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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1861: 17-24</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195316">http://hdl.handle.net/2433/195316</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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Borel sums of Voros coefficients of hypergeometric differential equations with a large parameter

By

Takashi AOKI* and Mika TANDA**

§ 1. Introduction

The notion of Voros coefficients was introduced by Voros [11] for some Schrödinger equations with irregular singularities. It plays a role in the analysis of Stokes phenomena for WKB solutions with respect to parameters which are contained in the potentials. For Weber equations and for Whittaker equations, concrete forms of the Voros coefficient were obtained by Shen-Silverstone [8], Takei [9] and by Koike-Takei [7].

Voros coefficients can be defined also for equations with regular singularities. In [2], the authors give a definition of them and a concrete form of a Voros coefficient for hypergeometric differential equations with a large parameter for a special case. As in the case of irregular singularities, we want to analyze the Stokes phenomena for WKB solutions in parameters by using Voros coefficients of hypergeometric equations. For this purpose, we must compute the Borel sums of them.

In this report, we give a concrete form of the Voros coefficient for each regular singular point and the Borel sums of it for hypergeometric equations. Detailed discussions and proofs will be given in our article in preparation.

§ 2. Voros coefficients

We consider the following Schrödinger-type equation with a large parameter $\eta$:

\[
\left( -\frac{d^2}{dx^2} + \eta^2 Q \right) \psi = 0
\]

2010 Mathematics Subject Classification(s): 33C05, 34M60, 34M40

Key Words: hypergeometric differential equation, WKB solution, Voros coefficient, Stokes curve

This work has been supported by KAKENHI No.22540210 and No.24·3612

*Department of Mathematics, School of Science and Engineering, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan.

**Interdisciplinary Graduate School of Science and Engineering, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan/JSPS Research Fellow (DC2).
with $Q = Q_0 + \eta^{-2}Q_1$, where we set

$$Q_0 = \frac{(\alpha - \beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x - 1)^2}$$  \hspace{1cm} (2.2)$$

and

$$Q_1 = -\frac{x^2 - x + 1}{4x^2(x - 1)^2}. \hspace{1cm} (2.3)$$

Then $\alpha, \beta$ and $\gamma$ are complex parameters. Equation (2.1) is obtained from the hypergeometric differential equation:

$$x(1-x)\frac{d^2w}{dx^2} + (c - (a + b + 1)x)\frac{dw}{dx} - abw = 0,$$

that is, we introduce a large parameter $\eta$ by setting $a = 1/2 + \eta\alpha$, $b = 1/2 + \eta\beta$, $c = 1 + \eta\gamma$ with complex parameters $\alpha, \beta$ and $\gamma$ and eliminate the first-order term by taking

$$\psi = x^{1/2 + \frac{\eta\alpha}{2}}(1-x)^{1/2 + \frac{\eta(\alpha + \beta - \gamma)}{2}}w$$

as unknown function. Then we have equation (2.1). Let

$$\psi_\pm = \frac{1}{\sqrt{S_{\text{odd}}}} \exp(\pm \int_{a_k}^{x} S_{\text{odd}} \, dx), \hspace{1cm} (2.5)$$

be WKB solutions of (2.1) (cf. [7]). Here $a_k(k = 0,1)$ is a turning points of (2.1), that is, zeros of $Q_0$ and $S_{\text{odd}}$ denotes the odd-order part of the formal solution $S = \sum_{h=-1}^{\infty} \eta^{-h}S_h$ in $\eta^{-1}$ of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2Q \hspace{1cm} (2.6)$$

associated with (2.1). We consider the following integrals which are called Voros coefficients:

$$V_0 = V_0(\alpha, \beta, \gamma) := \int_0^{a_k} (S_{\text{odd}} - \eta S_{-1}) \, dx,$$

$$V_1 = V_1(\alpha, \beta, \gamma) := \int_1^{a_k} (S_{\text{odd}} - \eta S_{-1}) \, dx$$

