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CONFORMALLY FLAT LORENTZ PARABOLIC MANIFOLD

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ABSTRACT. The purpose of this note is to introduce Lorentz parabolic structure on smooth manifolds. First we revisit \((G,X)\)-structure on manifolds. Secondly we study Lorentz similarity structure and Fefferman-Lorentz parabolic structure.

1. INTRODUCTION

In the first part of this paper we review \((G,X)\)-structure introduced by Thurston, Kulkarni et al. Many results are known when \((G,X)\) is a homogeneous Riemannian geometry. In 1980-90s non-Riemannian homogeneous geometries have been studied intensively. Specifically conformally flat geometry, spherical CR-geometry and flat quaternionic CR-geometry. Those geometries are obtained on the projective limit of the isometric actions of hyperbolic spaces. Similarly, another kind of non-Riemannian homogeneous geometry is obtained as the the boundary behavior of the isometric actions on pseudo-hyperbolic spaces. The typical example is conformally flat Lorentz geometry. In the second part of this paper, we introduce conformally flat Lorentz parabolic geometry. A Lorentz parabolic structure contains Lorentz similarity structure and Fefferman-Lorentz structure. It is explained that the fundamental group of a compact complete Lorentz similarity manifold \(M\) is virtually polycyclic. It turns out that a finite cover of \(M\) admits a Lorentz parabolic structure. We discuss Fefferman-Lorentz parabolic geometry. The conformally flat Lorentz geometry \((O(2n+2,2), S^1 \times S^{2n+1})\) contains this as a subgeometry \((U(n+1,1), S^1 \times S^{2n+1})\). Let \(\Gamma\) be a discrete subgroup of \(U(n+1,1)\) acting properly discontinuously on a domain \# of \(S^1 \times S^{2n+1}\). We present a classification of compact conformally flat Fefferman-Lorentz parabolic manifolds \#/\Gamma\) admitting a 1-parameter group \(H \leq \text{Conf}(\#/\Gamma)\).
class contains $S^1 \times \mathcal{N}/\Delta$ where $\mathcal{N}$ is a 3-dimensional Heisenberg nilmanifold. Finally we discuss the deformation space of conformally flat Fefferman-Lorentz parabolic structures on the product $S^1 \times \mathcal{N}/\Delta$.

2. $(G, x)$-structure

Our geometry is a pair $(G, X)$ where $G$ is a finite dimensional Lie group with finitely many components and $X$ is an $n$ dimensional homogeneous space of $G$. A geometric structure $((G, X)$-structure) on a smooth $n$ dimensional manifold $M$ is a maximal collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ whose coordinate changes belong to $G$. More precisely, $M = \bigcup_{\alpha \in \Lambda} U_\alpha$, $\phi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image. If $U_\alpha \cap U_\beta \neq \emptyset$ then it satisfies that there exists a unique element $g_{\alpha \beta} \in G$ such that $g_{\alpha \beta} \cdot \phi_\alpha = \phi_\beta$ on $U_\alpha \cap U_\beta$. We say that $M$ is uniformized over $X$ with respect to $G$ (or simply, $M$ is locally modelled on $(G, X)$). An $n$-manifold $M$ is called a $(G, X)$-manifold if $M$ is uniformized over $X$ with respect to $G$. Using a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ we can construct a geometric invariant $(\rho, \text{dev})$ called a developing pair of $M$. (See [5].)

Lemma 2.1. Given a $(G, X)$-structure on a smooth $n$-manifold $M$, there exists a pair $(\rho, \text{dev}) : (\pi_1(M), \tilde{M}) \to (G, X)$ unique up to conjugation of elements of $G$, where $\text{dev}$ is a $(G, X)$-structure preserving immersion and $\rho$ is a homomorphism such that the diagram is commutative for each element $\gamma \in \pi_1(M)$:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\text{dev}} & X \\
\gamma \downarrow & & \downarrow \rho(\gamma) \\
\tilde{M} & \xrightarrow{\text{dev}} & X.
\end{array}
\]

(2.1)

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ be a geometric structure on $M$. In the union $\bigcup_{\alpha \in \Lambda} (U_\alpha \times X)$, we define the following equivalence relation; for $(p, x) \in U_\alpha \times X$, $(q, y) \in U_\beta \times X$, then

\[
(p, x) \sim (q, y) \text{ if and only if } p = q \in U_\alpha \cap U_\beta, \ g_{\alpha \beta}x = y,
\]

(2.2)

$\exists g_{\alpha \beta} \in G$.

