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<th>Recent progress on Takhtajan-Zograf and Weil-Petersson metrics (Geometry of Moduli Space of Low Dimensional Manifolds)</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1862: 30-41</td>
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<tr>
<td>Issue Date</td>
<td>2013-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195323">http://hdl.handle.net/2433/195323</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Text version</td>
<td>publisher</td>
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Kyoto University
Recent progress on Takhtajan-Zograf and Weil-Petersson metrics

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Abstract

We will survey recent progress on Weil-Petersson and Takhtajan-Zograf metric. After reviewing the backgrounds and the known results for those metrics, a new estimate of the asymptotic behavior of the Takhtajan-Zograf metric near the boundary of the moduli space of punctured Riemann surfaces is stated without proof.

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1 Backgrounds on Weil-Petersson and Takhtajan-Zograf metrics

$T_{g,n}$ denotes the Teichmüller space of Riemann surfaces of genus $g$ with $n$ marked points $(2g - 2 + n > 0)$. Let $C_{g,n}$ be the Teichmüller curve over $T_{g,n}$ with the projection $\pi : C_{g,n} \to T_{g,n}$ which has $n$ sections $P_1, \ldots, P_n$ corresponding to $n$ marked points. Consider $\Omega^1_{C_{g,n}}$ (resp. $\Omega^1_{T_{g,n}}$) the sheaf of holomorphic 1-forms on $C_{g,n}$ (resp. $T_{g,n}$). The sheaf of relative differential forms on $C_{g,n}$ is defined as

$$\omega_{C_{g,n}/T_{g,n}} := \Omega^1_{C_{g,n}} / \pi^* \Omega^1_{T_{g,n}}.$$  \hfill (1.1)
Then the **determinant line bundle** \( \lambda_l \) on \( T_{g,n} (l \in \mathbb{N}) \) is defined as

\[
\lambda_l := \max R^0_{\pi_*}\omega_{C_{g,n}/T_{g,n}}^\otimes (l-1)(P_1 + \cdots + P_n).
\] (1.2)

For a point \( s \in T_{g,n} \), \( S := \pi^{-1}(s) \) is a compact Riemann surface. Set \( S^0 := S - \{P_1(s), \ldots, P_n(s)\} \) and \( P_p := P_p(s) (p = 1, \ldots, n) \).

Here we can see

\[
R^0_{\pi_*}\omega_{C_{g,n}/T_{g,n}}^\otimes ((l-1)(P_1 + \cdots + P_n))|_s = \Gamma(S, K_{S}^\otimes \otimes \mathcal{O}_S(P_1 + \cdots + P_n)^\otimes(l-1)) \approx \{\text{meromorphic} \ l \ \text{differentials on} \ S \ \text{with possibly poles of order at most} \ l-1 \ \text{only at the marked points}\}.
\]

Pick a basis of local holomorphic sections \( \phi_1, \ldots, \phi_{d(l)} \)
for \( R^0_{\pi_*}\omega_{C_{g,n}/T_{g,n}}^\otimes ((l-1)(P_1 + \cdots + P_n)) \), where

\[
d(l) = \begin{cases} 
g & (l = 1) \\
(2l - 1)(g - 1) + (l - 1)n & (l > 1) 
\end{cases}
\]

\[
\langle \phi_i, \phi_j \rangle := \iint_{S^0} \phi_i \overline{\phi_j} \rho_{S^0}^{(l-1)} (i, j = 1, \ldots, d(l))
\] (1.3)

is called the **Petersson product**, where \( \rho_{S^0} \) is the hyperbolic area element on \( S^0 \).

We set

\[
\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2} := |\det(\langle \phi_i, \phi_j \rangle)|^{1/2},
\] (1.4)

\[
\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_Q := \|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2} Z_{S^0}(l)^{-\frac{1}{2}}
\] (1.5)

\((l \geq 2)\). For \( l = 1 \), employ \( Z'_{S^0}(1) \) in place of \( Z_{S^0}(1) = 0 \). Here, \( Z_{S^0}(l) \) denotes the special value of \( Z_{S^0}(\cdot) \) on \( S^0 \) at \( l \) integer, which will be defined below. Then \( \lambda_l \rightarrow T_{g,n} \) is a Hermitian holomorphic line bundle equipped with the **Quillen metric** \( \| \cdot \|_Q \) (see [7]). Here

\[
Z_{S^0}(s) := \prod_{\{\gamma\}} \prod_{m=1}^{\infty} \left( 1 - e^{-(s+m)L(\gamma)} \right)
\] (1.6)
is the **Selberg Zeta function** for $S^0$, $\Re(s) > 1$, where $\gamma$ runs over all oriented primitive closed geodesics on $S^0$, and $L(\gamma)$ denotes the hyperbolic length of $\gamma$. It extends meromorphically to the whole plane in $s$.

