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Borel summability of WKB theoretic transformation to the Weber equation

By

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Abstract

We take a Schrödinger equation which has a pair of simple turning points connected by a Stokes line, and consider WKB theoretic transformation series to the Weber equation. Under suitable conditions, the transformation series is Borel summable, and analysis of so-called fixed singularities can be reduced to the Weber equation.

§ 1. Introduction

We consider a second order linear differential equation with a large parameter

\[(1.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q(x) \right) \psi = 0.\]

The coefficient $Q(x)$ is a holomorphic function (typically rational function or polynomial). The equation has formal solutions (WKB solutions) of the form

\[(1.2) \quad \psi(x, \eta) = \exp \left( \int^{x} S(x, \eta) \, dx \right),\]

where $S(x, \eta) = \eta S_{-1} + S_{0} + \eta^{-1} S_{1} + \cdots$ is a formal power series satisfying the Riccati equation

\[(1.3) \quad S^2 + \frac{dS}{dx} = \eta^2 Q(x).\]

The WKB solutions $\psi(x, \eta)$ (or $S(x, \eta)$) are divergent in general, and we apply Borel resummation method. (See e.g., [12], [9].) Under generic assumptions, the WKB solutions (of suitable normalization) is Borel summable (see [4], [8]), but in some cases not.

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For example if $Q(x) = E - x^2/4$ (namely the Weber equation) with a positive constant $E > 0$, WKB solutions are not Borel summable (depending on the normalization). See e.g., [11]. This is a general phenomenon if the equation has a pair of turning points (zeros of $Q(x)$) connected by a Stokes line $\Im \int^x \sqrt{Q(x)} \, dx = 0$. See e.g., [2], [3]. In Figure 1, we give examples of Stokes lines connecting turning points.

In such cases, the Borel transform of a WKB solution has singularities on the real axis (in the Borel plane), which we call “fixed singularites”. They are fixed in the sense that the location is independ of $x$. To analyze such singularities, in [1], WKB theoretic transformation to the Weber equation is constructed, and Borel transformability is given. Here transformation series is a formal power series $x(q, \eta) = x_0(q) + \eta^{-1}x_1(q) + \eta^{-2}x_2(q) + \cdots$ which transforms the equation

$$\left( \frac{d^2}{dq^2} - \eta^2 Q(q) \right) \psi = 0$$

(1.4)

to the Weber equation (with an infinite power series $E = E(\eta) = E_0 + \eta^{-1}E_1 + \eta^{-2}E_2 + \cdots$)

$$\left( \frac{d^2}{dx^2} - \eta^2 \left( E - \frac{x^2}{4} \right) \right) \phi = 0,$$

(1.5)

with a gauge transform $\psi = x^{-1/2}\phi$. This is equivalent to that $x(q, \eta)$ satisfies the following:

$$Q(q) = \left( \frac{dx}{dq} \right)^2 \left( E - \frac{x^2}{4} \right) - \frac{1}{2} \eta^{-2}\{x; q\}.$$

(1.6)

Here $\{x; q\}$ is the Schwarzian derivative. Though this is a transformation between equations, this also connect WKB solutions of certain normalization. (See [1] and the following section.)
The purpose of this paper is to present Borel summability of transformation series in a simple case, and add a brief explanation of the consequence (the following section).

In ending of this introduction, we refer to the work of Kamimoto and Koike ([5]), which shows Borel summability of transformation series to the Airy equation \((Q(x) = x)\). The Airy equation has only one simple turning point, and is the simplest equation whose (Borel transformed) WKB solution has so-called “movable singularities”. The basic idea of the proof of the Weber case follows the Airy case [5], while one additional problem arises which we should overcome. In this paper, we do not give a proof of Borel summability of transformation series to the Weber equation. A detailed proof will be given elsewhere.

§ 2. Borel summability of transformation series

In this section, for simplicity we assume that the coefficient \(Q(q)\) in (1.4) is polynomial. Let \(q_\pm\) be simple turning points of the equation (1.4). Assume \(q_\pm\) are connected by a Stokes line and the other Stokes lines emanating from the two points tend to infinity. For example, if \(Q(q) = q(q^2 - 1)\) and we take \(q_+ = 0\) and \(q_- = -1\), these conditions are satisfied (See Figure 1). Take a neighborhood

\[
D = \left\{ \left| \int_{q_+}^{q} \sqrt{Q} \, dq \right| < d \right\} \cup \left\{ \left| \int_{q_-}^{q} \sqrt{Q} \, dq \right| < d \right\}
\]

of \(\{q_\pm\}\) and set

\[
\hat{D} = \bigcup_{q \in D} \left\{ \Im \int_{q}^{q'} \sqrt{Q} \, dq = 0 \right\}.
\]

(cf. Figure 2.) We take \(d\) small enough so that \(\hat{D}\) does not contain any turning points except for \(q_\pm\). Then there exist formal power series \(x(q, \eta) = x_0(q) + \eta^{-1}x_1(q) + \eta^{-2}x_2(q) + \cdots\) and \(E(\eta) = E_0 + \eta^{-1}E_1 + \eta^{-2}E_2 + \cdots\) with \(x_j(q)\) being holomorphic on \(\hat{D}\) \((j = 0, 1, 2, \ldots)\) which satisfy the equation (1.6) and \(dx_0/dq \neq 0\). \(x(q, \eta)\) and \(E(\eta)\) are uniquely determined up to the choice of \(x_0(q)\). See [1], [9].

