A relation between instanton-type solutions of $P_J$ (J=I,II,III,IV)-hierarchies with a large parameter (Recent development of microlocal analysis and asymptotic analysis)

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A relation between instanton-type solutions of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter

By

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Abstract

We report a relation between instanton-type solutions for equations of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter. The content of these notes is a short summary of our forthcoming papers [2], [16] and [17].

§ 1. Definitions of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter $\eta$

We recall the definitions of equations of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter $\eta$ given in [15], [8] and [9].

(i) The $m$-th members $(P_I)_m$ and $(P_{34})_m$ of $P_I$, $P_{34}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ $(k = 1, 2, \ldots)$ be unknown functions of $t$ and $c_k$'s are constants. In what follows, $\delta_{jm}$ stands for Kronecker's delta.

• For $m = 1, 2, \ldots$, $(P_I)_m$ has the following form (see [15]):

\[
\begin{align*}
\eta^{-1} \frac{d u_j}{dt} &= 2v_j, \quad j = 1, 2, \ldots, m, \\
\eta^{-1} \frac{d v_j}{dt} &= 2(u_{j+1} + u_1 u_j + w_j), \quad j = 1, 2, \ldots, m,
\end{align*}
\]

(1.1)

with the assumption $u_{m+1} = 0$. Here $w_j$ is recursively defined by

\[
(1.2) \quad w_j = \frac{1}{2} \sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} u_k v_{j-k} + c_j + \delta_{jm}t.
\]

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For $m = 1, 2, \ldots$, $(P_{34})_m$ has the following form (see [9]):

\begin{align}
\begin{cases}
\eta^{-1} \frac{du_j}{dt} = 2v_j, & j = 1, 2, \ldots, m, \\
\eta^{-1} \frac{dv_j}{dt} = 2(u_{j+1} + u_1u_j + w_j), & j = 1, 2, \ldots, m,
\end{cases}
\end{align}

with

\begin{equation}
(1.4)
\quad u_{m+1} = -w_m + c_0 u_m + \frac{v_m^2 - \kappa^2}{2u_m}.
\end{equation}

Here $\gamma (\neq 0)$, $\kappa$ and $\{c_j\}_{j=0}^m$ are constants, and $w_j$ is recursively defined by

\begin{equation}
(1.5)
\quad w_j = \frac{1}{2} \sum_{k=1}^{j} u_{j-k+1}u_k + \sum_{k=1}^{j-1} u_{j-k}w_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k}v_k - \sum_{k=1}^{j-1} c_{j-k}w_k.
\end{equation}

Note that the form above has been slightly modified from the original form given by [9]. If we replace $u_m$ (resp. $v_m$) in (1.3) and (1.4) with $u_m - \gamma t$ (resp. $v_m - \eta^{-1} \frac{\gamma}{2}$), then we have the original form of $(P_{34})_m$.

(ii) The $m$-th members $(P_{II})_m$ and $(P_{IV})_m$ of $P_{II}$, $P_{IV}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ ($k = 1, 2, \ldots$) be unknown functions of $t$ and $c_k$'s are constants.

For $m = 1, 2, \ldots$, $(P_{II})_m$ has the following form (see [8]):

\begin{align}
\begin{cases}
\eta^{-1} \frac{du_j}{dt} = -2(u_1u_j + v_j + u_{j+1}) + 2c_ju_1, & j = 1, 2, \ldots, m, \\
\eta^{-1} \frac{dv_j}{dt} = 2(v_1u_j + v_{j+1} + w_j) - 2c_jv_1, & j = 1, 2, \ldots, m,
\end{cases}
\end{align}

with $u_{m+1} = \gamma t$ and $v_{m+1} = \kappa$. Here $\gamma (\neq 0)$, $\kappa$ and $c_j$'s are constants, and $w_j$ is recursively defined by

\begin{equation}
(1.7)
\quad w_j = \sum_{k=1}^{j-1} u_{j-k}w_k + \sum_{k=1}^{j} u_{j-k+1}v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k}v_k - \sum_{k=1}^{j-1} c_{j-k}w_k.
\end{equation}