In the late 80’s, we have discovered the following important formulas for the curvature forms of the determinant line bundles with respect to the Quillen metrics.

**Theorem 1.1** (Belavin-Knizhnik+Wolpert(1986), [1], [8]).

$$c_1(\lambda_l, \| \cdot \|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2} \omega_{WP} \ (n = 0).$$

**Theorem 1.2** (Takhtajan-Zograf (1988, 1991), [7]).

$$c_1(\lambda_l, \| \cdot \|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2} \omega_{WP} - \frac{1}{9} \omega_{TZ} \ (n > 0).$$

*Here, $\omega_{WP}, \omega_{TZ}$ are the Kähler forms of the Weil-Petersson, the Takhtajan-Zograf metrics respectively.*

Here remind us of the definitions of the Weil-Petersson and the Takhtajan-Zograf metrics. By the deformation theory of Kodaira-Spencer and the Hodge theory, for $[S^0] \in T_{g,n}$, we have

$$T_{[S^0]}T_{g,n} \simeq HB(S^0), \quad (1.7)$$

where $HB(S^0)$ is the space of harmonic Beltrami differentials on $S^0$.

By the Serre duality, one has

$$T_{[S^0]}^*T_{g,n} \simeq Q(S^0), \quad (1.8)$$

where $Q(S^0)$ is the space of holomorphic quadratic differentials on $S^0$ with finite the Petersson-norm, which is dual to $HB(S^0)$.

The inner product of the **Weil-Petersson** metric at $T_{[S^0]}T_{g,n}$ is defined to be

$$\langle \alpha, \beta \rangle_{WP}([S^0]) := \iint_{S^0} \alpha \overline{\beta} \ \rho_{S^0}, \quad (1.9)$$

where $\alpha, \beta$ are in $HB(S^0) \simeq T_{[S^0]}T_{g,n}$.

The inner products of the **Takhtajan-Zograf** metrics are defined to be

$$\langle \alpha, \beta \rangle_{p}([S^0]) := \iint_{S^0} \alpha \overline{\beta} E_p(\cdot, 2) \ \rho_{S^0}, \quad (p = 1, \ldots, n). \quad (1.10)$$
Here, $E_p(\cdot, 2)$ is the Eisenstein series associated with the $p$-th marked point with index 2. Moreover, we set

$$
\langle \alpha, \beta \rangle_{TZ}([S^0]) := \sum_{p=1}^{n} \langle \alpha, \beta \rangle_p([S^0]).
$$

(1.11)

The Eisenstein series associated with the $p$-th marked point with index 2 is defined to be

$$
E_p(z, 2) := \sum_{A \in \Gamma_p \backslash \Gamma} \{ \Im(\sigma^{-1}_p A(z)) \}^2, \quad \text{for } z \in \mathbb{H},
$$

(1.12)

where $\mathbb{H}$ is the upper-half plane, $\Gamma$ is a uniformizing Fuchsian group for $S^0$ and $\Gamma_p$ is the parabolic subgroup associated with the $p$-th marked point, and $\sigma_p \in \text{PSL}(2, \mathbb{R})$ is a normalizer. $E_p(z, 2)$ assumes the infinity at the $p$-th marked point and vanishes at the other marked points. In addition, the Eisenstein series satisfy

$$
\Delta_{hyp} E_p(z, 2) = 2E_p(z, 2),
$$

(1.13)

where $\Delta_{hyp}$ is the negative hyperbolic Laplacian on $S^0$. Especially $E_p(z, 2)$ is a positive subharmonic function on $S^0$.

$\text{Mod}_{g,n}$ denotes the mapping class group of surfaces of genus $g$ with $n$ marked points. Then the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points is described as $\mathcal{M}_{g,n} = T_{g,n}/\text{Mod}_{g,n}$. $\lambda_l$ and all metrics we defined are compatible with the action of $\text{Mod}_{g,n}$, thus they all naturally descend down to $\mathcal{M}_{g,n}$ as orbifold line sheaves and orbifold metrics respectively.

