exp(a|x|^\gamma) \quad \text{on } \mathbb{C}^N$$

for some $A > 0$ and $a > 0$. We denote by $\mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N)$ the set of all entire functions of order γ. Similarly, for $\delta > 0$ and $\gamma > 0$ we denote by $\mathrm{Exp}^{\{\gamma\}}(D_\delta \times \mathbb{C}^N)$ the set of all holomorphic functions $u(t,x)$ on $D_\delta \times \mathbb{C}^N$ satisying the estimate

$$|u(t,x)| \le B\exp(b|x|^\gamma) \quad \text{on } D_\delta \times \mathbb{C}^N$$

for some $B > 0$ and $b > 0$.

For $\gamma > 0$ we set $\sigma = 1 - 1/\gamma$; then we have $\sigma < 1$ and $\gamma = 1/(1-\sigma)$. As to the estimates of derivatives of entire function, we have the following

Proposition 4.1. *Let $\gamma > 0$ and let $f(x)$ be an entire function. Set $\sigma = 1 - 1/\gamma$. The following two conditions are equivalent:*

(1) *$f(x)$ belongs to the class $\mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N)$.*

(2) *For any compact subset K of $\mathbb{C}^N$ there are $A > 0$ and $h > 0$ such that the following estimates hold:*

$$\max_{x\in K} |\partial_x^\alpha f(x)| \le A h^{|\alpha|}(|\alpha|!)^\sigma, \quad \forall \alpha \in \mathbb{N}^N. \tag{4.1}$$

This result says that if $f(x)$ belongs to $\mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N)$, we have the estimates (4.1) which is the same as the estimates of a function in the Gevrey class of order $\sigma = 1 - 1/\gamma$. We note that $0 < \sigma < 1$ is equivalent to $\gamma > 1$. Therefore, Theorems 2.1, 3.1 and 3.3 are valid also in the case where we repace $G^{\{1,\sigma\}}(D_r \times V)$ by $\mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C}^N)$ with $\gamma > 1$. This leads us to the next §5.

Before proceeding to §5, let us explain how to use the $\mathrm{Exp}^{\{\gamma\}}$-version of Theorem 3.3. For example, let us consider

$$P(t^{k+1}\partial_t)u = F(t,x) + \sum_{(j,\alpha)\in\Lambda} A_{j,\alpha}(t)\big(t^{k\sigma|\alpha|}(t^{k+1}\partial_t)^j \partial_x^\alpha u\big) \tag{4.2}$$

where $k > 0$, $F(t,x) \in \mathrm{Exp}^{\{\gamma\}}(S_I(\delta) \times \mathbb{C}^N)$ with $|F(t,x)| \le A|t|^\mu \exp(a|x|^\gamma)$ for some $A > 0$, $\mu > 0$ and $a > 0$, and $A_{j,\alpha}(t)$ is a holomorphic function on $S_I(\delta)$ with $A_{j,\alpha}(t) = O(t^{q_{j,\alpha}})$ (as $t \longrightarrow 0$) for some $q_{i,j,\alpha} > 0$. If $|I| > \pi/k$ holds, by applying the k-Borel transform $\mathcal{B}_k[\cdot]$ to (4.2) we have

$$P(k\xi^k)w = f(\xi,x) + \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(\xi) *_k (\mathscr{M}_{j,\alpha}[\partial_x^\alpha w]) \tag{4.3}$$

with $w(\xi,x) = \mathcal{B}_k[u](\xi,x)$, $f(\xi,x) = \mathcal{B}_k[F](\xi,x)$, $a_{j,\alpha}(\xi) = \mathcal{B}_k[A_{j,\alpha}](\xi)$ $((j,\alpha) \in \Lambda)$. Thus, we can apply Theorem 3.3 with $\sigma = 1 - 1/\gamma$.

In the above calculation, we used the k-Borel transform $\mathcal{B}_k[F](\xi,x)$ of $F(t,x)$ etc. in the form

$$\mathcal{B}_k[F](\xi,x) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathscr{C}(\xi)} \exp((\xi/t)^k)F(t)dt^{-k}$$

where $\mathscr{C}(\xi)$ is a contour starting from $0e^{i(\arg\xi+\pi/2k+d)}$ and ending to $0e^{i(\arg\xi-\pi/2k-d)}$ with $0 < d < \min\{\arg\xi - \theta_1, \theta_2 - \arg\xi, \pi/k\}$ in $S_I(\delta)$.