• For $m = 1, 2, \ldots$, $(P_{IV})_m$ has the following form (see [8]):

\begin{align}
\begin{cases}
\eta^{-1} \frac{du_j}{dt} = -2(u_1u_j + v_j + u_{j+1}) + 2c_ju_1 - 2\delta_{j,m-1}\gamma t, \\
\eta^{-1} \frac{dv_j}{dt} = 2(v_1u_j + v_{j+1} + w_j) - 2c_jv_1, & j = 1, 2, \ldots, m,
\end{cases}
\end{align}

with

\[
\begin{align*}
    u_{m+1} &= -\alpha_1, \\
    v_{m+1} &= -w_m - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - c_m)}.
\end{align*}
\]

Here \(\gamma(\neq 0), \alpha_1, \alpha_2\) and \(c_j\)'s are constants, and \(w_j\) is defined by

\[
\begin{align*}
    w_j &= \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k + \delta_{jm} \gamma tv_1.
\end{align*}
\]

Note that the form above has been slightly modified from the original form given by [8]. If we replace \(u_m\) in (1.8) and (1.9) with \(u_m - \gamma t\), then we have the original form of \((P_{IV})_m\).

\S 2. \(P_J\) \((J = I, II, 34, IV)\)-hierarchies with \(\eta\) in terms of generating functions

In this note, we consider the represented forms with generating functions of unknown functions. Let \(\theta\) denotes an independent variable.

(i) The \(m\)-th members \((P_I)_m\) and \((P_{34})_m\) of \(P_I, P_{34}\)-hierarchies with \(\eta\)

We define generating functions \(U, V\) and \(C\) of \((P_I)_m\) (resp. \((P_{34})_m\)) by

\[
\begin{align*}
    U(\theta) &= \sum_{k=1}^{\infty} u_k \theta^k, \\
    V(\theta) &= \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \\
    C(\theta) &= \sum_{k=1}^{\infty} (c_k + \delta_{km} t) \theta^{k+1} \quad (\text{resp. } C(\theta) = \sum_{k=1}^{\infty} c_k \theta^{k+1}),
\end{align*}
\]

respectively. Here \(u_k, v_k, c_k\) \((k = 1, 2, \ldots)\) denote unknown functions and constants of \((P_I)_m\) (resp. \((P_{34})_m\)). In what follows, by \(A \equiv B\) we mean that \(A - B\) is zero modulo \(\theta^{m+2}\).

- \((P_I)_m\) is rewritten in the following form

\[
\eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta \\ V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ -(1 + 2u_1 \theta)(1 - U) + \frac{1 + 2C - \theta V^2}{1 - U} \end{pmatrix}
\]

with the condition that the coefficients of \(\theta^{m+1}\) of \(U\) and \(V\) are zero.

- \((P_{34})_m\) is rewritten in the following form

\[
\eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta \\ V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ -(1 + 2(u_1 + c_0) \theta)(1 - U) + \frac{1 + 2C - \theta(V^2 - 2c_0)}{1 - U} \end{pmatrix}
\]

\[
+ \begin{pmatrix} 0 \\ 2\gamma t \theta^m (1 + (u_1 + 2c_0) \theta) \end{pmatrix}
\]
with the condition that the coefficient of $\theta^{m+1}$ of $U$ (resp. $V$) is equal to the right hand side of (1.4) (resp. zero).

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m + 1$) on the both sides of (2.2) (resp. (2.3)), we obtain (1.1) (resp. (1.3)).

(ii) The $m$-th members $(P_{II})_m$ and $(P_{IV})_m$ of $P_{II}$, $P_{IV}$-hierarchies with $\eta$

We define generating functions $U$, $V$ and $C$ of $(P_{II})_m$ (resp. $(P_{IV})_m$) by

\begin{equation}
U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \quad C(\theta) = \sum_{k=1}^{\infty} c_k \theta^k,
\end{equation}

respectively. Here $u_k$, $v_k$, $c_k$ ($k = 1, 2, \ldots$) denote unknown functions and constants of $(P_{II})_m$ (resp. $(P_{IV})_m$).

- $(P_{II})_m$ is rewritten in the following form

\begin{equation}
\eta^{-1} \frac{d}{dt} \left( \begin{array}{c} U \theta \\ V \theta \end{array} \right) \equiv 2 \left( \begin{array}{c} u_1(1 - U + C) \theta - U - V \theta \\ -v_1(1 - U + C) \theta + \frac{2UV + V^2 \theta}{2(1 - U + C)} + V \end{array} \right)
\end{equation}

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are $\gamma t$ and $\kappa$, respectively.

