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A relation between instanton-type solutions of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter

By

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Abstract

We report a relation between instanton-type solutions for equations of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter. The content of these notes is a short summary of our forthcoming papers [2], [16] and [17].

§ 1. Definitions of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter $\eta$

We recall the definitions of equations of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter $\eta$ given in [15], [8] and [9].

(i) The $m$-th members $(P_I)_m$ and $(P_{34})_m$ of $P_I$, $P_{34}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ $(k = 1, 2, \ldots)$ be unknown functions of $t$ and $c_k$'s are constants. In what follows, $\delta_{jm}$ stands for Kronecker's delta.

- For $m = 1, 2, \ldots, (P_I)_m$ has the following form (see [15]):

\[
\begin{align*}
\eta^{-1} \frac{du_j}{dt} &= 2v_j, & j &= 1, 2, \ldots, m, \\
\eta^{-1} \frac{dv_j}{dt} &= 2(u_{j+1} + u_1 u_j + w_j), & j &= 1, 2, \ldots, m,
\end{align*}
\]

(1.1)

with the assumption $u_{m+1} = 0$. Here $w_j$ is recursively defined by

\[
w_j = \frac{1}{2} \sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} u_k v_{j-k} + c_j + \delta_{jm} t.
\]

(1.2)

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For $m = 1, 2, \ldots,$ $(P_{34})_m$ has the following form (see [9]):

$$(1.3) \quad \begin{cases} \eta^{-1} \frac{du_j}{dt} = 2v_j, & j = 1, 2, \ldots, m, \\ \eta^{-1} \frac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \ldots, m, \end{cases}$$

with

$$(1.4) \quad u_{m+1} = -w_m + c_0 u_m + \frac{v_m^2 - \kappa^2}{2u_m}.$$

Here $\gamma(\neq 0), \kappa$ and $\{c_j\}_{j=0}^m$ are constants, and $w_j$ is recursively defined by

$$(1.5) \quad w_j = \frac{1}{2} \sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \delta_{j,m-1} \gamma t + 2 \delta_{jm} \gamma t c_0.$$

Note that the form above has been slightly modified from the original form given by [9]. If we replace $u_m$ (resp. $v_m$) in (1.3) and (1.4) with $u_m - \gamma t$ (resp. $v_m - \eta^{-1} \frac{\gamma}{2}$), then we have the original form of $(P_{34})_m$.

(ii) The $m$-th members $(P_{II})_m$ and $(P_{IV})_m$ of $P_{II}, P_{IV}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ ($k = 1, 2, \ldots$) be unknown functions of $t$ and $c_k$'s are constants.

For $m = 1, 2, \ldots,$ $(P_{II})_m$ has the following form (see [8]):

$$(1.6) \quad \begin{cases} \eta^{-1} \frac{du_j}{dt} = -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1, & j = 1, 2, \ldots, m, \\ \eta^{-1} \frac{dv_j}{dt} = 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j = 1, 2, \ldots, m, \end{cases}$$

with $u_{m+1} = \gamma t$ and $v_{m+1} = \kappa$. Here $\gamma(\neq 0)$, $\kappa$ and $c_j$'s are constants, and $w_j$ is recursively defined by

$$(1.7) \quad w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k.$$

For $m = 1, 2, \ldots,$ $(P_{IV})_m$ has the following form (see [8]):

$$(1.8) \quad \begin{cases} \eta^{-1} \frac{du_j}{dt} = -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1 - 2 \delta_{j,m-1} \gamma t, \\ \eta^{-1} \frac{dv_j}{dt} = 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j = 1, 2, \ldots, m, \end{cases}$$
INSTANTON-TYPE SOLUTIONS OF $P_J$ ($J = I, II, 34, IV$)-HIERARCHIES WITH $\eta$

with

$$u_{m+1} = -\alpha_1, \quad v_{m+1} = -w_m - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - c_m)}.$$  \hspace{1cm} (1.9)

Here $\gamma(\neq 0)$, $\alpha_1$, $\alpha_2$ and $c_j$'s are constants, and $w_j$ is defined by

$$w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k + \delta_{jm} \gamma t v_1.$$  \hspace{1cm} (1.10)

Note that the form above has been slightly modified from the original form given by [8]. If we replace $u_m$ in (1.8) and (1.9) with $u_m - \gamma t$, then we have the original form of $(P_{IV})_m$.

§ 2. $P_J$ ($J = I, II, 34, IV$)-hierarchies with $\eta$ in terms of generating functions

In this note, we consider the represented forms with generating functions of unknown functions. Let $\theta$ denotes an independent variable.