§ 5. Multisummability of formal solutions

In this last section, we will give an application of results (with $G^{\{1,\sigma\}}(D_r \times V)$ replaced by $\mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C}^N)$) in §§2 and 3 to the problem of multisummability of formal solutions of the equation

$$P(t\partial_t)u = \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t)(t\partial_t)^j\partial_x^\alpha u + f(t,x), \tag{E}$$

where $(t,x) \in \mathbb{C}_t \times \mathbb{C}_x^N$,

$$P(\lambda) = \lambda^m + c_1\lambda^{m-1} + \cdots + c_{m-1}\lambda + c_m$$

is a polynomial of degree m, $a_{j,\alpha}(t)$ ($(j,\alpha) \in \Lambda$) are holomorphic functions on D_r and $f(t,x) \in \mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C}^N)$. As before, we denote by $\mathrm{ord}_t(a_{j,\alpha})$ the order of the zero of $a_{j,\alpha}(t)$ at $t = 0$. Without loss of generality we may suppose that $a_{j,\alpha}(t) \not\equiv 0$ for all $(j,\alpha) \in \Lambda$.

As is seen in §2, if the conditions

$$\begin{aligned} &P(n) \neq 0 \quad \text{for any } n = 0, 1, 2, \ldots, \quad \text{and} \\ &a_{j,\alpha}(0) = 0 \quad \text{for any } (j,\alpha) \in \Lambda \end{aligned}$$

are satisfied, we know that the equation (E) has a unique formal solution

$$\hat{u}(t,x) = \sum_{n=0}^{\infty} u_n(x)t^n \in \mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N)[\![t]\!]. \tag{5.1}$$

If $\Lambda \subset \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^N\,;\, j + |\alpha| \leq m\}$, by Theorem 2.1 and Lemma 3.2 we see that the formal solution $u(t,x)$ is convergent in a neighborhood of $(t,x) = (0,0)$. If otherwise, that is, if

$$j + |\alpha| > m \quad \text{for some } (j,\alpha) \in \Lambda, \tag{5.2}$$

this formal solution is not convergent in general. Thus, our problem is:

Problem 5.1. Under what condition on γ (in the assumption $f(t,x) \in \mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C}^N)$), is the formal solution (5.1) multisummable?

As to the definition of multisummability, we can refer to [5] and [1]. Standard arguments on the summability or multisummability of formal solutions in partial differential equations can be found in [7] and [3]. In the case of heat equation, the necessary and sufficient condition for the formal solution to be Borel summable is established in [4]. See also [6].

§ 5.1. Newton polygon with respect to t

For $(a,b) \in \mathbb{R}^2$, we write $C(a,b) = \{(x,y)\,;\, x \le a, y \ge b\}$. We define the t-Newton polygon $N_t(\mathrm{E})$ of the equation (E) by the convex hull of the union of sets $C(m,0)$ and $C(j, \mathrm{ord}_t(a_{j,\alpha}))$ $((j,\alpha) \in \Lambda)$; that is,

$$N_t(\mathrm{E}) = \text{the convex hull of } \Big[C(m,0) \cup \bigcup_{(j,\alpha)\in\Lambda} C(j, \mathrm{ord}_t(a_{j,\alpha})) \Big].$$

Note that the term $t^p (t\partial_t)^j \partial_x^\alpha$ corresponds to $C(j,p)$ (not $C(j+|\alpha|, p)$): therefore, our t-Newton polygon is different from the usual Newton polygon. We are observing only the t-variable. The figure of $N_t(\mathrm{E})$ can be drawn as follows:

