A relation between instanton-type solutions of $P_J$ (J=I, II, III, IV)-hierarchies with a large parameter (Recent development of microlocal analysis and asymptotic analysis)

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A relation between instanton-type solutions of $P_J$
($J = I, II, 34, IV$)-hierarchies with a large parameter

By

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Abstract

We report a relation between instanton-type solutions for equations of $P_J$ ($J = I, II, 34, IV$)-hierarchies with a large parameter. The content of these notes is a short summary of our forthcoming papers [2], [16] and [17].

§ 1. Definitions of $P_J$ ($J = I, II, 34, IV$)-hierarchies with a large parameter $\eta$

We recall the definitions of equations of $P_J$ ($J = I, II, 34, IV$)-hierarchies with a large parameter $\eta$ given in [15], [8] and [9].

(i) The $m$-th members $(P_I)_m$ and $(P_{34})_m$ of $P_I$, $P_{34}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ ($k = 1, 2, \ldots$) be unknown functions of $t$ and $c_k$'s are constants. In what follows, $\delta_{jm}$ stands for Kronecker's delta.

• For $m = 1, 2, \ldots$, $(P_I)_m$ has the following form (see [15]):

\begin{equation}
\begin{cases}
\eta^{-1} \frac{du_j}{dt} = 2v_j, & j = 1, 2, \ldots, m, \\
\eta^{-1} \frac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \ldots, m,
\end{cases}
\end{equation}

with the assumption $u_{m+1} = 0$. Here $w_j$ is recursively defined by

\begin{equation}
w_j = \frac{1}{2} \sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm}t.
\end{equation}

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• For $m = 1, 2, \ldots$, $(P_{34})_m$ has the following form (see [9]):

\[
\begin{aligned}
\eta^{-1} \frac{du_j}{dt} &= 2v_j, & j = 1, 2, \ldots, m, \\
\eta^{-1} \frac{dv_j}{dt} &= 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \ldots, m,
\end{aligned}
\]

with

\[
u_{m+1} = -w_m + c_0 u_m + \frac{v_m^2 - \kappa^2}{2u_m}.
\]

Here $\gamma(\neq 0)$, $\kappa$ and $\{c_j\}_{j=0}^m$ are constants, and $w_j$ is recursively defined by

\[
w_j = \frac{1}{2} \sum_{k=1}^{j-1} u_k u_{j-1-k} - \frac{1}{2} \sum_{k=1}^{j-1} v_k v_{j-k} + c_j + c_0 \left(2u_j - \sum_{k=1}^{j-1} u_k u_{j-k}\right) + \delta_{j,m-1} \gamma t + 2\delta_{jm} \gamma tc_0.
\]

Note that the form above has been slightly modified from the original form given by [9]. If we replace $u_m$ (resp. $v_m$) in (1.3) and (1.4) with $u_m - \gamma t$ (resp. $v_m - \eta^{-1} \frac{\gamma}{2}$), then we have the original form of $(P_{34})_m$.

(ii) The $m$-th members $(P_{II})_m$ and $(P_{IV})_m$ of $P_{II}$, $P_{IV}$-hierarchies with $\eta$

Let $u_k$ and $v_k$ ($k = 1, 2, \ldots$) be unknown functions of $t$ and $c_k$'s are constants.

• For $m = 1, 2, \ldots$, $(P_{II})_m$ has the following form (see [8]):

\[
\begin{aligned}
\eta^{-1} \frac{du_j}{dt} &= -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1, & j = 1, 2, \ldots, m, \\
\eta^{-1} \frac{dv_j}{dt} &= 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j = 1, 2, \ldots, m,
\end{aligned}
\]

with $u_{m+1} = \gamma t$ and $v_{m+1} = \kappa$. Here $\gamma(\neq 0)$, $\kappa$ and $c_j$'s are constants, and $w_j$ is recursively defined by

\[
w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k.
\]