Put $E = \bigcup_{\alpha} (U_\alpha \times X)/ \sim$. Let $\pi : E \to M$ be the map defined by $\pi([p, x]) = p$ if $p \in U_\alpha$. Then it is easy to see that $E \xrightarrow{\pi} M$ is a fiber bundle with fiber $X$. Recall that $E$ is determined by the transitive functions $\{g_\alpha\}$. Since $g_{\alpha \alpha} = 1$ and $g_{\alpha \beta} \cdot g_{\beta \gamma} = g_{\alpha \gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, $\{g_{\alpha \beta}\}$ is a 1-cocycle in the first cohomology $H^1(M; G)$. Here $G$ is
viewed as the sheaf of germs of $G$-valued functions. Since $H^1(M; G) \approx \text{Hom}(\pi_1(M), G)$, \{\$g_{\alpha\beta}\$\} determines a homomorphism $\rho : \pi_1(M) \to G$. More precisely it follows that $E \approx \tilde{M} \times X$ in which each element $\gamma \in \pi_1(M)$ acts on $\tilde{M} \times X$ by $(\gamma, (b, x)) = (\gamma b, \rho(\gamma)x)$. We construct a developing map. Let $s : M \to E$ be a section defined by $s(p) = [p, \phi_\alpha(p)]$ if $p \in U_\alpha$. Consider the pull back of the bundle:

\[
\begin{array}{ccc}
\pi_1(M) & \to & P^*E \to E \\
\downarrow & & \downarrow \\
\pi_1(M) & \to & \tilde{M} \to M.
\end{array}
\]

(2.3)

As before the bundle $P^*E$ is determined by a lift \{\$\tilde{g}_{\alpha\beta}\$\} of \{\$g_{\alpha\beta}\$\}. Since $H^1(\tilde{M}; G) = \{1\}$, the bundle $P^*E$ is trivial. Choose a trivialization $\Psi : P^*E \to \tilde{M} \times X$. The section $s$ extends to a section $\tilde{s} : \tilde{M} \to P^*E$. Put $\text{dev} = Pr_2 \cdot \Psi \cdot \tilde{s} : \tilde{M} \to X$. It is an immersion and preserves the $(G, X)$-structure. The map dev depends on the choice of sections and trivializations, however dev is unique up to elements of $G$.

On the other hand, we note that for $(\bar{p}, x) \in \tilde{U}_\alpha \times X, (\bar{q}, y) \in \tilde{U}_\beta \times X$ in $P^*E = \bigcup_\alpha (\tilde{U}_\alpha \times X)$, it follows that $(\bar{p}, x) \sim (\bar{q}, y)$ iff $\gamma \bar{p} = \bar{q}, \rho(\gamma)y = g_{\alpha\beta}x$ and $p = q \in U_\alpha \cap U_\beta$ for some $\gamma \in \pi_1(M)$ and $g_{\alpha\beta} \in G$. It is easy to see that

$$\text{dev} \cdot \gamma = \rho(\gamma) \cdot \text{dev} \text{ for every } \gamma \in \pi_1(M).$$

\[\square\]

If $\text{Aut}(\tilde{M})$ is the group of all $(G, X)$-structure preserving diffeomorphisms on $\tilde{M}$. Then note that $\pi_1(M) \leq \text{Aut}(\tilde{M})$ and $\rho$ extends naturally to a continuous homomorphism $\rho : \text{Aut}(\tilde{M}) \to G$.

**Definition 2.2.** The map $\text{dev}$ is called a developing map for a $(G, X)$-manifold $M$ and the map $\rho$ is called a holonomy homomorphism of $M$.

Let $\hat{\#}(M)$ be the space consisting of all possible developing pairs $(\rho, \text{dev})$. A topology on $\hat{\#}(M)$ is given by the following subbasis.

- $\mathcal{N}(U) = \{U\}$ where $U$ is an open subset of $\text{Map}(\tilde{M}, X)$ in the compact open topology of $\text{Map}(\tilde{M}, X)$.
- $\mathcal{N}(K) = \{\text{dev} \in \hat{\#}(M) | \text{dev}|K \text{ is embedding}\}$ for a compact subset $K \subset \tilde{M}$.