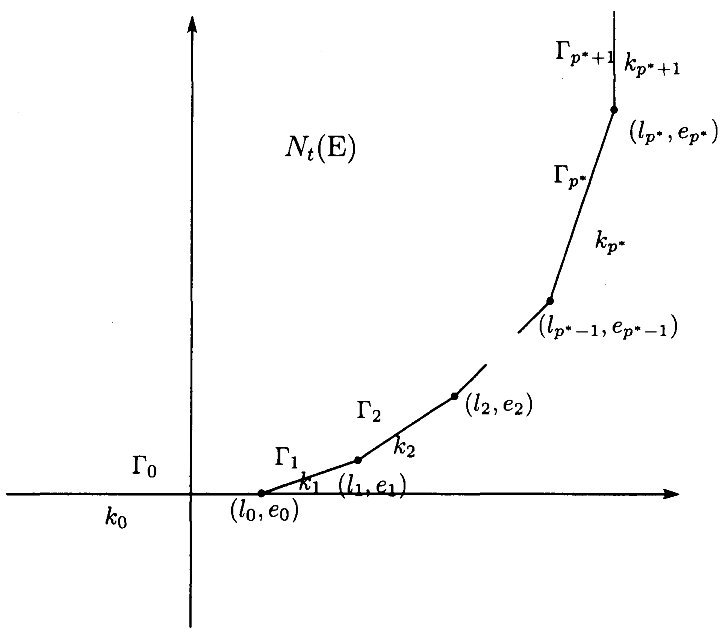

Figure 1. t-Newton polygon

As is seen in Figure 1, the verteces of $N_t(\mathrm{E})$ consists of p^*+1 points

$$(l_0, e_0),\ (l_1, e_1),\ (l_2, e_2), \ldots,\ (l_{p^*-1}, e_{p^*-1}),\ (l_{p^*}, e_{p^*});$$

the boundary of $N_t(\mathrm{E})$ consists of a horizontal half line Γ_0, p^*-segments $\Gamma_1, \Gamma_2, \ldots, \Gamma_{p^*}$, and a vertical half line Γ_{p^*+1}. We denote the slope of Γ_i by k_i $(i = 0, 1, 2, \ldots, p^*+1)$;

then we have

$$k_0 = 0 < k_1 < k_2 < \cdots < k_{p^*} < k_{p^*+1} = \infty.$$

Since $\mathrm{ord}_t(a_{j,\alpha}) \geq 1$ is supposed, we have $(l_0, e_0) = (m, 0)$.

We denote by $(N_t(\mathrm{E}))^\circ$ the interior of the set $N_t(\mathrm{E})$. From now, we suppose the following condition:

$$(j,\alpha) \in \Lambda \text{ and } |\alpha| > 0 \Longrightarrow (j, \mathrm{ord}_t(a_{j,\alpha})) \in (N_t(\mathrm{E}))^\circ \tag{5.3}$$

which is equivalent to

$$(j, \mathrm{ord}_t(a_{j,\alpha})) \in \bigcup_{i=1}^{p^*+1} \Gamma_i \Longrightarrow |\alpha| = 0.$$

§ 5.2. Singular directions

In the case $p^* \geq 1$, let us define the set of singular directions. For $i = 1, 2, \ldots, p^*$, we set

$$I_i = \{(j, 0) \in \Lambda\, ;\, (j, \mathrm{ord}_t(a_{j,0})) \in \Gamma_i\}, \quad i = 1, 2, \ldots, p^*.$$

For $(j, 0) \in I_1 \cup I_2 \cup \cdots \cup I_{p^*}$ we set $q_{j,0} = \mathrm{ord}_t(a_{j,0})$; then we have

$$a_{j,0}(t) = t^{q_{j,0}} a^0_{j,0}(t) \text{ with } a^0_{j,0}(0) \neq 0$$

for some holomorphic function $a^0_{j,0}(t)$. We set

$$P_1(\lambda) = \sum_{(j,0)\in I_1} a^0_{j,0}(0)\lambda^{j-m} - 1 = a^0_{l_1,0}(0)\lambda^{l_1-m} + \cdots - 1, \tag{5.4}$$

and for $2 \leq i \leq p^*$, we set

$$P_i(\lambda) = \sum_{(j,0)\in I_i} a^0_{j,0}(0)\lambda^{j-l_{i-1}} = a^0_{l_i,0}(0)\lambda^{l_i-l_{i-1}} + \cdots + a^0_{l_{i-1},0}(0). \tag{5.5}$$