• For $m = 1, 2, \ldots$, $(P_{IV})_m$ has the following form (see [8]):

\[
\begin{aligned}
\eta^{-1} \frac{du_j}{dt} &= -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1 - 2\delta_{j,m-1} \gamma t, \\
\eta^{-1} \frac{dv_j}{dt} &= 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j = 1, 2, \ldots, m,
\end{aligned}
\]
with
\[ u_{m+1} = -\alpha_1, \quad v_{m+1} = -w_m - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - c_m)}. \]

Here \( \gamma(\neq 0) \), \( \alpha_1 \), \( \alpha_2 \) and \( c_j \)'s are constants, and \( w_j \) is defined by
\[ w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^{j} u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k + \delta_{jm} \gamma t v_1. \]

Note that the form above has been slightly modified from the original form given by [8]. If we replace \( u_m \) in (1.8) and (1.9) with \( u_m - \gamma t \), then we have the original form of \((P_{IV})_m\).

\section{\( P_J \) \((J = I, II, 34, IV)\)-hierarchies with \( \eta \) in terms of generating functions}

In this note, we consider the represented forms with generating functions of unknown functions. Let \( \theta \) denotes an independent variable.

(i) The \( m \)-th members \( (P_I)_m \) and \( (P_{34})_m \) of \( P_I \), \( P_{34} \)-hierarchies with \( \eta \)

We define generating functions \( U \), \( V \) and \( C \) of \( (P_I)_m \) (resp. \( (P_{34})_m \)) by
\[ U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \quad C(\theta) = \sum_{k=1}^{\infty} (c_k + \delta_{km} t) \theta^{k+1} \]
respectively. Here \( u_k, v_k, c_k \) \((k = 1, 2, \ldots)\) denote unknown functions and constants of \( (P_I)_m \) (resp. \( (P_{34})_m \)). In what follows, by \( A \equiv B \) we mean that \( A - B \) is zero modulo \( \theta^{m+2} \).

- \( (P_I)_m \) is rewritten in the following form
\[ \eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ -(1+2u_1 \theta)(1-U) + \frac{1+2C - \theta V^2}{1-U} \end{pmatrix} \]

with the condition that the coefficients of \( \theta^{m+1} \) of \( U \) and \( V \) are zero.
- \( (P_{34})_m \) is rewritten in the following form
\[ \eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta V \theta \end{pmatrix} \equiv \begin{pmatrix} 2V \theta \\ -(1+2(u_1+c_0) \theta)(1-U) + \frac{1+2C - \theta (V^2-2c_0)}{1-U} \end{pmatrix} + \begin{pmatrix} 0 \\ 2\gamma \theta^m (1+(u_1+2c_0) \theta) \end{pmatrix} \]
with the condition that the coefficient of $\theta^{m+1}$ of $U$ (resp. $V$) is equal to the right hand side of (1.4) (resp. zero).

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m+1$) on the both sides of (2.2) (resp. (2.3)), we obtain (1.1) (resp. (1.3)).

(ii) The $m$-th members $(P_{II})_m$ and $(P_{IV})_m$ of $P_{II}, P_{IV}$-hierarchies with $\eta$

We define generating functions $U$, $V$ and $C$ of $(P_{II})_m$ (resp. $(P_{IV})_m$) by

\begin{align}
U(\theta) &= \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \\
C(\theta) &= \sum_{k=1}^{\infty} c_k \theta^k,
\end{align}

respectively. Here $u_k, v_k, c_k$ ($k = 1, 2, \ldots$) denote unknown functions and constants of $(P_{II})_m$ (resp. $(P_{IV})_m$).

- $(P_{II})_m$ is rewritten in the following form

\begin{align}
\eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta V \theta \end{pmatrix} &\equiv 2 \begin{pmatrix} u_1(1-U+C)\theta - U - V \theta \\
-v_1(1-U+C)\theta + \frac{2UV + V^2\theta}{2(1-U+C)} + V \end{pmatrix} 
\end{align}

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are $\gamma t$ and $\kappa$, respectively.