and

$$V_2 = V_2(\alpha, \beta, \gamma) := \int_{a_k}^{\infty} (S_{\text{odd}} - \eta S_{-1}) \, dx$$

of equation (2.1). Since the residues of $S_{\text{odd}}$ and $\eta S_{-1}$ at the singular points coincide (See [6] for the computation of residues of $S_{\text{odd}}$), these integrals are well-defined for every homotopy class of the path of integration and we have a formal series $V_j(\alpha, \beta, \gamma)$.
(j = 0, 1, 2) in η⁻¹. Note that, there are two turning points a₀, a₁ in general, however, V₀, V₁ and V₂ are independent of the choice of a_k (k = 0, 1).

For j = 0, 1 and 2, V_j(α, β, γ) describes the discrepancy between WKB solutions normalized at a_k and those normalized at singular points b₀ = 0, b₁ = 1 and b₂ = ∞, respectively, that is, when we set

(2.7) \[ \psi_\pm = \frac{1}{\sqrt{S_{odd}}} \exp \left( \pm \int_{a_k}^{x} S_{odd} dx \right) \]

and

(2.8) \[ \psi^{(b_j)}_\pm = \frac{1}{\sqrt{S_{odd}}} \exp \left( \pm \int_{b_j}^{x} (S_{odd} - \eta S_{-1}) dx \pm \eta \int_{a_k}^{x} S_{-1} dx \right) \]

we have

(2.9) \[ \psi^{(b_j)}_\pm = \exp(\pm V) \psi_\pm. \]

Here the paths of integration should be chosen suitably. Voros coefficient V_j satisfies a system difference equations with respect to parameters α, β and γ. Solving the system we have the following Theorem.

**Theorem 2.1.** Voros coefficients V_j have the following forms:

V₀ = \[ \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} + \frac{1}{(\gamma - \alpha)^{n-1}} + \frac{1}{(\gamma - \beta)^{n-1}} \right) \right\} + \frac{2}{\gamma^{n-1}} \]

V₁ = \[ \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma - \alpha)^{n-1}} - \frac{1}{(\gamma - \beta)^{n-1}} \right) \right\} \]

and

V₂ = \[ \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} - \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma - \alpha)^{n-1}} + \frac{1}{(\gamma - \beta)^{n-1}} \right) \right\} - \frac{2}{(\beta - \alpha)^{n-1}} \}

Here B_n are Bernoulli numbers defined by

\[ \frac{te^{t}}{e^{t}-1} = \sum_{n=0}^{\infty} \frac{B_n t^{n}}{n!}. \]
§ 3. Stokes graphs

A characterization of Stokes graphs in term of parameters of (2.1) is given in [2]. A Stokes curve emanating from the turning point \(a_k\) \((k = 0, 1)\) is a curve defined by

\[
\text{Im} \int_{a_k}^{x} \sqrt{Q_0} \, dx = 0.
\]

A Stokes curve flows into a singular point or a turning point. The Stokes graph ([1]) of (2.1) is, by definition, a two-colored sphere graph consisting of all Stokes curves (emanating from \(a_0\) and \(a_1\)) as edges, \(\{a_0, a_1\}\) as vertices of the first color and \(\{b_0, b_1, b_2\}\) as vertices of the second color. The Stokes graph of (2.4) is, by definition, that of (2.1). We define that the sets \(H_j\) \((j = 0, 1, 2)\) of the parameters \(\alpha, \beta, \gamma\) as follows:

\[
\begin{align*}
H_0 &= \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha \cdot \beta \cdot \gamma \cdot (\alpha - \beta) \cdot (\alpha - \gamma) \cdot (\beta - \gamma) \cdot (\alpha + \beta - \gamma) \neq 0 \}, \\
H_1 &= \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \text{Re} \alpha \cdot \text{Re} \beta \cdot \text{Re} (\gamma - \alpha) \cdot \text{Re} (\gamma - \beta) \neq 0 \}, \\
H_2 &= \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \text{Re} (\alpha - \beta) \cdot \text{Re} (\alpha + \beta - \gamma) \cdot \text{Re} \gamma \neq 0 \}.
\end{align*}
\]

If \((\alpha, \beta, \gamma)\) is contained in \(H_0\), the turning points and the singular points of (2.4) are mutually distinct. Moreover, if \((\alpha, \beta, \gamma)\) is not contained in \(H_1 \cup H_2\), then the Stokes geometry is degenerate.