(Compare [1].) Recall that the deformation space $\mathcal{T}(M)$ is a space of $(G, X)$-structures on marked manifolds homeomorphic to $M$. $\mathcal{T}(M)$ consists of equivalence classes of diffeomorphisms $f : M \to M'$ from
$M$ to a $(G, X)$-manifolds $M'$. Two such diffeomorphisms $f_i : M \to M_i$ $(i = 1, 2)$ are equivalent if and only if there is an isomorphism (i.e. a $(G, X)$-structure preserving diffeomorphism) $h : M_1 \to M_2$ such that $h \circ f_1$ is isotopic to $f_2$.

\[ M \xrightarrow{f_1} M_1 \]
\[ f_2 \searrow \simeq \downarrow h \]
\[ M_2 \]

(2.4)

Denote by $\text{Diff}^0(M)$ the subgroup of $\text{Diff}(M)$ whose elements are isotopic to the identity map. Put $\pi = \pi_1(M)$. Consider the following exact sequences of the diffeomorphism groups, where $N_{\text{Diff}(\tilde{M})}(\pi)$ (resp. $C_{\text{Diff}(\tilde{M})}(\pi)$) is the normalizer (resp. centralizer) of $\pi$ in $\text{Diff}(\tilde{M})$

\[
1 \longrightarrow \pi \longrightarrow N_{\text{Diff}(\tilde{M})}(\pi) \xrightarrow{\eta} \text{Diff}(M) \longrightarrow 1
\]

\[
C_{\text{Diff}(\tilde{M})}(\pi) \longrightarrow \text{Diff}^0(M)
\]

Put $\tilde{\text{Diff}}(M) = \eta^{-1}(\text{Diff}(M))$ and let $\tilde{\text{Diff}}^0(M)$ be the identity component. Then $\eta(\tilde{\text{Diff}}(M)) = \text{Diff}^0(M)$ and $\tilde{\text{Diff}}^0(M) \leq C_{\text{Diff}(\tilde{M})}(\pi)$. The natural right action of $\tilde{\text{Diff}}(M)$ and the left action of $G$ on $\#(M)$ are given by

\[
(\rho, \text{dev}) \circ f = (\rho \circ \mu(\tilde{f}), \text{dev} \circ \tilde{f}),
\]

\[
g \circ (\rho, \text{dev}) = (g \circ \rho \circ g^{-1}, g \circ \text{dev}),
\]

where $\mu(\tilde{f}) : \pi \to \pi$ is an isomorphism defined by $\mu(\tilde{f})(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$. Obviously both actions commute.

It is noted that two developing pairs $(\rho_i, \text{dev}_i)$ $(i = 1, 2)$ represent the same structure on $M$ if and only if there exists an element $g \in G$ such that $g \circ \text{dev}_1 = \text{dev}_2$. Put

\[
\#(M) = \hat{\#}(M) / \tilde{\text{Diff}}^0(M).
\]

The action of $G$ induces an action of $\#(M)$. Then it is easy to show that

Lemma 2.3. The elements of $\mathcal{T}(M)$ are in one-to-one correspondence with the orbits of $G \setminus \#(M)$.

If $f : M \to M'$ is a representative element of $\mathcal{T}(H, M)$ then there is a developing pair $(\rho, \text{dev}) : (\pi_1(M'), \tilde{M}') \to (G, X)$. We have the
holonomy representation $\rho \circ f_\#: \pi \to G$ up to conjugate by an element of $G$. We then obtain a map $\text{hol} : \mathcal{T}(M) \to \text{Hom}(\pi, G)/G$ which assigns to a marked structure its holonomy representation. By the definition $\text{hol}$ lifts to a map $\hat{\text{hol}} : \#(M) \to \text{Hom}(\pi, G)$ which makes the following diagram commute.

$$
\begin{array}{ccc}
\#(M) & \xrightarrow{\text{hol}} & \text{Hom}(\pi, G) \\
\downarrow & & \downarrow \\
\mathcal{T}(M) & \xrightarrow{\text{hol}} & \text{Hom}(\pi, G)/G.
\end{array}
$$

Thurston has shown the following. (See [Lo],[J-M],[Th] for the proof.)