- $(P_{IV})_m$ is rewritten in the following form

\begin{align}
\eta^{-1} \frac{d}{dt} \begin{pmatrix} U \theta V \theta \end{pmatrix} &\equiv 2 \begin{pmatrix} u_1(1-U+C)\theta - U - V \theta - \gamma t \theta^m \\
-v_1(1-U+C)\theta + \frac{2UV + V^2\theta}{2(1-U+C)} + V + \gamma tv_1 \theta^{m+1} \end{pmatrix} 
\end{align}

with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are equal to the right hand sides of (1.9), respectively.

Remark that, if we compare the coefficients with respect to $\theta^j$ ($2 \leq j \leq m+1$) on the both sides of (2.5) (resp. (2.6)), we obtain (1.6) (resp. (1.8)).

§ 3. The generating functions of the leading terms of 0-parameter solutions

As is shown in [4], each $P_J$-hierarchy has a formal power series of $\eta^{-1}$ in the form

\begin{align}
(3.1) \quad u_k(t) &= \sum_{j=0}^{\infty} \eta^{-j} \hat{u}_{k,j}(t), \quad v_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{v}_{k,j}(t), \quad j = 1, \ldots, m.
\end{align}

The solution taking the form of (3.1) is often called a 0-parameter solution, as the form does not have any free parameters. Let us define the generating functions of the leading terms $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ of their 0-parameter solutions of $(P_J)_m$ ($J = I, II, 34, IV$) by

\begin{align}
(3.2) \quad \hat{u}_0(\theta) &= \sum_{i=1}^{\infty} \hat{u}_{i,0}\theta^i, \quad \hat{v}_0(\theta) = \sum_{i=1}^{\infty} \hat{v}_{i,0}\theta^i.
\end{align}
Each explicit form of (3.2) for \((P_J)_{m}\) \((J = I, II, 34, IV)\) is given as follows.

<table>
<thead>
<tr>
<th>(\text{(P}<em>J\text{)}</em>{m})</th>
<th>(\hat{u}<em>0 = 1 - \frac{1 + 2C}{1 + 2\hat{u}</em>{1,0}\theta} ), (\hat{v}<em>0 = 0). Here (\hat{u}</em>{1,0}) is taken so that the coefficient of (\theta^{m+1}) in (\hat{u}_0) is zero.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{(P}<em>{34}\text{)}</em>{m})</td>
<td>(\hat{u}<em>0 \equiv 1 - \sqrt{\frac{1 + 2(C + c_0\theta)}{(1 + 2(\hat{u}</em>{1,0} + c_0)\theta)(1 - 2\gamma t\theta^m)}} ), (\hat{v}<em>0 = 0), where (\hat{u}</em>{1,0}) and (\hat{v}<em>{1,0}) are taken so that the coefficients of (\theta^{m+1}) in (\hat{u}<em>0) and (\hat{v}<em>0) are equal to (-\hat{w}</em>{m,0} + c_0\hat{u}</em>{m,0} + \frac{(\hat{v}</em>{m,0}^2 - \kappa^2)}{2\hat{u}<em>{m,0}}) and 0, respectively. Here (\hat{w}</em>{m,0}) is defined by (1.5) with (u_k), (v_k) and (w_k) being replaced by (\hat{u}<em>{k,0}), (\hat{v}</em>{k,0}) and (\hat{w}_{k,0}).</td>
</tr>
<tr>
<td>(\text{(P}<em>{11}\text{)}</em>{m})</td>
<td>(\hat{u}<em>0 = (1 + C)\left(1 - \sqrt{\frac{1}{(1 + \hat{u}</em>{1,0}\theta)^2 - 2\hat{v}<em>{1,0}\theta^2}} \right)), (\hat{v}<em>0 = (1 + C)\left(-1 + (1 + \hat{u}</em>{1,0}\theta)\sqrt{\frac{1}{(1 + \hat{u}</em>{1,0}\theta)^2 - 2\hat{v}<em>{1,0}\theta^2}} \right)). Here (\hat{u}</em>{1,0}) and (\hat{v}_{1,0}) are taken so that the coefficients of (\theta^{m+1}) in (\hat{u}_0) and (\hat{v}_0) are (\gamma t) and (\kappa), respectively.</td>
</tr>
<tr>
<td>(\text{(P}<em>{1}\text{)}</em>{m})</td>
<td>(\hat{u}<em>0 \equiv (1 + C)(1 - \sqrt{1/f(t, \theta)})), (\hat{v}<em>0 \equiv (1 + C)(-1 + (1 + \hat{u}</em>{1,0}\theta)\sqrt{1/f(t, \theta)}) - \gamma t\theta^m), (f(t, \theta) := (1 + \hat{u}</em>{1,0}\theta)^2 - 2\hat{v}<em>{1,0}\theta^2 - 2\gamma t\theta^m(1 + 2\hat{u}</em>{1,0}\theta - c_1\theta)), where (\hat{u}<em>{1,0}) and (\hat{v}</em>{1,0}) are taken so that the coefficients of (\theta^{m+1}) in (\hat{u}<em>0) and (\hat{v}<em>0) are equal to (-\alpha_1) and (-\hat{w}</em>{m,0} - \frac{(\hat{v}</em>{m,0} - \alpha_1)^2 - \alpha_2^2}{2(\hat{u}<em>{m,0} - c_m)}), respectively. Here (\hat{w}</em>{m,0}) is defined by (1.10) with (u_k), (v_k) and (w_k) being replaced by (\hat{u}<em>{k,0}), (\hat{v}</em>{k,0}) and (\hat{w}_{k,0}).</td>
</tr>
</tbody>
</table>
§ 4. Instanton-type solutions of \((P_J)_m\) (\(J = I, II, 34, IV\))

