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<th>A relation between instanton-type solutions of $P_J$ (J=I,II,34,IV)-hierarchies with a large parameter (Recent development of microlocal analysis and asymptotic analysis)</th>
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Kyoto University
A relation between instanton-type solutions of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter

By

Yoko UMETA*

Abstract

We report a relation between instanton-type solutions for equations of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter. The content of these notes is a short summary of our forthcoming papers [2], [16] and [17].

§ 1. Definitions of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter $\eta$

We recall the definitions of equations of $P_J$ $(J = I, II, 34, IV)$-hierarchies with a large parameter $\eta$ given in [15], [8] and [9].

(i) The $m$-th members $(P_I)_m$ and $(P_{34})_m$ of $P_I$, $P_{34}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ $(k = 1, 2, \ldots)$ be unknown functions of $t$ and $c_k$'s are constants. In what follows, $\delta_{jm}$ stands for Kronecker's delta.

- For $m = 1, 2, \ldots$, $(P_I)_m$ has the following form (see [15]):

\begin{equation}
\eta^{-1} \frac{du_j}{dt} = 2v_j, \quad j = 1, 2, \ldots, m,
\end{equation}

\begin{equation}
\eta^{-1} \frac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), \quad j = 1, 2, \ldots, m,
\end{equation}

with the assumption $u_{m+1} = 0$. Here $w_j$ is recursively defined by

\begin{equation}
w_j = \frac{1}{2} \sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} u_k v_{j-k} + c_j + \delta_{jm} t.
\end{equation}

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For $m=1,2,\ldots$, $(P_{34})_m$ has the following form (see [9]):

$$
\begin{align*}
\eta^{-1} \frac{du_j}{dt} &= 2v_j, & j &= 1,2,\ldots,m, \\
\eta^{-1} \frac{dv_j}{dt} &= 2(u_{j+1} + u_1 u_j + w_j), & j &= 1,2,\ldots,m,
\end{align*}
$$

with

$$
u_{m+1} = -w_m + c_0 u_m + \frac{v_m^2 - \kappa^2}{2u_m}.
$$

Here $\gamma(\neq 0)$, $\kappa$ and $\{c_j\}_{j=0}^{m}$ are constants, and $w_j$ is recursively defined by

$$
w_j = \frac{1}{2} \sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} v_k v_{j-k} + c_j
$$

$$
+ c_0 \left( 2u_j - \sum_{k=1}^{j-1} u_k u_{j-k} \right) + \delta_{j,m-1} \gamma t + 2\delta_{jm} \gamma tc_0.
$$

Note that the form above has been slightly modified from the original form given by [9]. If we replace $u_m$ (resp. $v_m$) in (1.3) and (1.4) with $u_m - \gamma t$ (resp. $v_m - \eta^{-1} \frac{\gamma}{2}$), then we have the original form of $(P_{34})_m$.

(ii) The $m$-th members $(P_{II})_m$ and $(P_{IV})_m$ of $P_{II}$, $P_{IV}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ ($k = 1,2,\ldots$) be unknown functions of $t$ and $c_k$'s are constants.

For $m=1,2,\ldots$, $(P_{II})_m$ has the following form (see [8]):

$$
\begin{align*}
\eta^{-1} \frac{du_j}{dt} &= -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1, & j &= 1,2,\ldots,m, \\
\eta^{-1} \frac{dv_j}{dt} &= 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j &= 1,2,\ldots,m,
\end{align*}
$$

with $u_{m+1} = \gamma t$ and $v_{m+1} = \kappa$. Here $\gamma(\neq 0)$, $\kappa$ and $c_j$'s are constants, and $w_j$ is recursively defined by

$$
w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k.
$$

For $m=1,2,\ldots$, $(P_{IV})_m$ has the following form (see [8]):

$$
\begin{align*}
\eta^{-1} \frac{du_j}{dt} &= -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1 - 2\delta_{j,m-1} \gamma t, \\
\eta^{-1} \frac{dv_j}{dt} &= 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j &= 1,2,\ldots,m,
\end{align*}
$$
with
\[
(1.9) \quad u_{m+1} = -\alpha_1, \quad v_{m+1} = -w_m - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - c_m)}.
\]

Here $\gamma(\neq 0)$, $\alpha_1$, $\alpha_2$ and $c_j$'s are constants, and $w_j$ is defined by
\[
(1.10) \quad w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} v_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k + \delta_{jm} \gamma tv_1.
\]

Note that the form above has been slightly modified from the original form given by [8]. If we replace $u_m$ in (1.8) and (1.9) with $u_m - \gamma t$, then we have the original form of $(P_{IV})_m$.