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$. We have known the relations of the $L^2$—cohomology of $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson metric and the second cohomology of $\overline{\mathcal{M}}_{g,n}$.

**Theorem 1.3** (Saper (1993) [6]). For $g > 1$, $n = 0$, $H^*_2(\mathcal{M}_g, \omega_{WP}) \simeq H^*(\overline{\mathcal{M}}_g, \mathbb{R})$.

Here, the left hand side is the $L^2$—cohomology with respect to the Weil-Petersson metric.
2 Known results for the asymptotic behaviors of the Weil-Petersson and Takhtajan-Zograf metrics

The proof of Theorem 1.3 is based on the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space which we will review now.

Here we set \( D := \overline{\mathcal{M}_{g,n}} \setminus \mathcal{M}_{g,n} \) the compactification divisor. Now take \( X_0 \in D \) a degenerate Riemann surface of genus \( g \) with \( n \) marked points and \( k \) nodes (we regard the marked points as deleted from the surface).

Each node \( q_i \) \((i = 1, 2, \ldots, k)\) has a neighborhood

\[ N_i = \{ (z_i, w_i) \in \mathbb{C}^2 \mid |z_i|, |w_i| < 1, z_i w_i = 0 \}. \]

\( X_t \) denotes the smooth surface gotten from \( X_0 \) after cutting and pasting \( N_i \) under the relation \( z_i w_i = t_i |t_i| \) small. Then, \( D \) is locally described as \( \{ t_1 \cdots t_k = 0 \} \) (see 3. in more details).

\( D \) has locally the pinching coordinate \((t, s) = (t_1, \ldots, t_k, s_{k+1}, \ldots, s_{3g-3+n})\) around \([X_0]\). Set \( \alpha_i = \partial/\partial t_i, \beta_\mu = \partial/\partial s_\mu \in T_{(t, s)}(T_{g,n}) \). We define the Riemannian tensors for the Weil-Petersson metric

\[
\begin{align*}
g_{ij}(t, s) & := \langle \alpha_i, \alpha_j \rangle_{WP}(t, s), \\
g_{i\overline{\mu}}(t, s) & := \langle \alpha_i, \beta_\mu \rangle_{WP}(t, s), \\
g_{\mu\overline{\nu}}(t, s) & := \langle \beta_\mu, \beta_\nu \rangle_{WP}(t, s), 
\end{align*}
\]

\((i, j = 1, 2, \ldots, k, \mu, \nu = k + 1, \ldots, 3g - 3 + n)\).

Furthermore, we define the Riemannian tensors for the Takhtajan-Zograf metric

\[
\begin{align*}
h_{ij}(t, s) & := \langle \alpha_i, \alpha_j \rangle_{TZ}(t, s), \\
h_{i\overline{\mu}}(t, s) & := \langle \alpha_i, \beta_\mu \rangle_{TZ}(t, s), \\
h_{\mu\overline{\nu}}(t, s) & := \langle \beta_\mu, \beta_\nu \rangle_{TZ}(t, s), 
\end{align*}
\]

\((i, j = 1, 2, \ldots, k, \mu, \nu = k + 1, \ldots, 3g - 3 + n)\).

The following theorem is a pioneering result for the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space.
**Theorem 2.1** (Masur (1976), [2]). As $t_i, s_\mu \to 0$,

1. $g_{ii}(t, s) \approx \frac{1}{|t_i|^2(-\log|t_i|)^3}$ for $i \leq k$,
2. $g_{ij}(t, s) = O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)$ for $i, j \leq k, i \neq j$,
3. $g_{i\mu}(t, s) = O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right)$ for $i \leq k, \mu \geq k+1$,
4. $g_{\mu\nu}(t, s) \to g_{\mu\nu}(O,0)$ for $\mu, \nu \geq k+1$.

Recently, we updated Masur's result by improving Wolpert's formula for the asymptotic of the hyperbolic metric for degenerating Riemann surfaces.