(i) The $m$-th members $(P_I)_m$ and $(P_{34})_m$ of $P_I, P_{34}$-hierarchies with $\eta$

We define generating functions $U, V$ and $C$ of $(P_I)_m$ (resp. $(P_{34})_m$) by

$$U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and}$$

$$C(\theta) = \sum_{k=1}^{\infty} (c_k + \delta_{km} t) \theta^{k+1} \quad \text{(resp. } C(\theta) = \sum_{k=1}^{\infty} c_k \theta^{k+1})\text{,}$$

respectively. Here $u_k$, $v_k$, $c_k$ ($k = 1, 2, \ldots$) denote unknown functions and constants of $(P_I)_m$ (resp. $(P_{34})_m$). In what follows, by $A \equiv B$ we mean that $A - B$ is zero modulo $\theta^{m+2}$.

• $(P_I)_m$ is rewritten in the following form

$$\eta^{-1} \frac{d}{dt}\begin{pmatrix} U \theta \\ V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ - (1 + 2u_1 \theta)(1 - U) + \frac{1 + 2C - \theta V^2}{1 - U} \end{pmatrix}$$

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are zero.

• $(P_{34})_m$ is rewritten in the following form

$$\eta^{-1} \frac{d}{dt}\begin{pmatrix} U \theta \\ V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ - (1 + 2(u_1 + c_0) \theta)(1 - U) + \frac{1 + 2C - \theta(V^2 - 2c_0)}{1 - U} \end{pmatrix}$$

$$+ \begin{pmatrix} 2\gamma t \theta^{m+1} (1 + (u_1 + 2c_0) \theta) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 2\gamma t \theta^{m+1} (1 + (u_1 + 2c_0) \theta) \end{pmatrix}$$
with the condition that the coefficient of $\theta^{m+1}$ of $U$ (resp. $V$) is equal to the right hand side of (1.4) (resp. zero).

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m + 1$) on the both sides of (2.2) (resp. (2.3)), we obtain (1.1) (resp. (1.3)).

(ii) The $m$-th members $(P_{\text{II}})_m$ and $(P_{\text{IV}})_m$ of $P_{\text{II}}$, $P_{\text{IV}}$-hierarchies with $n$

We define generating functions $U$, $V$ and $C$ of $(P_{\text{II}})_m$ (resp. $(P_{\text{IV}})_m$) by

\[
U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \quad C(\theta) = \sum_{k=1}^{\infty} c_k \theta^k,
\]

respectively. Here $u_k$, $v_k$, $c_k$ ($k = 1, 2, \ldots$) denote unknown functions and constants of $(P_{\text{II}})_m$ (resp. $(P_{\text{IV}})_m$).

- $(P_{\text{II}})_m$ is rewritten in the following form

\[
\eta^{-1} \frac{d}{dt} \left( \begin{array}{l} U \theta V \theta \\ U \theta \end{array} \right) \equiv 2 \left( \begin{array}{l} u_1 (1 - U + C) \theta - U - V \theta \\ -v_1 (1 - U + C) \theta + \frac{2UV + V^2 \theta}{2(1 - U + C)} + V \end{array} \right)
\]

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are $\gamma t$ and $\kappa$, respectively.

- $(P_{\text{IV}})_m$ is rewritten in the following form

\[
\eta^{-1} \frac{d}{dt} \left( \begin{array}{l} U \theta V \theta \\ U \theta \end{array} \right) \equiv 2 \left( \begin{array}{l} u_1 (1 - U + C) \theta - U - V \theta - \gamma t \theta^m \\ -v_1 (1 - U + C) \theta + \frac{2UV + V^2 \theta}{2(1 - U + C)} + V + \gamma tv_1 \theta^{m+1} \end{array} \right)
\]

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are equal to the right hand side of (1.9), respectively.

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m + 1$) on the both sides of (2.5) (resp. (2.6)), we obtain (1.6) (resp. (1.8)).