**Theorem 2.4 (Holonomy Theorem).** $\hat{\text{hol}} : \#(M) \to \text{Hom}(\pi, G)$ is a local homeomorphism.

3. **Examples of non-Riemannian homogeneous geometry**

3.1. **Homogeneous Riemannian geometry.** Let $X = G_x \backslash G$ be the simply connected homogeneous space $(x \in X)$. If $G_x$ is compact, then $(G, X)$ is called homogeneous Riemannian geometry. If $M$ is a compact manifold which admits a $(G, X)$-structure, then it follows that $M = X/\rho(\pi)$ where $\rho : \pi = \pi_1(M) \to G$ is a discrete faithful representation. This is obtained by the following lemma.

**Lemma 3.1.** If $f : M \to N$ is a Riemannian immersion and $M$ is complete, then $f$ is a covering map.

A Riemannian manifold is complete if every Cauchy sequence converges relative to the Riemannian metric. Specifically a compact Riemannian manifold is complete.

Thus the deformation space $\mathcal{T}(M)$ is identified with the set of equivalence classes of discrete faithful representations $\text{R}(\pi, G)/G$. For example, when we take $G = \text{Isom}(\mathbb{H}_K^n)$ the full isometry group of the $K$-hyperbolic space where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. The Mostow rigidity theorem says that $\text{R}(\pi, G)/G$ is a single point. By Margulis-Mostow rigidity, the same result holds for a noncompact semisimple Lie group $G$ of $\mathbb{R}$-rank $\geq 2$. If $X = K \backslash G$, then $X/T$ is a compact nonpositively curved Riemannian manifold. On the other hand, if $M$ is noncompact, there occurs a remarkably distinct feature, one is Thurston bending while the other is Margulis super rigidity. After Thurston's hyperbolization theory several non-Riemannian homogeneous geometry surrounding hyperbolic geometry came to our interest in 1980s~1990s. The $K$-hyperbolic space $\mathbb{H}_K^{n+1}$ has the projective compactification $\partial \mathbb{H}_K^{n+1}$ which...
is diffeomorphic to the sphere $S^{[K](n+1)-1}$. It is well known that the isometric action $\text{Isom}(\mathbb{H}_K^n)$ extends to a smooth action on $S^{[K](n+1)-1}$. This phenomenon occurs also for Hadamard manifolds (complete simply connected Riemannian manifold of nonpositive curvature). In general, an extended action on the boundary sphere is \textit{topological}. But the above actions on $\partial \mathbb{H}_K^{n+1}$ are known to be analytic. Denote $\text{Aut}(S^{[K](n+1)-1})$ the (extended) action of $\text{Isom}(\mathbb{H}_K^n)$ on $S^{[K](n+1)-1}$. It is known that $\text{Aut}(S^{[K](n+1)-1})$ acts transitively on $S^{[K](n+1)-1}$ with \textit{noncompact} stabilizer $\text{Aut}(S^{[K](n+1)-1})_\infty$ such that

$$S^{[K](n+1)-1} = \text{Aut}(S^{[K](n+1)-1})_\infty \backslash \text{Aut}(S^{[K](n+1)-1})$$

where $\infty \in S^{[K](n+1)-1}$. Hence we have a non-Riemannian homogeneous geometry $(\text{Aut}(S^{[K](n+1)-1}), S^{[K](n+1)-1})$. According to whether $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, it is said to be

\begin{align*}
\text{Conformally flat geometry} & \quad (\text{PO}(n + 1, 1), S^n) \\
\text{Spherical CR-geometry} & \quad (\text{PU}(n + 1, 1), S^{2n+1}) \\
\text{Quaternionic flat CR-geometry} & \quad (\text{PSp}(n + 1, 1), S^{4n+3})
\end{align*}

(3.1)

It is an excellent result by Gromov-Lawson-Yau that a nontrivial $S^1$-bundle $M^3$ over a closed surface $\Sigma_g$ of genus $g > 1$ admits a conformally flat structure. It is trivial that the product $S^1 \times \Sigma_g$ is a conformally flat manifold. On the other hand, in spherical CR-geometry $(\text{PU}(2, 1), S^3)$, the complement of geometric circle $S^3 - S^1$ has an invariant subgroup $U(1, 1) = P(U(1, 1) \times U(1))$. Choosing a discrete cocompact subgroup $\Gamma \leq U(1, 1)$, we get a spherical CR-manifold $S^3 - S^1/\Gamma$ which is a nontrivial $S^1$-bundle: $S^1 \to U(1) \backslash U(1, 1)/\Gamma \to U(1) \backslash \text{PU}(1, 1)/P(\Gamma)$. Here $U(1) \backslash \text{PU}(1, 1)/P(\Gamma) = \mathbb{H}_C^1/P(\Gamma) = \Sigma_g$. However, to our knowledge, the following problem hasn’t been yet proved rigorously.