§ 2. $P_J$ ($J = I, II, 34, IV$)-hierarchies with $\eta$ in terms of generating functions

In this note, we consider the represented forms with generating functions of unknown functions. Let $\theta$ denote an independent variable.

(i) The $m$-th members $(P_I)_m$ and $(P_{34})_m$ of $P_I, P_{34}$-hierarchies with $\eta$

We define generating functions $U$, $V$ and $C$ of $(P_I)_m$ (resp. $(P_{34})_m$) by
\[
(2.1) \quad U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \\
C(\theta) = \sum_{k=1}^{\infty} (c_k + \delta_{km} t) \theta^{k+1} \quad (\text{resp. } C(\theta) = \sum_{k=1}^{\infty} c_k \theta^{k+1}),
\]
respectively. Here $u_k, v_k, c_k$ ($k = 1, 2, \ldots$) denote unknown functions and constants of $(P_I)_m$ (resp. $(P_{34})_m$). In what follows, by $A \equiv B$ we mean that $A - B$ is zero modulo $\theta^{m+2}$.

• $(P_I)_m$ is rewritten in the following form
\[
(2.2) \quad \eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta V \theta \\ V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ -(1 + 2u_1 \theta)(1 - U) + \frac{1 + 2C - \theta V^2}{1 - U} \end{pmatrix}
\]
with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are zero.

• $(P_{34})_m$ is rewritten in the following form
\[
(2.3) \quad \eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta V \theta \\ V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ -(1 + 2(u_1 + c_0) \theta)(1 - U) + \frac{1 + 2C - \theta(V^2 - 2c_0)}{1 - U} \end{pmatrix} + \begin{pmatrix} 0 \\ 2\gamma \theta^m (1 + (u_1 + 2c_0) \theta) \end{pmatrix}
\]
with the condition that the coefficient of $\theta^{m+1}$ of $U$ (resp. $V$) is equal to the right hand side of (1.4) (resp. zero).

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m + 1$) on the both sides of (2.2) (resp. (2.3)), we obtain (1.1) (resp. (1.3)).

(ii) The $m$-th members $(P_{II})_m$ and $(P_{IV})_m$ of $P_{II}$, $P_{IV}$-hierarchies with $\eta$

We define generating functions $U$, $V$ and $C$ of $(P_{II})_m$ (resp. $(P_{IV})_m$) by

$$(2.4) \quad U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \quad C(\theta) = \sum_{k=1}^{\infty} c_k \theta^k,$$

respectively. Here $u_k$, $v_k$, $c_k$ ($k = 1, 2, \ldots$) denote unknown functions and constants of $(P_{II})_m$ (resp. $(P_{IV})_m$).

- $(P_{II})_m$ is rewritten in the following form

$$(2.5) \quad \eta^{-1} \frac{d}{dt} \left[ \eta \theta U(\theta) V(\theta) \right] \equiv 2 \left( \frac{u_1(1 - U + C)\theta - U - V\theta}{v_1(1 - U + C)\theta + 2UV + V^2\theta} + V \right)$$

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are $\gamma t$ and $\kappa$, respectively.

- $(P_{IV})_m$ is rewritten in the following form

$$(2.6) \quad \eta^{-1} \frac{d}{dt} \left[ \eta \theta U(\theta) V(\theta) \right] \equiv 2 \left( \frac{u_1(1 - U - C)\theta - U - V\theta - \gamma t\theta^m}{v_1(1 - U + C)\theta + 2UV + V^2\theta} + V + \gamma tv_1\theta^{m+1} \right)$$

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are equal to the right hand sides of (1.9), respectively.

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m + 1$) on the both sides of (2.5) (resp. (2.6)), we obtain (1.6) (resp. (1.8)).

§ 3. The generating functions of the leading terms of 0-parameter solutions

As is shown in [4], each $P_J$-hierarchy has a formal power series of $\eta^{-1}$ in the form

$$(3.1) \quad u_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{u}_{k,j}(t), \quad v_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{v}_{k,j}(t), \quad j = 1, \ldots, m.$$

The solution taking the form of (3.1) is often called a 0-parameter solution, as the form does not have any free parameters. Let us define the generating functions of the leading terms $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ of their 0-parameter solutions of $(P_J)_m$ ($J = I, II, 34, IV$) by

$$(3.2) \quad \hat{u}_0(\theta) = \sum_{i=1}^{\infty} \hat{u}_{i,0} \theta^i, \quad \hat{v}_0(\theta) = \sum_{i=1}^{\infty} \hat{v}_{i,0} \theta^i.$$
Each explicit form of (3.2) for \((P_J)_m\) \((J = I, II, 34, IV)\) is given as follows.