We assume that \((\alpha, \beta, \gamma)\) is contained in the sets \(H_0 \cap H_1 \cap H_2\). Stokes graphs can be classified by its order sequence \(\hat{n} = (n_0, n_1, n_2)\), where \(n_0, n_1\) and \(n_2\) are numbers of Stokes curves that flow into 0, 1 and \(\infty\), respectively. Next we define the sets \(\omega_k\) \((k = 1, 2, 3, 4)\) of the parameters \(\alpha, \beta\) and \(\gamma\) as follows:

\[
\begin{align*}
\omega_1 &= \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \text{Re} \alpha < \text{Re} \gamma < \text{Re} \beta \}, \\
\omega_2 &= \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \text{Re} \alpha < \text{Re} \beta < \text{Re} \gamma < \text{Re} \alpha + \text{Re} \beta \}, \\
\omega_3 &= \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \text{Re} \gamma < \text{Re} \alpha < \text{Re} \beta \}, \\
\omega_4 &= \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \text{Re} \gamma - \text{Re} \beta < \text{Re} \alpha < 0 \}
\end{align*}
\]

and involutions \(\iota_j\) \((j = 0, 1, 2)\) in the space of parameters as follows:

\[
\begin{align*}
\iota_0 : (\alpha, \beta, \gamma) &\mapsto (\beta, \alpha, \gamma), \\
\iota_1 : (\alpha, \beta, \gamma) &\mapsto (\gamma - \beta, \gamma - \alpha, \gamma), \\
\iota_2 : (\alpha, \beta, \gamma) &\mapsto (-\alpha, -\beta, -\gamma).
\end{align*}
\]

The potential \(Q\) is invariant under those involutions. Moreover, we define \(\Pi_k\) as follows:

\[
\Pi_k = \bigcup_{r \in G} r(\omega_k) \quad (k = 1, 2, 3, 4).
\]
Here $G$ is the group generated by $\iota_j$ ($j = 0, 1, 2$). We characterize the types of Stokes graphs in terms of the parameters. The following Theorem is proved in [2] (Theorem 3.2) (See all so [3], [10].)

**Theorem 3.1.** Let $\hat{n}$ denote the order sequence of the Stokes graph with parameters $(\alpha, \beta, \gamma)$.

(1) If $(\alpha, \beta, \gamma) \in \Pi_1$, then $\hat{n} = (2, 2, 2)$.
(2) If $(\alpha, \beta, \gamma) \in \Pi_2$, then $\hat{n} = (4, 1, 1)$.
(3) If $(\alpha, \beta, \gamma) \in \Pi_3$, then $\hat{n} = (1, 4, 1)$.
(4) If $(\alpha, \beta, \gamma) \in \Pi_4$, then $\hat{n} = (1, 1, 4)$.

**Remark.** For a fixed $\text{Re} \gamma > 0$, configurations of $\omega_k$’s and $\Pi_k$’s in the real $\alpha-\beta$ plane are shown in Fig. 3.1.

We will consider the Borel sums of Voros coefficients in $\omega_1$ and in $\omega_3$ in the next section. We show some example of Stokes curves in Fig. 3.2.
Here bullets and white bullets designate turning points and singular points, respectively and \( \varepsilon > 0 \). If we take \((\alpha, \beta, \gamma) = (1, 2, 1)\), which is located on the boundary between \( \omega_1 \) and \( \omega_3 \), turning points coincide (cf. Fig. 3.2). If we take \((\alpha, \beta, \gamma) = (1 - \varepsilon, 2, 1)\) (resp. \((1 + \varepsilon, 2, 1)\)), i.e., parameters are contained in \( \omega_1 \) (resp. \( \omega_3 \)), we have \( \hat{n} = (2, 2, 2) \) (resp. \( \hat{n} = (1, 4, 1) \)) in left-hand side (resp. right-hand side) of Fig. 3.2.