We call $P_i(\lambda)$ the characteristic polynomial on Γ_i and we denote by

$$\lambda_{i,1}\ , \ldots ,\ \lambda_{i,\, l_i - l_{i-1}}$$

the roots of $P_i(\lambda) = 0$ that are called the characteristic roots on Γ_i. Since $a^0_{l_i}(0) \neq 0$ and $a^0_{l_{i-1}}(0) \neq 0$ hold, we have

$$\lambda_{i,d} \neq 0 \quad \text{for all } 1 \leq i \leq p^* \text{ and } 1 \leq d \leq l_i - l_{i-1}.$$

Definition 5.2. We define the set Ξ of singular directions by

$$\Xi = \bigcup_{i=1}^{p^*} \bigcup_{d=1}^{l_i - l_{i-1}} \left\{ \frac{\arg \lambda_{i,d} + 2\pi j}{k_i}\, ;\, j = 0, \pm 1, \pm 2, \ldots \right\}.$$

§ 5.3. Statement of main result

In the equation (E), we have supposed:

$$f(t,x) \in \mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C}^N). \tag{5.6}$$

In order to state our condition on the exponent γ, we need to define the set $\mathscr{C}$ of admissible exponents. We set

$$\Lambda^* = \{(j,\alpha) \in \Lambda\,;\, (j+|\alpha|, \mathrm{ord}_t(a_{j,\alpha})) \notin N_t(E)\}. \tag{5.7}$$

If $(j,\alpha) \in \Lambda^*$, by the definition of $N_t(E)$ we have $|\alpha| > 0$ and by the assumption we have $\mathrm{ord}_t(a_{j,\alpha}) \geq 1 > e_0\,(=0)$. Therefore, if we set $\Lambda_i^* = \{(j,\alpha) \in \Lambda^*\,;\, e_{i-1} < \mathrm{ord}_t(a_{j,\alpha}) \leq e_i\}$ $(i = 1, 2, \ldots, p^*+1$ with $e_{p^*+1} = \infty)$, we have

$$\Lambda^* = \Lambda_1^* \cup \cdots \cup \Lambda_{p^*}^* \cup \Lambda_{p^*+1}^*.$$

We note:

Lemma 5.3. *If $(j,\alpha) \in \Lambda_i^*$ for some $1 \leq i \leq p^*+1$, we have*

$$0 < j + |\alpha| - l_{i-1} - (\mathrm{ord}_t(a_{j,\alpha}) - e_{i-1})/k_i < |\alpha|.$$

Let us define:

Definition 5.4 (Definition of $\mathscr{C}$). We define the set $\mathscr{C}$ of admissible exponents for (E) in the following way.

(1) In the case $1 \leq i \leq p^*$: if $\Lambda_i^* = \emptyset$ we set $\gamma_i = \infty$ and $\mathscr{C}_i = (0,\infty)$; if $\Lambda_i^* \neq \emptyset$ we set

$$\gamma_i = \min_{(j,\alpha)\in\Lambda_i^*} \left(\frac{|\alpha|}{j + |\alpha| - l_{i-1} - (\mathrm{ord}_t(a_{j,\alpha}) - e_{i-1})/k_i} \right)$$

and set $\mathscr{C}_i = (0, \gamma_i)$ which is a nonempty open interval.

(2) In the case $i = p^*+1$: if $\Lambda_{p^*+1} = \emptyset$ we set $\gamma_{p^*+1} = \infty$ and $\mathscr{C}_{p^*+1} = (0,\infty)$; if $\Lambda_{p^*+1} \neq \emptyset$ we set

$$\gamma_{p^*+1} = \min_{(j,\alpha)\in\Lambda_{p^*+1}^*} \left(\frac{|\alpha|}{j + |\alpha| - l_{p^*}} \right)$$

and set $\mathscr{C}_{p^*+1} = (0, \gamma_{p^*+1}]$ which is a nonempty half-open and half-closed interval.

(3) Then, we define $\mathscr{C}$ by

$$\mathscr{C} = \bigcap_{i=1}^{p^*+1} \mathscr{C}_i.$$

By Lemma 5.3 we have

$$1 < \gamma_i \leq \infty, \quad i = 1, 2, \ldots, p^* + 1$$

and so we have $(0, 1+\epsilon) \subset \mathscr{C}$ for some $\epsilon > 0$.