**Theorem 2.2** (Obitsu and Wolpert (2008), [5]). We can improve iv) in Theorem 2.1 as follows;

\[ iv') \quad g_{\mu\nu}(t, s) = g_{\mu\nu}(0, s) + \frac{4\pi^4}{3} \sum_{i=1}^{k} (\log|t_i|)^{-3} \left\langle \beta_\mu, (E_{i,1} + E_{i,2})\beta_\nu \right\rangle_{WP}(0, s) + O\left(\sum_{i=1}^{k} (\log|t_i|)^{-3}\right) \]

as $t \to 0$, for $\mu, \nu \geq k+1$.

Here, $E_{i,1}, E_{i,2}$ denote a pair of the Eisenstein series with index 2 associated with the $i$-th node of the limit surface $X_0$.

That is, the Takhtajan-Zograf metrics have appeared from degeneration of the Weil-Petersson metric. On the other hand, we have a result for asymptotics of the Takhtajan-Zograf metric near the boundary of the moduli space. Before stating the result, we need the following definition.

**Definition 2.3.** Let $X_0$ be a degenerate Riemann surface with $n$ punctures $p_1, \ldots, p_n$ and $m$ nodes $q_1, \ldots, q_m$.

A node $q_i$ is said to be adjacent to punctures (resp. a puncture $p_j$) if the component of $X_0 \setminus \{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m\}$ containing $q_i$ also contains at least one of the $p_j$'s (resp. the puncture $p_j$). Otherwise, it is said to be non-adjacent to punctures (resp. the puncture $p_j$).

**Theorem 2.4** (Obitsu-To-Weng (2008), [3]). As $(t, s) \to 0$, we observe the followings:

1. For any $\epsilon > 0$, there exists a constant $C_{1,\epsilon}$ such that
\[ h_{ii}(t, s) \leq \frac{C_{1,\epsilon}}{|t_{i}|^{2}(-\log|t_{i}|)^{4-\epsilon}} \quad \text{for } i \leq k; \]

For any \( \epsilon > 0 \), there exists a constant \( C_{2,\epsilon} \) such that
\[ h_{ii}(t, s) \geq \frac{C_{2,\epsilon}}{|t_{i}|^{2}(-\log|t_{i}|)^{4+\epsilon}} \quad \text{for } i \leq k \]

and the node \( q_{i} \) adjacent to punctures;
\[ \text{ii) } h_{ij}(t, s) = O\left( \frac{1}{|t_{i}||t_{j}|(|\log|t_{i}|)^{3}(|\log|t_{j}|)^{3}} \right) \quad \text{for } i, j \leq k, i \neq j; \]
\[ \text{iii) } h_{i\bar{\mu}}(t, s) = O\left( \frac{1}{|t_{i}|(-\log|t_{i}|)^{3}} \right) \quad \text{for } i \leq k, \mu \geq k + 1; \]
\[ \text{iv) } h_{\mu\bar{\nu}}(t, s) \rightarrow h_{\mu\bar{\nu}}(0, 0) \quad \text{for } \mu, \nu \geq k + 1. \]

3 Degenerate families of punctured Riemann surfaces and A test Eisenstein series

First of all, let us review the construction of degenerating punctured hyperbolic surfaces. We recall the construction of the plumbing family (see 2 [5]). Considerations begin with the plumbing variety \( \mathcal{V} = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\} \). The defining function \( zw - t \) has differential \( zdw + wdz - dt \). Consequences are that \( \mathcal{V} \) is a smooth variety, \( (z, w) \) are global coordinates, while \( (z, t) \) and \( (w, t) \) are not. Consider the projection \( \Pi : \mathcal{V} \rightarrow D \) onto the \( t \)-unit disc. The projection \( \Pi \) is a submersion, except at \((z, w) = (0, 0)\); we consider \( \Pi : \mathcal{V} \rightarrow D \) as a (degenerate) family of open Riemann surfaces. The \( t \)-fiber, \( t \neq 0 \), is the hyperbola germ \( zw = t \) or equivalently the annulus \( \{|t| < |z| < 1, w = t/z\} = \{|t| < |w| < 1, z = t/w\} \). The 0-fiber is the intersection of the unit ball with the union of the coordinate axes in \( \mathbb{C}^{2} \); on removing the origin the union becomes \( \{0 < |z| < 1\} \cup \{0 < |w| < 1\} \). Each fiber of \( \mathcal{V}_{0} = \mathcal{V} - \{0\} \rightarrow D \) has a complete hyperbolic metric.