- $(P_{IV})_m$ is rewritten in the following form

\begin{equation}
\eta^{-1} \frac{d}{dt} \left( \begin{array}{c} U \theta \\ V \theta \end{array} \right) \equiv 2 \left( \begin{array}{c} u_1(1 - U + C) \theta - U - V \theta - \gamma t \theta^m \\ -v_1(1 - U + C) \theta + \frac{2UV + V^2 \theta}{2(1 - U + C)} + V + \gamma tv_1 \theta^{m+1} \end{array} \right)
\end{equation}

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are equal to the right hand sides of (1.9), respectively.

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m + 1$) on the both sides of (2.5) (resp. (2.6)), we obtain (1.6) (resp. (1.8)).

§ 3. The generating functions of the leading terms of 0-parameter solutions

As is shown in [4], each $P_J$-hierarchy has a formal power series of $\eta^{-1}$ in the form

\begin{equation}
\begin{array}{c}
\hat{u}_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{u}_{k,j}(t), \quad v_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{v}_{k,j}(t), \quad j = 1, \ldots, m.
\end{array}
\end{equation}

The solution taking the form of (3.1) is often called a 0-parameter solution, as the form does not have any free parameters. Let us define the generating functions of the leading terms $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ of their 0-parameter solutions of $(P_J)_m$ ($J = I, II, 34, IV$) by

\begin{equation}
\begin{array}{c}
\hat{u}_0(\theta) = \sum_{i=1}^{\infty} \hat{u}_{i,0} \theta^i, \quad \hat{v}_0(\theta) = \sum_{i=1}^{\infty} \hat{v}_{i,0} \theta^i.
\end{array}
\end{equation}
Each explicit form of (3.2) for \((P_J)_m\) \((J = I, II, 34, IV)\) is given as follows.

\[
\begin{array}{|c|l|}
\hline
\((P_I)_m\) & \hat{u}_0 = 1 - \sqrt{\frac{1 + 2C}{1 + 2\hat{u}_{1,0} \theta}}, \quad \hat{v}_0 = 0. \\
\hline
\((P_{34})_m\) & \hat{u}_0 \equiv 1 - \sqrt{\frac{1 + 2(C + c_0 \theta)}{(1 + 2(\hat{u}_{1,0} + c_0) \theta)(1 - 2\gamma t \theta^m)}}, \quad \hat{v}_0 = 0, \\
& \text{where } \hat{u}_{1,0} \text{ and } \hat{v}_{1,0} \text{ are taken so that the coefficients of } \theta^{m+1} \\
& \text{in } \hat{u}_0 \text{ and } \hat{v}_0 \text{ are equal to } -\hat{w}_{m,0} + c_0 \hat{u}_{m,0} + \frac{(\hat{v}_{m,0}^2 - \kappa^2)}{2\hat{u}_{m,0}} \text{ and } 0, \text{ respectively. Here } \hat{w}_{m,0} \text{ is defined by (1.5) with } u_k, v_k \text{ and } w_k \\
& \text{being replaced by } \hat{u}_{k,0}, \hat{v}_{k,0} \text{ and } \hat{w}_{k,0}. \\
\hline
\((P_{11})_m\) & \hat{u}_0 = (1 + C) \left( 1 - \sqrt{\frac{1}{(1 + \hat{u}_{1,0} \theta)^2 - 2\hat{v}_{1,0} \theta^2}} \right), \\
\hat{v}_0 \theta = (1 + C) \left( -1 + (1 + \hat{u}_{1,0} \theta) \sqrt{\frac{1}{(1 + \hat{u}_{1,0} \theta)^2 - 2\hat{v}_{1,0} \theta^2}} \right). \\
& \text{Here } \hat{u}_{1,0} \text{ and } \hat{v}_{1,0} \text{ are taken so that the coefficients of } \theta^{m+1} \\
& \text{in } \hat{u}_0 \text{ and } \hat{v}_0 \text{ are } \gamma t \text{ and } \kappa, \text{ respectively.} \\
\hline
\((P_{IV})_m\) & \hat{u}_0 \equiv (1 + C)(1 - \sqrt{1/f(t, \theta)}), \\
\hat{v}_0 \theta \equiv (1 + C)(-1 + (1 + \hat{u}_{1,0} \theta) \sqrt{1/f(t, \theta)}) - \gamma t \theta^m, \\
& f(t, \theta) := (1 + \hat{u}_{1,0} \theta)^2 - 2\hat{v}_{1,0} \theta^2 - 2\gamma t \theta^m(1 + 2\hat{u}_{1,0} \theta - c_1 \theta), \\
& \text{where } \hat{u}_{1,0} \text{ and } \hat{v}_{1,0} \text{ are taken so that the coefficients of } \theta^{m+1} \\
& \text{in } \hat{u}_0 \text{ and } \hat{v}_0 \text{ are equal to } -\alpha_1 \text{ and } -\hat{w}_{m,0} - \frac{(\hat{v}_{m,0} - \alpha_1)^2 - \alpha_2^2}{2(\hat{u}_{m,0} - c_m)}, \text{ respectively. Here } \hat{w}_{m,0} \text{ is defined by (1.10) with } u_k, v_k \text{ and } w_k \\
& \text{being replaced by } \hat{u}_{k,0}, \hat{v}_{k,0} \text{ and } \hat{w}_{k,0}. \\
\hline
\end{array}
\]
§ 4. Instanton-type solutions of \((P)_m\) (J = I, II, 34, IV)