We prepare some notation. Let \(\nu_{\pm 1}(t), \ldots, \nu_{\pm m}(t)\) of \((P_J)_m\) denote the roots of the following algebraic equation \(\Lambda_J(\lambda, t) = 0\) of \(\lambda\) with \(\nu_k = -\nu_{|k|}\) if \(k < 0\):

- If \(J = I\) (resp. 34), then \(\Lambda_J(\lambda, t)\) is defined by
  \[
  g_J(\lambda)^m - \sum_{k=1}^{m} \hat{u}_{k,0} g_J(\lambda)^{m-k}
  \]
  with
  \[
  g_J(\lambda) = \frac{\lambda^2 - 8\hat{u}_{1,0}}{4} \quad \text{(resp. } g_{34}(\lambda) = \frac{\lambda^2 - 8(\hat{u}_{1,0} + c_0)}{4}).
  \]
  Here \(\hat{u}_{k,0}\)'s are defined by (3.2) of \((P_J)_m\) \((J = I, 34)\) respectively.

- If \(J = II\) or 4, then \(\Lambda_J(\lambda, t)\) is defined by
  \[
  g_J(\lambda)^m - \sum_{k=1}^{m} (\hat{u}_{k,0} - c_k) g_J(\lambda)^{m-k}
  \]
  with
  \[
  g_J(\lambda) = -\hat{u}_{1,0} - \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}.
  \]

Here \(\hat{u}_{k,0}, \hat{v}_{1,0}\) are defined by (3.2) of \((P_J)_m\) \((J = II, IV)\) respectively and so are \(c_k\).

Let \(\Omega\) be an open subset in \(\mathbb{C}_t\) and the two conditions are always assumed:

(A1) The roots \(\nu_i(t)\)'s \((1 \leq |i| \leq m)\) are mutually distinct for each \(t \in \Omega\).

(A2) The function \(p_1\nu_1(t) + \cdots + p_m\nu_m(t)\) does not vanish identically on \(\Omega\) for any \((p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}.

Then we have the following theorem.