§ 3. The generating functions of the leading terms of $0$-parameter solutions

As is shown in [4], each $P_J$-hierarchy has a formal power series of $\eta^{-1}$ in the form

\[
\begin{align*}
 u_k(t) &= \sum_{j=0}^{\infty} \eta^{-j} \hat{u}_{k,j}(t), \quad v_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{v}_{k,j}(t), \quad j = 1, \ldots, m.
\end{align*}
\]

The solution taking the form of (3.1) is often called a $0$-parameter solution, as the form does not have any free parameters. Let us define the generating functions of the leading terms $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ of their $0$-parameter solutions of $(P_J)_m$ ($J = I, II, 34, IV$) by

\[
\hat{u}_0(\theta) = \sum_{i=1}^{\infty} \hat{u}_{i,0} \theta^i, \quad \hat{v}_0(\theta) = \sum_{i=1}^{\infty} \hat{v}_{i,0} \theta^i.
\]
Each explicit form of (3.2) for \((P_j)_m\) \((j = I, II, 34, IV)\) is given as follows.

\[
(P_1)_m \quad \hat{u}_0 = 1 - \sqrt{\frac{1 + 2C}{1 + 2\hat{u}_{1,0}\theta}} , \quad \hat{v}_0 = 0 .
\]

Here \(\hat{u}_{1,0}\) is taken so that the coefficient of \(\theta^{m+1}\) in \(\hat{u}_0\) is zero.

\[
(P_{34})_m \quad \hat{u}_0 \equiv 1 - \sqrt{\frac{1 + 2(C + c_0\theta)}{(1 + 2(\hat{u}_{1,0} + c_0)\theta)(1 - 2\gamma t\theta^m)}}, \quad \hat{v}_0 = 0 ,
\]

where \(\hat{u}_{1,0}\) and \(\hat{v}_{1,0}\) are taken so that the coefficients of \(\theta^{m+1}\) in \(\hat{u}_0\) and \(\hat{v}_0\) are equal to \(-\hat{w}_{m,0} + c_0 \hat{u}_{m,0} + \frac{\hat{v}_{m,0}^2 - \kappa^2}{2\hat{u}_{m,0}}\) and 0, respectively. Here \(\hat{w}_{m,0}\) is defined by (1.5) with \(u_k, v_k\) and \(w_k\) being replaced by \(\hat{u}_{k,0}, \hat{v}_{k,0}\) and \(\hat{w}_{k,0}\).

\[
(P_{11})_m \quad \hat{u}_0 = (1 + C)
\left(1 - \sqrt{\frac{1}{(1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\right),
\]

\[
(P_{11})_m \quad \hat{v}_0\theta = (1 + C)
\left(-1 + (1 + \hat{u}_{1,0}\theta)\sqrt{\frac{1}{(1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\right).
\]

Here \(\hat{u}_{1,0}\) and \(\hat{v}_{1,0}\) are taken so that the coefficients of \(\theta^{m+1}\) in \(\hat{u}_0\) and \(\hat{v}_0\) are \(\gamma t\) and \(\kappa\), respectively.

\[
(P_{IV})_m \quad \hat{u}_0 \equiv (1 + C)(1 - \sqrt{1/f(t, \theta)}),
\]

\[
(P_{IV})_m \quad \hat{v}_0\theta \equiv (1 + C)(-1 + (1 + \hat{u}_{1,0}\theta)\sqrt{1/f(t, \theta)}) - \gamma t\theta^m ,
\]

\[
f(t, \theta) := (1 + \hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2 - 2\gamma t\theta^m(1 + 2\hat{u}_{1,0}\theta - c_1\theta) ,
\]

where \(\hat{u}_{1,0}\) and \(\hat{v}_{1,0}\) are taken so that the coefficients of \(\theta^{m+1}\) in \(\hat{u}_0\) and \(\hat{v}_0\) are equal to \(-\alpha_1\) and \(-\hat{w}_{m,0} - \frac{\hat{v}_{m,0}^2 - \alpha_1^2 - \alpha_2^2}{2(\hat{u}_{m,0} - c_m)}\), respectively. Here \(\hat{w}_{m,0}\) is defined by (1.10) with \(u_k, v_k\) and \(w_k\) being replaced by \(\hat{u}_{k,0}, \hat{v}_{k,0}\) and \(\hat{w}_{k,0}\).
§ 4. Instanton-type solutions of $(P_J)_m$ ($J=I, II, 34, IV$)

We prepare some notation. Let $\nu_{\pm 1}(t), \ldots, \nu_{\pm m}(t)$ of $(P_J)_m$ denote the roots of the following algebraic equation $\Lambda_J(\lambda, t) = 0$ of $\lambda$ with $\nu_k = -\nu_{|k|}$ if $k < 0$:

- If $J = I$ (resp. 34), then $\Lambda_J(\lambda, t)$ is defined by
  
  $$ g_J(\lambda)^{m} - \sum_{k=1}^{m} \hat{u}_{k,0} g_J(\lambda)^{m-k} $$

  with
  
  $$ g_J(\lambda) = \frac{\lambda^2 - 8\hat{u}_{1,0}}{4} $$

  (resp. $g_{34}(\lambda) = \frac{\lambda^2 - 8(\hat{u}_{1,0} + c_0)}{4}$).