Consider \( X_{0} \) a finite union of hyperbolic surfaces with cusps. A plumbing family is the fiberwise gluing of the complement of cusp neighborhoods in \( X_{0} \) and the plumbing variety \( \mathcal{V} = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\} \). For a positive constant \( c_{*} < 1 \) and initial surface \( X_{0} \), with puncture \( p \) with cusp coordinate \( z \) and puncture \( q \) with cusp coordinate \( w \), we construct a family \( \{X_{t}\} \). For \( |t| < c_{*}^{4} \) the resulting surface \( X_{t} \) will be independent of \( c_{*} \);
the constant $c_*$ will serve to specify the overlap of coordinate charts and to define a collar in each $X_t$.

We first describe the gluing of fibers. For $|t| < c_*^4$, remove from $X_0$ the punctured discs $\{0 < |z| \leq |t|/c_*\}$ about $p$ and $\{0 < |w| \leq |t|/c_*\}$ about $q$ to obtain a surface $X_{t/c_*}^*$. For $t \neq 0$, form an identification space $X_t$ by identifying the annulus $\{|t|/c_* < |z| < c_*\} \subset X_{t/c_*}^*$ with the annulus $\{|t|/c_* < |w| < c_*\} \subset X_{t/c_*}^*$ by the rule $zw = t$. The resulting surface $X_t$ is the plumbing for the prescribed value of $t$. We note for $|t| < |t'|$ that there is an inclusion of $X_{t/c_*}^*$ in $X_{t/c_*}^*$; the inclusion maps provide a way to compare structures on the surfaces. The inclusion maps are a basic feature of the plumbing construction. We next describe the plumbing family. Consider the variety $\mathcal{V}_c = \{(z, w, t) | zw = t, |z|, |w| < c_*, |t| < c_*^4\}$ and the disc $D_c = \{|t| < c_*^4\}$. The complex manifolds $M = X_{t/c_*}^* \times D_c$ and $\mathcal{V}_c$ have holomorphic projections to the disc $D_c$. The variables $z, w$ denote prescribed coordinates on $X_{t/c_*}^*$ and on $\mathcal{V}_c$. There are holomorphic maps of subsets of $M$ to $\mathcal{V}_c$, commuting with the projections to $D_c$, as follows

$$(z, t) \rightarrow (z/t, z, t) \text{ and } (w, t) \rightarrow (w/t, w, t).$$

The identification space $\mathcal{F} = M \cup \mathcal{V}_c / \{\hat{F}, \hat{G} \text{ equivalence}\}$ is the plumbing family $\{X_t\}$ with projection to $D_c$ (an analytic fiber space of Riemann surfaces in the sense of Kodaira. For $0 < |t| < c_*^4$, the $t$-fiber of $\mathcal{F}$ is the surface $X_t$ constructed by overlapping annuli $N_t$.

We set two anului

$$\Omega_1^1 := \left\{ z \in \mathbb{C} \mid \frac{|t|}{e^{a_0}c_*} < |z| < e^{a_0}c_* \right\} \text{ for } |t| < (c^*)^4, \quad (3.1)$$

$$\Omega_2^2 := \left\{ w \in \mathbb{C} \mid \frac{|t|}{e^{a_0}c_*} < |w| < e^{a_0}c_* \right\} \text{ for } |t| < (c^*)^4. \quad (3.2)$$

Here $0 < c^* < 1, a_0 < 0$ are the constants in [5].

When $t \neq 0$, one can identify as an annulus via coordinate projections as

$$N_t \leftrightarrow \Omega_1^1 \leftrightarrow \Omega_2^2. \quad (3.3)$$

And we may write $N_t = N_1^1 \cup N_2^2$, where

$$N_1^1 = \{z \in \mathbb{C} \mid |t|^{\frac{1}{2}} \leq |z| < e^{a_0}c_*\}, N_2^2 = \{w \in \mathbb{C} \mid |t|^{\frac{1}{2}} \leq |w| < e^{a_0}c_*\}. \quad (3.4)$$

For $t = 0$, define the cusp neighborhood

$$N_0 := \Omega_1^0 \cup \Omega_2^0. \quad (3.5)$$
In another word, we may consider that $\Omega_t^1$ embed into $X_t$ holomorphically for $t, z$. (See 2 in [5])