We prepare some notation. Let \(\nu_{\pm 1}(t), \ldots, \nu_{\pm m}(t)\) of \((P)_m\) denote the roots of the following algebraic equation \(\Lambda_J(\lambda, t) = 0\) of \(\lambda\) with \(\nu_k = -\nu_{|k|}\) if \(k < 0\):
- If \(J = I\) (resp. 34), then \(\Lambda_J(\lambda, t)\) is defined by
  \[
  g_J(\lambda)^m - \sum_{k=1}^{m} \hat{u}_{k,0} g_J(\lambda)^{m-k}
  \]
  with
  \[
  g_J(\lambda) = \frac{\lambda^2 - 8\hat{u}_{1,0}}{4} \quad \text{(resp. } g_{34}(\lambda) = \frac{\lambda^2 - 8(\hat{u}_{1,0} + c_0)}{4}).
  \]
  Here \(\hat{u}_{k,0}\)'s are defined by (3.2) of \((P)_m\) (J = I, 34) respectively.
- If \(J = II\) or IV, then \(\Lambda_J(\lambda, t)\) is defined by
  \[
  g_J(\lambda)^m - \sum_{k=1}^{m} (\hat{u}_{k,0} - c_k) g_J(\lambda)^{m-k}
  \]
  with
  \[
  g_J(\lambda) = -\hat{u}_{1,0} - \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}.
  \]
  Here \(\hat{u}_{k,0}, \hat{v}_{1,0}\) are defined by (3.2) of \((P)_m\) (J = II, IV) respectively and so are \(c_k\).

Let \(\Omega\) be an open subset in \(\mathbb{C}_t\) and the two conditions are always assumed:
(A1) The roots \(\nu_i(t)\)'s \((1 \leq |i| \leq m)\) are mutually distinct for each \(t \in \Omega\).
(A2) The function \(p_1\nu_1(t) + \cdots + p_m\nu_m(t)\) does not vanish identically on \(\Omega\) for any \((p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}.

Then we have the following theorem.

**Theorem 4.1** ([2], [17]). We have instanton-type solutions of \((P)_m\) (J = I, 34) with free \(2m\)-parameters \((\beta_{-m}, \ldots, \beta_{m}) \in \mathbb{C}^{2m}[\eta^{-1}]\) of the form

\[
U = \hat{u}_0 + (1 - \hat{u}_0)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0)v,
\]