**Remark.** \(x_0(q)\) is a map which maps a turning point to a turning point, a level curve (Stokes line) \(\Im \int_{q}^{q'} \sqrt{Q} \, dq = 0\) to a level curve (Stokes line) \(\Im \int_{E_0 - x^2/4}^{x} \sqrt{E_0 - x^2/4} \, dx = 0\). There are two turning points \(q_\pm\), and we have two choices of \(x_0(q)\).

The Borel summability of \(E(\eta)\) is known. See [8]. In addition we have the following theorem.

**Theorem 2.1.** Under the assumptions above, the transformation series \(x(q, \eta)\) is Borel summable uniformly on \(\hat{D}\).
Figure 2. Domains $D$ and $\hat{D}$.

Thus the equation (1.4) on $\hat{D}$ is transformed to the canonical equation (1.5) by two Borel summable series $x(q, \eta)$ and $E(\eta)$. Then as is explained in [1] and [9] (though mainly Airy case, not Weber case), a WKB solution of (1.4) is also transformed into a WKB solution of (1.5); Let $\psi(q, \eta)$ be a WKB solution of (1.4) normalized at $q_+$ $\phi(x, E, \eta)$ be a WKB solution of (1.5) normalized at $2\sqrt{E}$. Here we assume $x_0(q_+) = 2\sqrt{E_0}$. (For normalization, see e.g., [9].) Then the following relation holds:

$$
\psi(q, \eta) = (\frac{dx}{dq}(q, \eta))^{-1/2} \phi(x(q, \eta), E(\eta), \eta).
$$

(2.3)

Though this is a formal relation, if Borel transformed, this becomes an analytic relation. Set $x(q, \eta) = x_0(q) + X(q, \eta)$ and $E(\eta) = E_0 + F(\eta)$. By Taylor expansion, we have

$$
\psi(q, \eta) = (\frac{dx}{dq}(q, \eta))^{-1/2} \sum_{n=0}^{\infty} \frac{X^n(q, \eta)}{n!} \frac{\partial^n \phi}{\partial x^n}(x_0(q), E(\eta), \eta)
$$

(2.4)

$$
\psi_B(q, y) = \left( (\frac{dx}{dq})^{-1/2} \right)_B(q, y) \sum_{n=0}^{\infty} \frac{X_B^n(q, y)}{n!} \left( \sum_{m=0}^{\infty} \frac{F_B^m(y)}{m!} \frac{\partial^{n+m} \phi_B}{\partial E^m \partial x^n}(x_0(q), E_0, y) \right).
$$

(2.5)

where the subscript B means Borel transform and * is convolution. Now let us take one term

$$
\left( (\frac{dx}{dq})^{-1/2} \right)_B(q, y) \frac{X_B^n(q, y)}{n!} \frac{F_B^m(y)}{m!} \frac{\partial^{n+m} \phi_B}{\partial E^m \partial x^n}(x_0(q), E_0, y).
$$
Figure 3. Stokes lines connecting a pair of a simple turning point and a simple pole (left), a pair of simple poles (right).

Since $X$ and $F$ are Borel summable, the front part

$$
\left(\left(\frac{dx}{dq}\right)^{-1/2}\right)_B^{(q, y)\ast}\frac{X_B^{*n}(q, y)}{n!}\ast\frac{F_B^{*m}(y)}{m!}
$$

is holomorphic in a strip region containing the positive real axis. Since we know well about $\phi_B$ (see e.g., [11], [10]), for this single term, we can see continuability avoiding singularities, discontinuity at a singularity, etc. Then by summing up with respect to $m$ and $n$ (with care on convergence), we see continuability etc. also for $\psi_B(q, y)$.

**Remark.** $\phi_B(x_0(q), E_0, y)$ has infinitely many singularities in the $y$-plane with real period $2\pi E_0$, and with Borel summability we can analyze all singularities through transformation. Thus Borel summability of transformation is important in the analysis of fixed singularities.

**Remark.** In this paper, we considered only two simple turning points problem. On the other hand, simple poles (of $Q$) are known to play a role similar to simple turning points ([6], [7]), and a pair of a simple turning point and a simple pole, or a pair of simple poles causes fixed singularities as well (cf. Figure 3). The former one can be treated in the same manner as a pair of simple turning points. However the latter one is difficult to treat with. Also, a sole simple turning point makes a pair in some sense, generating a loop of Stokes line (cf. Figure 4), and this has the same difficulty.
Figure 4. A loop of Stokes line ending a sole simple turning point.

References