**Problem.** Does the product $S^1 \times \Sigma_g$ admit a spherical CR-structure?

4. CONFORMALLY FLAT LORENTZ GEOMETRY

It is natural to consider how the isometry group of the pseudo-hyperbolic space acts on the compactification. Put $V_{-m+2,2}^1 = \{ x \in \mathbb{R}^{m+4} \mid B(x, x) = x_1^2 + \cdots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 < 0 \}$. If $P_{\mathbb{R}} : \mathbb{R}^{m+4} - \{0\} \to \mathbb{R}^{m+3}$ is the canonical projection, then the real pseudo-hyperbolic space $\mathbb{H}_\mathbb{R}^{m+2,1}$ is defined to be $P_{\mathbb{R}}(V_{-m+2,2}^1)$. For this reason, the $m+3$-dimensional quadrics $V_{-1}^{m+2,2} = \{ x \in \mathbb{R}^{m+4} \mid x_1^2 + \cdots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 = -1 \}$ with Lorentz metric $g$ is the complete pseudo-Riemannian manifold of signature $(m + 1, 1)$ and of constant curvature $-1$ such that $P_{\mathbb{R}}(V_{-m+2,2}^1) = P_{\mathbb{R}}(V_{-1}^{m+2,2})$. Since $P_{\mathbb{R}} : V_{-1}^{m+2,2} \to \mathbb{H}_\mathbb{R}^{m+2,1}$
is a two-fold covering, so $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is a complete pseudo-hyperbolic space form. The action $O(m + 2, 2)$ on $V_{-}^{m+2,2}$ induces an action on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$. The kernel of this action is the center $\mathbb{Z}/2 = \{\pm 1\}$ whose quotient is called real pseudo-hyperbolic group $PO(m + 2, 2)$. The projective compactification of $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is obtained by taking the closure $\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}}$ in $\mathbb{R}P^{m+3}$. Consider the commutative diagram:

$$
\begin{align*}
(GL(m + 4, \mathbb{R}), \mathbb{R}^{m+4} - \{0\}) & \longrightarrow (PGL(m + 4, \mathbb{R}), \mathbb{R}P^{m+3}) \\
\cup & \\
(O(m + 2, 2), V_{-}^{m+2,2} \cup V_{0}) & \longrightarrow (PO(m + 2, 2), \mathbb{H}_{\mathbb{R}}^{m+2,1} \cup S^{m+1,1})
\end{align*}
$$

Here $V_{0} = V_{0}^{m+2,1} = \{x \in \mathbb{R}^{m+4} | x_{1}^{2} + \cdots + x_{m+2}^{2} - x_{m+3}^{2} - x_{m+4}^{2} = 0\}$. It follows that $\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}} = \mathbb{H}_{\mathbb{R}}^{m+2,1} \cup S^{m+1,1}$.

From this viewpoint, the pseudo-hyperbolic action of $PO(m + 2, 2)$ on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ extends to conformal action of $S^{m+1,1}$. We obtain conformally flat Lorentz geometry $(PO(m + 2, 2), S^{m+1,1})$. This is of course non-Riemannian homogeneous geometry.