**Theorem 4.1** ([2], [17]). We have instanton-type solutions of \((P_J)_m\) \((J = I, 34)\) with free \(2m\)-parameters \((\beta_{-m}, \ldots, \beta_m) \in \mathbb{C}^{2m}[\eta^{-1}]\) of the form

\[
U = \hat{u}_0 + (1 - \hat{u}_0)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0)v,
\]

\[
\begin{pmatrix}
u \\
\end{pmatrix} = \sum_{1 \leq |k| \leq m} f_k^J(\tau; \eta) A(\nu_k), \quad A(\nu_k) = \left( \frac{a(\nu_k)}{2a(\nu_k)} \right),
\]

\[
f_k^J(\tau; \eta) = \sum_{j=1}^{\infty} \left( \sum_{\ell \geq 0, p \in \mathbb{Z}^m, |p|=j} f_{k,p,\ell}^J(t) e^{p \cdot \tau} \right) \eta^{-j/2},
\]

with \(a(\nu_k) = \frac{\theta}{1 - \theta g_j(\nu_k)} = \sum_{j=0}^{\infty} g_j(\nu_k)^j \theta^{j+1}\). Here \((\hat{u}_0, \hat{v}_0), \nu_k\) and \(g_j\) of \((P_J)_m\) \((J = I, 34)\) have been defined in the previous section respectively. For the explicit forms of \(f_{k,p,\ell}^J(t)\) \((J = I, 34)\), see [2] and [17].
The following describes the relation between instanton-type solutions of \((P_{I})_{m}\) and \((P_{34})_{m}\).

**Theorem 4.2** ([17]). An instanton-type solution of \((P_{34})_{m}\) is transformed algebraically to that of \((P_{I})_{m}\) by the replacements of all terms which are depending on their 0-parameter solutions.

**Theorem 4.3** ([16]). We have instanton-type solutions of \((P_{J})_{m}\) \((J = I, III, IV)\) with free 2m-parameters \((\beta_{-m}, \ldots, \beta_{m}) \in \mathbb{C}^{2m}[\eta^{-1}]\) of the form

\[
U = \hat{u}_0 + (1 - \hat{u}_0 + C)u, \quad V = \hat{v}_0 + (1 - \hat{u}_0 + C)v,
\]

\[
\begin{pmatrix}
u 
\end{pmatrix} = \sum_{1 \leq |k| \leq m} f^{J}_{k}(\tau, t; \eta)A(\nu_{k}), \quad A(\nu_{k}) := \begin{pmatrix} a(\nu_{k}) \\ g_{3}(\nu_{k})a(\nu_{k}) \end{pmatrix}
\]

\[
f^{J}_{k}(\tau, t; \eta) = \sum_{j=1}^{\infty} \left( \sum_{\ell \geq 0, p \in \mathbb{Z}^{m}} f^{J}_{k,p,\ell}(t)e^{\ell \cdot \tau} \right) \eta^{-j/2},
\]

where \(a(\nu_{k}) = \frac{\theta}{1 - g_{3}(\nu_{k})} = \sum_{j=0}^{\infty} g_{3}(\nu_{k})^{j+1}\) and \(g_{3}(\nu_{k}) := -\frac{\nu_{k}}{2} + \sqrt{\frac{\nu_{k}^{2}}{4} + 2\hat{v}_{1,0}}\). Here \((\hat{u}_0, \hat{v}_0), \nu_{k}\) and \(g_{3}\) of \((P_{3})_{m}\) \((J = II, IV)\) have been defined in the previous section respectively. For the explicit forms of \(f^{J}_{k,p,\ell}(t)\) \((J = II, IV)\), see [16].

The following describes the relations between instanton-type solutions of \((P_{II})_{m}\) and \((P_{IV})_{m}\).

**Theorem 4.4** ([16]). An instanton-type solution of \((P_{II})_{m}\) is transformed algebraically to that of \((P_{IV})_{m}\) by the replacements of all terms which are depending on their 0-parameter solutions.

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**References**