\section{Borel sums of Voros coefficients}

In this section we consider the relation between Borel sums of Voros coefficients in \( \omega_1 \) and \( \omega_3 \). Let \( V_{0,B}^{1} \) (resp. \( V_{0,B}^{3} \)) and \( V_{j}^{1} \) (resp. \( V_{j}^{3} \) \((j = 0, 1, 2)\)) denote the Borel transforms and the Borel sums of the Voros coefficients \( V_{j} \) in \( \omega_1 \) (resp. \( \omega_3 \)), respectively. To clarify the relations between Borel sums \( V_{j}^{1} \) and \( V_{j}^{3} \)(\( j = 0, 1, 2 \)), we need the concrete forms of \( V_{j}^{1} \) and \( V_{j}^{3} \). They are given as follows.

\textbf{Theorem 4.1.} Borel sums \( V_{j}^{1} \) (resp. \( V_{j}^{3} \)) of Voros coefficients have following forms:

\begin{align*}
V_{0}^{1} &= \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)\Gamma^{2}(\gamma)\alpha^{\alpha}\beta^{\beta}\gamma^{(\gamma - \alpha)\eta}}{\Gamma^{2}(\alpha + \beta - \gamma)\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)\Gamma^{2}(\gamma)\alpha^{\alpha}\beta^{\beta}\gamma^{(\gamma - \alpha)\eta}}; \tag{4.1} \\
V_{0}^{3} &= \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + (\alpha - \gamma)\eta)\Gamma^{2}(\alpha)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}{\Gamma^{2}(\alpha + \beta - \gamma)\Gamma(\frac{1}{2} + (\alpha - \gamma)\eta)\Gamma^{2}(\alpha)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}; \tag{4.2} \\
V_{1}^{1} &= \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + (\alpha - \gamma)\eta)\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}{\Gamma^{2}(\alpha + \beta - \gamma)\Gamma(\frac{1}{2} + (\alpha - \gamma)\eta)\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}; \tag{4.3} \\
V_{1}^{3} &= \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}{\Gamma^{2}(\alpha + \beta - \gamma)\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}; \tag{4.4} \\
V_{2}^{1} &= \frac{1}{2} \log \frac{2\pi\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}{\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}; \tag{4.5} \\
V_{2}^{3} &= \frac{1}{2} \log \frac{2\pi\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}{\Gamma^{2}((\alpha + \beta - \gamma)\eta)\alpha^{\alpha}\beta^{\beta}\gamma^{(\alpha - \gamma)\eta}}. \tag{4.6}
\end{align*}

\textbf{Outline of the proof.} To compute the Borel sums \( V_{0}^{1} \) and \( V_{0}^{3} \), we first take the Borel transforms \( V_{0,B}^{1} \) and \( V_{0,B}^{3} \) of Voros coefficient \( V_{0} \).

\textbf{Proposition 4.2.} Borel transforms \( V_{0,B}^{1} \) and \( V_{0,B}^{3} \) of Voros coefficients \( V_{0} \) have following forms:

\begin{align*}
V_{0,B}^{1} &= -g(\alpha) - g(\beta) - g(\gamma - \alpha) + g(\beta - \gamma) + \frac{1}{y}(\frac{1}{\exp_{\frac{\gamma}{y}} - 1} - \frac{\gamma}{y} + \frac{1}{2}) \\
\text{and} \\
V_{0,B}^{3} &= -g(\alpha) - g(\beta) + g(\alpha - \gamma) + g(\beta - \gamma) + \frac{1}{y}(\frac{1}{\exp_{\frac{\gamma}{y}} - 1} - \frac{\gamma}{y} + \frac{1}{2})
\end{align*}

Here \( g(s) = \frac{1}{2y} \exp(-\frac{y}{2s})(\frac{1}{\exp_{\frac{\gamma}{y}} - 1} + \frac{1}{2} - \frac{s}{2}) \).
The Borel sums of $V_0$, are obtained by using the following integral representation of the logarithm of the $\Gamma$-function.