Let $(1, 0, \ldots, 0, 1) \in V_{0}$ be a null vector. Put $\infty = P(1, 0, \ldots, 0, 1) \in S^{m+1,1}$ which is called the point at infinity. The stabilizer $PO(m+2, 2)_{\infty}$ is $\mathbb{R}^{m+2} \times (O(m+1, 1) \times \mathbb{R}^{+})$ up to conjugacy. When $h \in PO(m+2, 2)_{\infty}$, the differential map $h_{\ast} : T_{\infty}S^{m+1,1} \to T_{\infty}S^{m+1,1}$ is an isomorphism, $h_{\ast} \in \text{Aut}(T_{\infty}S^{m+1,1}) = O(m+1, 1) \times \mathbb{R}^{+}$. Thus the structure group of $(PO(m + 2, 2), S^{m+1,1})$ is $O(m + 1, 1) \times \mathbb{R}^{+}$. Originally as a $G$-structure, conformal Lorentz structure is an $O(m+1, 1) \times \mathbb{R}^{+}$-structure. In addition, an integrable $O(m+1, 1) \times \mathbb{R}^{+}$-structure is conformally flat Lorentz structure. (Equivalently, the Weyl conformal curvature tensor vanishes.) When $\{\infty\}$ is the point at infinity of $S^{m} = \partial \mathbb{H}_{\mathbb{R}}^{m+1}$, we can consider the minimal parabolic group $O(m + 1, 1)_{\infty}$ which is an amenable Lie subgroup of $O(m + 1, 1)$. We remark that $O(m + 1, 1)_{\infty}$ is isomorphic to the similarity group $\text{Sim}(\mathbb{R}^{m})$.

**Definition 4.1.** If the structure group of a conformally flat Lorentz $(m + 2)$-manifold $M$ belongs to $O(m + 1, 1)_{\infty} \times \mathbb{R}^{+}$, then $M$ is said to be a conformally flat Lorentz parabolic manifold.

We study a special class of conformally flat Lorentz parabolic manifolds called Lorentz similarity manifold of dimension $m+2$ and Fefferman-Lorentz manifold of dimension $2n + 2$. 
5. LORENTZIAN SIMILARITY GEOMETRY

Recall that $\mathbb{R}^{m+2}$ is the euclidean space with Lorentz inner product sitting in $S^{m+1,1} - \{\infty\}$. Then $\text{PO}(m+2,2)_{\infty} = \mathbb{R}^{m+2} \ltimes (O(m+1,1) \times \mathbb{R}^+)$. We define $\text{Sim}_L(\mathbb{R}^{m+2}) = \mathbb{R}^{m+2} \ltimes (O(m+1,1) \times \mathbb{R}^+)$. The pair $(\text{Sim}_L(\mathbb{R}^{m+2}), \mathbb{R}^{m+2})$ is said to be Lorentz similarity geometry. In [3] we proved the following.

**Theorem 5.1.** If $M$ is a compact complete Lorentz similarity manifold of dimension $m + 2$, then the fundamental group of $M$ is virtually polycyclic. Furthermore, $M$ is diffeomorphic to an infrasolvmanifold.

This theorem is originally proved by T. Aristide. Once $\pi_1(M)$ turns out to be virtually polycyclic, the holonomy group $L(\pi_1(M))$ belongs to either $O(m+1,1)_{\infty} \ltimes \mathbb{R}^+$ or $O(m+1) \times (O(m) \times \mathbb{R}^+)$. Here $O(m+1,1)_{\infty} = \text{Sim}(\mathbb{R}^m) = \mathbb{R}^m \ltimes (O(m) \times \mathbb{R}^+)$. Since $\Gamma$ acts freely as a\'e motions on $\mathbb{R}^{m+2}$, the matrix of holonomy group has no eigenvalue 1. The latter case shows that $L(\pi_1(M)) \leq O(m+1) \times O(1)$ so that $M$ reduces to a compact euclidean space form. Then $\pi_1(M)$ is a Bieberbach group.

**Corollary 5.2.** A finite cover of a compact complete Lorentz similarity manifold $M$ is a conformally flat Lorentz parabolic manifold.

We shall give a sketch of proof of Theorem 5.1. Put $M = \mathbb{R}^{m+2}/\Gamma$ where $\Gamma \leq \text{Sim}_L(\mathbb{R}^{m+2})$. There is the exact sequence: $1 \rightarrow \mathbb{R}^{m+2} \rightarrow \text{Sim}_L(\mathbb{R}^{m+2}) \overset{L}{\rightarrow} O(m+1,1) \times \mathbb{R}^+ \rightarrow 1$. If $\mathbb{R}^{m+2} \cap \Gamma$ is nontrivial, say $\mathbb{Z}^k$, then a properly discontinuous action of $\Gamma$ induces a properly discontinuous action of $L(\Gamma)$ on $\mathbb{R}^{m-k}$ as in the same argument of [3, (1) Proposition 2.2]. Then $\Gamma$ is virtually polycyclic by induction. So we assume