<table>
<thead>
<tr>
<th>( (P_J)_m )</th>
<th>( \hat{u}<em>0 = 1 - \sqrt{\frac{1 + 2C}{1 + 2\hat{u}</em>{1,0}\theta}} ), ( \hat{v}_0 = 0 ).</th>
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<tr>
<td>( \hat{u}<em>0 \equiv 1 - \sqrt{\frac{1 + 2(C + c_0\theta)}{(1 + 2(\hat{u}</em>{1,0} + c_0)\theta)(1 - 2\gamma t\theta^m)}} ), ( \hat{v}_0 = 0 ),</td>
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<tr>
<td>( \hat{u}<em>{1,0} ) and ( \hat{v}</em>{1,0} ) are taken so that the coefficients of ( \theta^{m+1} ) in ( \hat{u}<em>0 ) and ( \hat{v}<em>0 ) are equal to (-\hat{w}</em>{m,0} + c_0\hat{u}</em>{m,0} + \frac{(\hat{v}<em>{1,0}^2 - \kappa^2)}{2\hat{u}</em>{m,0}}) and (0), respectively. Here ( \hat{w}<em>{m,0} ) is defined by (1.5) with ( u_k, v_k ) and ( w_k ) being replaced by ( \hat{u}</em>{k,0}, \hat{v}<em>{k,0} ) and ( \hat{w}</em>{k,0} ).</td>
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<tr>
<td>( \hat{u}<em>0 = (1 + C)(1 - \sqrt{\frac{1}{(1 + \hat{u}</em>{1,0}\theta)^2 - 2\hat{v}<em>{1,0}\theta^2}}), \hat{v}<em>0 \theta = (1 + C)(-1 + (1 + \hat{u}</em>{1,0}\theta))\sqrt{\frac{1}{(1 + \hat{u}</em>{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}} ).</td>
<td></td>
</tr>
<tr>
<td>( \hat{u}<em>{1,0} ) and ( \hat{v}</em>{1,0} ) are taken so that the coefficients of ( \theta^{m+1} ) in ( \hat{u}_0 ) and ( \hat{v}_0 ) are ( \gamma t ) and ( \kappa ), respectively.</td>
<td></td>
</tr>
<tr>
<td>( \hat{u}_0 \equiv (1 + C)(1 - \sqrt{1/f(t, \theta)}), \hat{v}<em>0 \theta \equiv (1 + C)(-1 + (1 + \hat{u}</em>{1,0}\theta))\sqrt{1/f(t, \theta)}) - \gamma t\theta^m, )</td>
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<tr>
<td>( f(t, \theta) := (1 + \hat{u}<em>{1,0}\theta)^2 - 2\hat{v}</em>{1,0}\theta^2 - 2\gamma t\theta^m(1 + 2\hat{u}_{1,0}\theta - c_1\theta), )</td>
<td></td>
</tr>
<tr>
<td>( \hat{u}<em>{1,0} ) and ( \hat{v}</em>{1,0} ) are taken so that the coefficients of ( \theta^{m+1} ) in ( \hat{u}<em>0 ) and ( \hat{v}<em>0 ) are equal to (-\alpha_1 ) and (-\hat{w}</em>{m,0} - \frac{(\hat{v}</em>{m,0} - \alpha_1)^2 - \alpha_2^2}{2(\hat{u}<em>{m,0} - c_m)}), respectively. Here ( \hat{w}</em>{m,0} ) is defined by (1.10) with ( u_k, v_k ) and ( w_k ) being replaced by ( \hat{u}<em>{k,0}, \hat{v}</em>{k,0} ) and ( \hat{w}_{k,0} ).</td>
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§ 4. Instanton-type solutions of \((P_J)_m\) \((J = I, II, 34, IV)\)

We prepare some notation. Let \(\nu_{\pm 1}(t), \ldots, \nu_{\pm m}(t)\) of \((P_J)_m\) denote the roots of the following algebraic equation \(\Lambda_J(\lambda, t) = 0\) of \(\lambda\) with \(\nu_k = -\nu_{|k|}\) if \(k < 0\):

- If \(J = I\) (resp. 34), then \(\Lambda_J(\lambda, t)\) is defined by

\[
g_J(\lambda)^m - \sum_{k=1}^{m} \hat{u}_{k,0} g_J(\lambda)^{m-k}
\]

with

\[
g_J(\lambda) = \frac{\lambda^2 - 8\hat{u}_{1,0}}{4} \quad \text{(resp. } g_{34}(\lambda) = \frac{\lambda^2 - 8(\hat{u}_{1,0} + c_0)}{4}).
\]

Here \(\hat{u}_{k,0}\)'s are defined by (3.2) of \((P_J)_m\) \((J = I, 34)\) respectively.