The following theorem is the main result of this note.

Theorem 5.5 (Tahara-Yamazawa [9]). *Suppose the condition*

$$\gamma \in \mathscr{C} \tag{5.8}$$

and let

$$\hat{u}(t,x) = \sum_{n=0}^{\infty} u_n(x) t^n \in \mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N)[[t]]$$

be the unique formal solution of (E). *Then we have:*

(1) *If* $p^* = 0$, $\hat{u}(t,x)$ *is convergent on* $D_\delta \times \mathbb{C}^N$ *for some* $\delta > 0$.

(2) *If* $p^* \geq 1$, *for any* $d \in \mathbb{R} \setminus \Xi$ *we can find* $\varepsilon > 0$, $\delta > 0$ *and a holomorphic solution* $u(t,x)$ *of* (E) *on* $S(d, \pi/2k_{p^*} + \epsilon; \delta) \times \mathbb{C}^N$ *with* $S(d, \pi/2k_{p^*} + \epsilon; \delta) = \{t \in \mathcal{R}(\mathbb{C}_t \setminus \{0\})\,;\, 0 < |t| < \delta, |\arg t - d| < \pi/2k_{p^*} + \varepsilon\}$ *such that the following asymptotic relation holds*:

$$\left| u(t,x) - \sum_{n=0}^{N-1} u_n(x) t^n \right| \leq A H^N N!^{1/k_1} |t|^N \exp(b|x|^\gamma) \tag{5.9}$$

$$\text{on } S(d, \pi/2k_{p^*} + \epsilon; \delta) \times \mathbb{C}^N \text{ for any } N = 0, 1, 2, \ldots$$

for some $A > 0$, $H > 0$ *and* $b > 0$.

Example 5.6. (1) Let us consider

$$(t\partial_t + 1)u = f(t,x) + t\partial_x^2 u + t^2 (t\partial_t)^3 u, \tag{5.10}$$

where $(t,x) \in \mathbb{C}^2$ and $f(t,x) \in \mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C})$. In this case, we have a unique formal solution $\hat{u}(t,x) \in \mathrm{Exp}^{\{\gamma\}}(\mathbb{C})[[t]]$ and

$$\Xi = \{\pi j\,;\, j = 0, \pm 1, \pm 2, \ldots\}, \quad \mathscr{C} = (0, \infty).$$

Therefore, for any $\gamma > 0$, the formal solution $\hat{u}(t,x)$ is Borel summable in any direction $d \in \mathbb{R} \setminus \Xi$.

(2) Let us consider

$$(t\partial_t + 1)u = f(t,x) + t\partial_x^3 u + t^2 (t\partial_t)^3 u, \tag{5.11}$$

where $(t,x) \in \mathbb{C}^2$ and $f(t,x) \in \mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C})$. In this case, we have a unique formal solution $\hat{u}(t,x) \in \mathrm{Exp}^{\{\gamma\}}(\mathbb{C})[[t]]$ and

$$\Xi = \{\pi j\,;\, j = 0, \pm 1, \pm 2, \ldots\}, \quad \mathscr{C} = (0, 3).$$

Therefore, if $0 < \gamma < 3$ holds, the formal solution $\hat{u}(t,x)$ is Borel summable in any direction $d \in \mathbb{R} \setminus \Xi$.

(3) Let us consider

$$(5.12) \qquad (t\partial_t + 1)u = f(t,x) + t^3\partial_x^4 u + t^2(t\partial_t)^3 u,$$

where $(t,x) \in \mathbb{C}^2$ and $f(t,x) \in \mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C})$. In this case, we have a unique formal solution $\hat{u}(t,x) \in \mathrm{Exp}^{\{\gamma\}}(\mathbb{C})[\![t]\!]$ and

$$\Xi = \{\pi j \,;\, j = 0, \pm 1, \pm 2, \dots\}, \quad \mathscr{C} = (0, 4].$$

Therefore, for any $0 < \gamma \le 4$, the formal solution $\hat{u}(t,x)$ is Borel summable in any direction $d \in \mathbb{R} \setminus \Xi$.

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