$$
(5.1) \quad \mathbb{R}^{m+2} \cap \Gamma = \{1\}.
$$

Note also that $(\mathbb{R}^{m+2} \times \mathbb{R}^+) \cap \Gamma = \{1\}$ because each element has the form $(a, \lambda \cdot I)$. As $\Gamma$ acts freely on $\mathbb{R}^{m+2}$, $\lambda = 1$. It follows $(\mathbb{R}^{m+2} \times \mathbb{R}^+) \cap \Gamma = \mathbb{R}^{m+2} \cap \Gamma$.

Consider the following exact sequence:

$$
(5.2) \quad 1 \rightarrow \mathbb{R}^{m+2} \times \mathbb{R}^+ \rightarrow \text{Sim}_L(\mathbb{R}^{m+2}) \overset{p}{\rightarrow} O(m+1,1) \rightarrow 1.
$$

If $p(\Gamma)$ is discrete in $O(m+1,1)$, then the cohomological dimension $\text{cd} \ p(\Gamma) \leq m + 1$. As $\mathbb{R}^{m+2}/\Gamma$ is compact, $\text{cd} \ \Gamma = m + 2$. On the other hand, $\Gamma \cong p(\Gamma)$ by (5.1), $\text{cd} \ \Gamma = \text{cd} \ p(\Gamma)$ which yields a contradiction.

Suppose that $p(\Gamma)$ is indiscrete in $O(m+1,1)$. Then the identity component of the closure $\overline{p(\Gamma)}^0$ is solvable in $O(m+1,1)$. 

Case I. If it is noncompact, then it belongs to the maximal amenable subgroup $\text{Sim}(\mathbb{R}^m)$ up to conjugate. The normalizer of $\overline{p(\Gamma)}^0$ is contained in $\text{Sim}(\mathbb{R}^m)$. In particular, $p(\Gamma) \leq \text{Sim}(\mathbb{R}^m)$. (5.2) induces an exact sequence:

$$1 \to \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \to p^{-1}(\text{Sim}(\mathbb{R}^m)) \xrightarrow{p} \text{Sim}(\mathbb{R}^m) \to 1$$

in which $p^{-1}(\text{Sim}(\mathbb{R}^m))$ is an amenable Lie subgroup. Any discrete subgroup of an amenable Lie group is virtually polycyclic so is $\Gamma$.

Case II. Suppose that $\overline{p(\Gamma)}^0$ is compact, say $T^\ell$. We consider actions of subgroups of $O(m+1,1)$ on $\mathbb{H}_\mathbb{R}^{m+1} \cup S^m$. If $T^\ell$ has no fixed point in $S^m$, then $T^\ell$ has a unique fixed point $0 \in \mathbb{H}_\mathbb{R}^{m+1}$ so that $p(\Gamma) \leq O(m+1) \times O(1)$. Thus $\Gamma \leq \text{Sim}(\mathbb{R}^{m+2})$. $\mathbb{R}^{m+2}/\Gamma$ turns out to be a compact complete similarity manifold and so $\Gamma$ is virtually abelian (a Bieberbach group).

Suppose that $T^\ell$ has the fixed point set $S^k$ in $S^m$ for some $k < m$. As $p(\Gamma)$ leaves invariant the complement $S^m - S^k = \mathbb{H}_\mathbb{R}^{k+1} \times S^{m-k-1}$. It follows $\overline{p(\Gamma)} \leq O(k+1,1) \times O(m-k)$ for which $T^\ell = \overline{p(\Gamma)}^0 \leq O(m-k)$. If $\text{Pr} : O(k+1,1) \times O(m-k) \to O(k+1,1)$ is the canonical projection, then $\text{Pr}(p(\Gamma))$ is discrete. Note that $\text{Ker Pr} \circ p = \mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+)$. Put

$$\Delta = (\mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+)) \cap \Gamma.$$

It is nontrivial, because if trivial, $\Gamma \cong \text{Pr}(p(\Gamma))$ so $m + 2 = \text{cd} \Gamma = \text{cd} \text{Pr}(p(\Gamma))$ but $\text{cd} \text{Pr}(p(\Gamma)) \leq k + 1$ which is impossible by the inequality $k < m$.