\[
\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \leq |k| \leq m} f^J_k(\tau, t; \eta) A(\nu_k), \quad A(\nu_k) = \left( \frac{a(\nu_k)}{\nu_k} \right) = \frac{a(\nu_k)}{\nu_k} = \frac{\theta}{1 - \theta g_J(\nu_k)} = \sum_{j=0}^{\infty} g_J(\nu_k) \theta^j + 1. \quad \text{Here} \ (\hat{u}_0, \hat{v}_0), \ \nu_k \ \text{and} \ g_J \ \text{of} \ (P)_m \ \text{(J = I, 34)} \ \text{have been defined in the previous section respectively. For the explicit forms of} \ f^J_{k,p,l}(t) \ (J = I, 34), \ \text{see [2] and [17].}
\]

\[
\begin{pmatrix} f^J_k(\tau, t; \eta) \end{pmatrix} = \sum_{j=0}^{\infty} g_J(\nu_k) \theta^j + 1. \quad \text{Here} \ (\hat{u}_0, \hat{v}_0), \ \nu_k \ \text{and} \ g_J \ \text{of} \ (P)_m \ \text{(J = I, 34)} \ \text{have been defined in the previous section respectively. For the explicit forms of} \ f^J_{k,p,l}(t) \ (J = I, 34), \ \text{see [2] and [17].}
\]

\[
\begin{pmatrix} f^J_k(\tau, t; \eta) \end{pmatrix} = \sum_{j=0}^{\infty} g_J(\nu_k) \theta^j + 1. \quad \text{Here} \ (\hat{u}_0, \hat{v}_0), \ \nu_k \ \text{and} \ g_J \ \text{of} \ (P)_m \ \text{(J = I, 34)} \ \text{have been defined in the previous section respectively. For the explicit forms of} \ f^J_{k,p,l}(t) \ (J = I, 34), \ \text{see [2] and [17].}
\]

\[
\begin{pmatrix} f^J_k(\tau, t; \eta) \end{pmatrix} = \sum_{j=0}^{\infty} g_J(\nu_k) \theta^j + 1. \quad \text{Here} \ (\hat{u}_0, \hat{v}_0), \ \nu_k \ \text{and} \ g_J \ \text{of} \ (P)_m \ \text{(J = I, 34)} \ \text{have been defined in the previous section respectively. For the explicit forms of} \ f^J_{k,p,l}(t) \ (J = I, 34), \ \text{see [2] and [17].}
\]
The following describes the relation between instanton-type solutions of \((P_I)_m\) and \((P_{34})_m\).

**Theorem 4.2** ([17]). An instanton-type solution of \((P_{34})_m\) is transformed algebraically to that of \((P_I)_m\) by the replacements of all terms which are depending on their 0-parameter solutions.

**Theorem 4.3** ([16]). We have instanton-type solutions of \((P_J)_m\) (\(J = II, IV\)) with free 2m-parameters \((\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m}[\eta^{-1}]\) of the form

\[
U = \hat{u}_0 + (1 - \hat{u}_0 + C)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0 + C)v,
\]

\[
\begin{pmatrix}
u 
\end{pmatrix} = \sum_{1 \leq |k| \leq m} f^J_k(\tau, t; \eta)A(\nu_k), \quad A(\nu_k) := \left( \begin{array}{c} a(\nu_k) \\ g_3(\nu_k)a(\nu_k) \end{array} \right)
\]

\[
f^J_k(\tau, t; \eta) = \sum_{j=1}^{\infty} \left( \sum_{\sum_{l+|p|=j} f^J_{k,p}(t)e^{p\cdot\tau}} \right) \eta^{-j/2},
\]

where \(a(\nu_k) = \frac{\theta}{1 - \theta g_3(\nu_k)} = \sum_{j=0}^{\infty} g_3(\nu_k)^j \theta^{j+1}\) and \(g_3(\nu_k) := -\frac{\nu_k}{2} + \sqrt{\frac{\nu_k^2}{4} + 2\hat{v}_{1,0}}\). Here \((\hat{u}_0, \hat{v}_0), \nu_k\) and \(g_3\) of \((P_J)_m\) (\(J = II, IV\)) have been defined in the previous section respectively. For the explicit forms of \(f^J_{k,p}(t)\) (\(J = II, IV\)), see [16].

The following describes the relations between instanton-type solutions of \((P_{II})_m\) and \((P_{IV})_m\).

**Theorem 4.4** ([16]). An instanton-type solution of \((P_{II})_m\) is transformed algebraically to that of \((P_{IV})_m\) by the replacements of all terms which are depending on their 0-parameter solutions.

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