- If \(J = II\) or IV, then \(\Lambda_J(\lambda, t)\) is defined by

\[
g_J(\lambda)^m - \sum_{k=1}^{m} (\hat{u}_{k,0} - c_k) g_J(\lambda)^{m-k}
\]

with

\[
g_J(\lambda) = -\hat{u}_{1,0} - \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}.
\]

Here \(\hat{u}_{k,0}, \hat{v}_{1,0}\) are defined by (3.2) of \((P_J)_m\) \((J = II, IV)\) respectively and so are \(c_k\).

Let \(\Omega\) be an open subset in \(\mathbb{C}_t\) and the two conditions are always assumed:

(A1) The roots \(\nu_i(t)'s (1 \leq |i| \leq m)\) are mutually distinct for each \(t \in \Omega\).

(A2) The function \(p_{1}\nu_{1}(t) + \cdots + p_{m}\nu_{m}(t)\) does not vanish identically on \(\Omega\) for any \((p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}\).

Then we have the following theorem.

**Theorem 4.1** ([2], [17]). We have instanton-type solutions of \((P_J)_m\) \((J = I, 34)\) with free \(2m\)-parameters \((\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m}[\eta^{-1}]\) of the form

\[
U = \hat{u}_0 + (1 - \hat{u}_0)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0)v,
\]

(4.1)

\[
\begin{pmatrix}
u \\
\end{pmatrix} = \sum_{1 \leq |k| \leq m} f^J_k(\tau, t; \eta) A(\nu_k), \quad A(\nu_k) = \left(\frac{a(\nu_k)}{2\nu_k} \frac{\theta^{j+1}}{a(\nu_k)}\right),
\]

(4.2)

\[
f^J_k(\tau, t; \eta) = \sum_{j=1}^{\infty} \left( \sum_{l \geq 0, p \in \mathbb{Z}^m \atop 2|p| = j} f^J_{k,p,l}(t)e^{p \cdot \tau} \right) \eta^{-j/2},
\]

with \(a(\nu_k) = \frac{\theta}{1 - \theta g_J(\nu_k)} = \sum_{j=0}^{\infty} g_j(\nu_k) j^j \theta^{j+1}\). Here \((\hat{u}_0, \hat{v}_0), \nu_k\) and \(g_j\) of \((P_J)_m\) \((J = I, 34)\) have been defined in the previous section respectively. For the explicit forms of \(f^J_{k,p,l}(t)\) \((J = I, 34)\), see [2] and [17].
The following describes the relation between instanton-type solutions of \((P_{I})_m\) and \((P_{34})_m\).

**Theorem 4.2** ([17]). An instanton-type solution of \((P_{34})_m\) is transformed algebraically to that of \((P_{I})_m\) by the replacements of all terms which are depending on their 0-parameter solutions.

**Theorem 4.3** ([16]). We have instanton-type solutions of \((P_{J})_m\) \((J = II, IV)\) with free \(2m\)-parameters \((\beta_{-m}, \ldots, \beta_{m}) \in \mathbb{C}^{2m}[\eta^{-1}]\) of the form

\[
U = \hat{u}_0 + (1 - \hat{u}_0 + C)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0 + C)v,
\]

\[(4.3) \quad \begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \leq |k| \leq m} f^J_k(\tau, t; \eta)A(\nu_k), \quad A(\nu_k) := \begin{pmatrix} a(\nu_k) \\ g_3(\nu_k)a(\nu_k) \end{pmatrix} \]

\[(4.4) \quad f^1_k(\tau, t; \eta) = \sum_{j=1}^{\infty} \left( \sum_{l_2 \geq 0, p \in \mathbb{Z}^m} f^{l_1}_{k,p,\ell}(t)e^{p \cdot \tau} \right)\eta^{-j/2}, \]

where \(a(\nu_k) = \frac{\theta}{1 - \theta g_3(\nu_k)} = \sum_{j=0}^{\infty} g_3(\nu_k)^j\theta^{i+1} \) and \(g_3(\nu_k) := -\frac{\nu_k}{2} + \sqrt{\frac{\nu_k^2}{4} + 2\hat{v}_{1,0}}\). Here \((\hat{u}_0, \hat{v}_0), \nu_k \) and \(g_3\) of \((P_{J})_m\) \((J = II, IV)\) have been defined in the previous section respectively. For the explicit forms of \(f^1_k(t)\) \((J = II, IV)\), see [16].

The following describes the relations between instanton-type solutions of \((P_{II})_m\) and \((P_{IV})_m\).

**Theorem 4.4** ([16]). An instanton-type solution of \((P_{II})_m\) is transformed algebraically to that of \((P_{IV})_m\) by the replacements of all terms which are depending on their 0-parameter solutions.

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**References**


[17] , Instanton-type solutions of $P_{34}$-hierarchy with a large parameter, in preparation.