Here, remember the test function which is defined in [3]. For $t \neq 0$ one defines for $z \in \Omega_t^1$,

$$E_t^*(z) := \frac{-\pi}{\log |t| \sin \left( \frac{\pi \log |z|}{\log |t|} \right)}, \quad \rho_t^*(z) := \frac{\pi^2}{|z|^2 \log^2 |t| \sin^2 \left( \frac{\pi \log |z|}{\log |t|} \right)},$$

for $t = 0, z \in \Omega_t^1$,

$$E_0^*(z) := \frac{-1}{\log |z|}, \quad \rho_0^*(z) := \frac{1}{|z|^2 \log |z|}.$$

It is easy to see that for $t \neq 0$, $E_t^*, \rho_t^*$ have similar expressions for $w$ in $\Omega_t^2$ via the rule $zw = t$. Thus, $E_t^*, \rho_t^*$ can be considered as functions on the manifolds $N_t$ for $t \neq 0$. And one defines for $w \in \Omega_0^2$, $E_0^*(w), \rho_0^*(w)$ as the same expression as $E_t^*(z), \rho_t^*(z)$. Furthermore, we can easily observe that

$$\rho_0^* \leq \rho_t^* \quad \text{on} \quad N_t \quad \text{for} \quad |t| < (c^*)^4. \quad (3.6)$$

Masur showed in (6.5) [2] that there exists a positive constant $K$ such that

$$\rho_t^* \leq K \rho_0^* \quad \text{on} \quad N_t \quad \text{for} \quad |t| < (c^*)^4. \quad (3.7)$$

From now, we always assume that the smooth surfaces $X_t$ have at least one punctures. We are ready to consider a function

$$\varphi_t := \frac{E_t}{E_t^*}, \quad \text{on} \quad N_t, \quad \text{for} \quad |t| < (c^*)^4,$$

where $E_t$ is the intrinsic Eisenstein series on a punctured hyperbolic surface $X_t$ associated with a puncture.

We have already seen in the proof of Proposition 4.2.2 in [3] that on $\Omega_t^1$,

$$\Delta E_t(z) = 2\rho_t(z)E_t(z), \quad (3.8)$$

$$\Delta E_t^*(z) = \left(1 + \cos^2 \left( \frac{\pi \log |z|}{\log |t|} \right) \right) \rho_t^*(z)E_t^*(z), \quad (3.9)$$

where $\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}}$, $\rho_t(z)$ is the intrinsic hyperbolic area element on $X_t$, and $\rho_t^*(z)$ is the restriction to $\Omega_t^1$ of the complete hyperbolic metric $r(z)|dz|^2$ of an annulus $\{z \in \mathbb{C} \mid |t| < |z| < 1\}$. It should be noted that $\rho_t^*(z)$ on $\Omega_t^1$ is strictly smaller than the complete hyperbolic metric of $\Omega_t^1$. Now a straightforward calculation leads the following proposition (see [4] for the proof).
Proposition 3.1. The function \( \varphi_t(z) \) satisfies the following equation on \( \Omega_t^1 \)

\[
-\Delta \varphi_t(z) + \frac{\pi}{\log |t|} \cot \left( \frac{\pi \log |z|}{\log |t|} \right) \left( \frac{2}{z} \frac{\partial \varphi_t(z)}{\partial \overline{z}} + \frac{2}{\overline{z}} \frac{\partial \varphi_t(z)}{\partial z} \right) \\
+ \left\{ 2 \rho_t(z) - \left( 1 + \cos^2 \left( \frac{\pi \log |z|}{\log |t|} \right) \right) \rho_t^*(z) \right\} \varphi_t(z) = 0.
\]

We need the following result which is a special case of [5] Theorem 1.

Theorem 3.2. On \( N_t \), \( \rho_t \) has the expansion for \( t \to 0 \),

\[
\rho_t = \rho_t^* \left( 1 + \frac{4\pi^4}{3} \frac{1}{(\log |t|)^2} + Q(t) \right),
\]

where \( Q(t) \) has the estimate

\[
Q(t) = O \left( \frac{1}{(\log |t|)^3} \right) \quad \text{for} \ t \to 0.
\]

The function \( E^\dagger_{t,1}, E^\dagger_{t,2} \) is the modified Eisenstein series. The \( O \)-term refers to the intrinsic \( C^1 \)-norm of a function on \( X_t \). The bounds depend on the choice of \( c^*, a_0 \) and a lower bound for the injectivity radius for the complement of the cusp regions in \( X_0 \).