**Lemma 4.3.** We have the formula:

\[
\int_0^\infty \left( \frac{1}{e^t - 1} + \frac{1}{2} - \frac{1}{t} \right) \frac{e^{-\theta t}}{t} \, dt = \log \frac{\Gamma(\theta)}{\sqrt{2\pi}} - (\theta - \frac{1}{2}) \log \theta + \theta.
\]

Next we consider the relation between $V_0^1$ and $V_0^3$. Borel sums of Voros coefficient $V_0^1$ is analytically continued over $\omega_3$. We compare it with $V_0^3$. If $\Im(\alpha - \gamma) > 0$, then we rewrite $V_0^1$ as follows.

\[
V_0^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + (\beta - \gamma)\eta\right)\Gamma^2(\gamma\eta\alpha^\alpha \beta^\beta (\gamma - \alpha)^{\gamma - \alpha} \eta\gamma}{\Gamma\left(\frac{1}{2} + \alpha\eta\right)\Gamma\left(\frac{1}{2} + \beta\eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)(\beta - \gamma)\eta \gamma^2 \gamma - 1}.
\]

(4.7)

Subtracting (4.7) from (4.2), we have

\[
V_0^1 - V_0^3 = \frac{1}{2} \log \frac{2\pi}{\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)\Gamma\left(\frac{1}{2} + (\alpha - \gamma)\eta\right)} + \frac{(\gamma - \alpha)\eta \pi i}{2}.
\]

On other hand, if $\Im(\alpha - \gamma) < 0$, we rewrite $V_0^1$ as follows.

\[
V_0^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + (\beta - \gamma)\eta\right)\Gamma^2(\gamma\eta\alpha^\alpha \beta^\beta (\alpha - \gamma)^{(\gamma - \alpha)\eta\gamma}}{\Gamma\left(\frac{1}{2} + \alpha\eta\right)\Gamma\left(\frac{1}{2} + \beta\eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)(\beta - \gamma)\eta \gamma^2 \gamma - 1} - \frac{(\gamma - \alpha)\eta \pi i}{2}.
\]

Hence we have the following relation:

\[
V_0^1 - V_0^3 = \frac{1}{2} \log(e^{2(\alpha - \gamma)\eta \pi i} + 1).
\]

In the same way, we obtain formulas for the other cases. Summing up, we have the following.

**Theorem 4.4.** The relations between Borel sums $V_j^1$ and $V_j^3$ ($j = 0, 1, 2$) of Voros coefficients have following forms:

(1) If $\Im(\alpha - \gamma) > 0$, then we have

\[
V_0^1 = V_0^3 + \frac{1}{2} \log(e^{2(\gamma - \alpha)\eta \pi i} + 1),
\]

\[
V_1^1 = V_1^3 - \frac{1}{2} \log(e^{2(\gamma - \alpha)\eta \pi i} + 1).
\]
and

\[ V_2^1 = V_2^3 - \frac{1}{2} \log(e^{2(\gamma - \alpha)\eta\pi i} + 1). \]

(2) If \( \text{Im} (\alpha - \gamma) < 0 \), then we have

\[ V_0^1 = V_0^3 + \frac{1}{2} \log(e^{2(\alpha - \gamma)\eta\pi i} + 1), \]
\[ V_1^1 = V_1^3 - \frac{1}{2} \log(e^{2(\alpha - \gamma)\eta\pi i} + 1) \]

and

\[ V_2^1 = V_2^3 - \frac{1}{2} \log(e^{2(\alpha - \gamma)\eta\pi i} + 1). \]

References