  Here $\hat{u}_{k,0}$'s are defined by (3.2) of $(P_J)_m$ ($J=I, 34$) respectively.

- If $J = II$ or IV, then $\Lambda_J(\lambda, t)$ is defined by
  
  $$ g_J(\lambda)^{m} - \sum_{k=1}^{m} (\hat{u}_{k,0} - c_k) g_J(\lambda)^{m-k} $$

  with
  
  $$ g_J(\lambda) = -\hat{u}_{1,0} - \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}. $$

  Here $\hat{u}_{k,0}, \hat{v}_{1,0}$ are defined by (3.2) of $(P_J)_m$ ($J=II, IV$) respectively and so are $c_k$.

Let $\Omega$ be an open subset in $\mathbb{C}_t$ and the two conditions are always assumed:

(A1) The roots $\nu_i(t)$'s ($1 \leq |i| \leq m$) are mutually distinct for each $t \in \Omega$.

(A2) The function $p_1\nu_1(t) + \cdots + p_m\nu_m(t)$ does not vanish identically on $\Omega$ for any $(p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}$.

Then we have the following theorem.

**Theorem 4.1** ([2], [17]). We have instanton-type solutions of $(P_J)_m$ ($J=I, 34$) with free $2m$-parameters $(\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m}[\eta^{-1}]$ of the form

$$ U = \hat{u}_0 + (1 - \hat{u}_0)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0)v, $$

(4.1) $$

$$ \begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \leq |k| \leq m} f_k^J(\tau, t; \eta) A(\nu_k), \quad A(\nu_k) = \left( \frac{a(\nu_k)}{2} \frac{\nu_k}{a(\nu_k)} \right), $$

(4.2) $$

$$ f_k^J(\tau, t; \eta) = \sum_{j=1}^{\infty} \left( \sum_{\ell \geq 0, p \in \mathbb{Z}^m} \sum_{|\ell| = j, p \vdash \ell} f_{k,p,\ell}^J(t) e^{p \cdot \tau} \right) \eta^{-j/2}, $$

with $a(\nu_k) = \frac{\theta}{1 - \theta g_J(\nu_k)} = \sum_{j=0}^{\infty} g_J(\nu_k)^j \theta^{j+1}$. Here $(\hat{u}_0, \hat{v}_0)$, $\nu_k$ and $g_J$ of $(P_J)_m$ ($J = I, 34$) have been defined in the previous section respectively. For the explicit forms of $f_{k,p,\ell}^J(t)$ ($J = I, 34$), see [2] and [17].
The following describes the relation between instanton-type solutions of \((P_I)_m\) and \((P_{34})_m\).

**Theorem 4.2** ([17]). An instanton-type solution of \((P_{34})_m\) is transformed algebraically to that of \((P_I)_m\) by the replacements of all terms which are depending on their 0-parameter solutions.

**Theorem 4.3** ([16]). We have instanton-type solutions of \((P_J)_m\) \((J = \text{II, IV})\) with free \(2m\)-parameters \((\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m} \eta^{-1}\) of the form

\[
U = \hat{u}_0 + (1 - \hat{u}_0 + C)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0 + C)v,
\]

\[
\begin{pmatrix}
u_k\end{pmatrix} = \sum_{\ell \geq 0, \mu \in \mathbb{Z}^m} f_{k,p,\ell}^J(t)e^{p \cdot \tau}\eta^{-j/2} \quad j = 1, \ell \geq 0, p \in \mathbb{Z}^m |p| = j,
\]

where \(a(\nu_k) = \frac{\theta}{1 - \theta g_3(\nu_k)} = \sum_{j=0}^{\infty} g_3(\nu_k)^j \theta^{j+1}\) and \(g_3(\nu_k) := -\nu_k^2/4 + 2\hat{v}_{1,0}\). Here \((\hat{u}_0, \hat{v}_0), \nu_k\) and \(g_3\) of \((P_J)_m\) \((J = \text{II, IV})\) have been defined in the previous section respectively. For the explicit forms of \(f_{k,p,\ell}^J(t)\) \((J = \text{II, IV})\), see [16].

The following describes the relations between instanton-type solutions of \((P_{\text{II}})_m\) and \((P_{\text{IV}})_m\).

**Theorem 4.4** ([16]). An instanton-type solution of \((P_{\text{II}})_m\) is transformed algebraically to that of \((P_{\text{IV}})_m\) by the replacements of all terms which are depending on their 0-parameter solutions.

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**References**