The functions \( E^\dagger_{t,1}, E^\dagger_{t,2} \) are constructed as follows (see Definition 1 in [5]). First, consider the case where the pinching curve is non-dividing. Now we may assume that for \( t = 0 \), our coordinates \( z, w \) are so-called the standard coordinate (see Remark-Definition 2.1.2 in [3]). Take the two Eisenstein series \( E_{0,1}, E_{0,2} \) on \( X_0 \) associated with the node. Set \( E^\dagger_{0,1} = E_{0,1} - (\log |z|)^2 \) on \( \Omega_0^1 \), \( E^\dagger_{0,1} = E_{0,1} \) otherwise. \( E^\dagger_{0,2} = E_{0,2} - (\log |w|)^2 \) on \( \Omega_0^2 \), \( E^\dagger_{0,2} = E_{0,2} \) otherwise. Set \( E^\dagger_{t,1} = E^\dagger_{0,1}(z) + E^\dagger_{0,1}(\frac{t}{z}) \) on \( N_t \). Similarly set \( E^\dagger_{t,2} = E^\dagger_{0,2}(w) + E^\dagger_{0,2}(\frac{t}{w}) \) on \( N_t \). These functions are smooth, bounded and strictly positive on \( N_t \) for \( |t| < (c^*)^4 \). In the dividing case, we consider \( E_{0,1} \) be just 0 on the other component, follow the construction in the non-dividing case. It should be noted that \( E^\dagger_{0,1}, E^\dagger_{0,2} \) on \( N_t \) is independent of \( t \). Furthermore, we should remark that in the construction of [5], \( E^\dagger_{0,1}, E^\dagger_{0,2} \) are modified except for the factor \( (\log |z|)^2, (\log |w|)^2 \) just on \( \{ e^{a_0} c^* < |z| < c^* \} \simeq \{ \frac{|t|}{c^*} < |z| < \frac{|t|}{e^{a_0} c^*} \} \) and \( \{ \frac{|t|}{c^*} < |z| < \frac{|t|}{e^{a_0} c^*} \} \simeq \{ e^{a_0} c^* < |w| < c^* \} \) so that the modified function be smooth, thus in our case, \( E^\dagger_{0,1}, E^\dagger_{0,2} \) is exactly \( E_{0,1}, E_{0,2} \) on \( X_0 \) except for the factor \( (\log |z|)^2, (\log |w|)^2 \) respectively.
Remark 3.3. As mentioned before, $\rho_{t}^{*}$ is strictly smaller than the complete hyperbolic metric of $\Omega_{t}^{1}$. Thus, the claim of Theorem 3.2 does not contradict the implication of the classical Schwarz lemma.

4 A new estimate for the Takhtajan-Zograf metric

We are ready to state a new estimate of the intrinsic Eisenstein series which is an improvement of Proposition 4.2.2 in [3]. Detailed proofs will appear in [4]. Here we quote a lemma ( Lemma 1[5]).

Lemma 4.1. There exist a positive constant $C^*$ such that

$$E_0 \leq C^* E_0^* \quad \text{on } \Omega_0^1.$$ 

We are now in a position to generalize Lemma 4.1 for any $t$.

Proposition 4.2. Assume that in the family $\{X_t\}$, $N_0$ has the intersection with the component attached to the cusp where the Eisenstein series $E_0$ has a singularity. Then there exists a positive constant $C, C'$ independent of $t$ such that

$$E_t \leq CE_t^* \quad \text{on } N_t \quad \text{for } |t| \text{ sufficiently small}, \quad (4.1)$$

$$E_t \leq C'E_0^* \quad \text{on } N_t \quad \text{for } |t| \text{ sufficiently small}. \quad (4.2)$$

Applying Proposition 4.2, we can improve (i) of Theorem 1 in [3].

Theorem 4.3. For the simplicity of description, we assume that the degenerating family of a punctured hyperbolic surface $X_t$ has only one pinching curve. Then there exists a positive constant $C$ such that the Takhtajan-Zograf inner product has the estimate

$$g^{TZ} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) \leq \frac{C}{|t|^2 (\log |t|)^4} \quad \text{for } t \to 0.$$ 

That is, we have removed , in (i) of Theorem 1 in [3].

References


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