Since $T^\ell$ is a maximal torus in $O(m-k)$, $N_{O(m-k)}(T^\ell)/T^\ell$ is finite for the normalizer $N_{O(m-k)}(T^\ell)$. As $\overline{p(\Gamma)}$ normalizes $T^\ell$, there exists a finite index normal subgroup $H$ of $\overline{p(\Gamma)}$ which centralizes $T^\ell$ with $H^0 = T^\ell$. Note that $H \cap p(\Gamma)$ is of finite index in $p(\Gamma)$. Put $\Gamma_1 = p^{-1}(H \cap p(\Gamma))$ which is a finite index subgroup of $\Gamma$.

Let $p_1 : \mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+) \to O(m-k)$ be the projection. Then $p_1(\Delta) \leq O(m-k)$ such that $\overline{p_1(\Delta)}^0$ is a torus in $O(m-k)$ from (5.3). Since $\overline{p_1(\Delta)}$ is a finite extension of $\overline{p_1(\Delta)}^0$, we choose $\Delta_1$ such that $p_1(\Delta_1) = \overline{p_1(\Delta)}^0 \cap p_1(\Gamma)$. As $p_1(\Delta_1) \leq p(\Gamma)$, $p_1(\Delta_1)^0 \leq p(\Gamma)^0$. Noting that $p(\Gamma_1) \leq H$ and $p(\Delta_1) \leq p(\Gamma)^0$ for which $H$ centralizes $T^\ell = \overline{p(\Gamma)}^0$ as above, it follows that $p(\Gamma_1)$ centralizes $p(\Delta_1)$.

Note that $p_1 : \Delta \to p_1(\Delta)$ is injective. In fact, if not, then $\Delta \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\}$, so $\Gamma \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\}$ which is impossible by the remark below (5.1). Since $\Gamma$ normalizes $\Delta$, it is easy to see that $\Gamma_1$ centralizes $\Delta_1$. Consider the exact sequences:
$\mathbb{R}^{m+2} \rtimes O(m-k) \rightarrow O(m-k) \rightarrow 1$

where $p(\Delta_1) = T^s$ for some $s \leq m-k$. It is well known that the abelian discrete subgroup belongs to the following group (cf. [8]):

$$\Delta_1 \leq V \times T^s = \{ \left( \begin{array}{c} a \\ 0 \end{array} \right), \left( \begin{array}{cc} I & 0 \\ 0 & C \end{array} \right) \} | C \in T^s, a \in V \}$$

such that $V \times T^s/\Delta_1$ is compact. Here $V \cong \mathbb{R}^{k+2}$.

Let $\Gamma_1 \leq \mathbb{R}^{m+2} \rtimes (O(k+1,1) \times O(m-k) \times \mathbb{R}^+)$ be as before and choose an arbitrary element $\gamma = \left( \begin{array}{c} x \\ y \end{array} \right), \lambda \cdot \left( \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right)$ and take an element $\alpha = \left( \begin{array}{c} a \\ 0 \end{array} \right), \left( \begin{array}{cc} I & 0 \\ 0 & C \end{array} \right)$ from $\Delta_1$. As $\Gamma_1$ centralizes $\Delta_1$, the equation $\gamma \alpha \gamma^{-1} = \alpha$ implies that

$$\lambda \cdot Aa = a, BCB^{-1} = C$$

and $y - BCB^{-1}y = 0$.

The projection $P : \mathbb{R}^{k+2} - \{0\} \rightarrow \mathbb{R}P^{k+1}$ maps the cone $V_0$ onto $S^k$. We observe that if $(a, \lambda) = 0$ with respect to the Lorentz inner product, then $P(a) = [a] \in S^k$. Put $[a] = \infty \in S^k$ up to conjugacy. The equality $\lambda \cdot Aa = a$ implies $A\infty = \infty$ so $A \in O(k+1,1)_{\infty}$. This holds for arbitrary elements of $\Gamma_1$. It follows

$$\Gamma \leq \mathbb{R}^{m+2} \rtimes (O(k+1,1)_{\infty} \times O(m-k) \times \mathbb{R}^+)$$

which is an amenable Lie subgroup. Thus $\Gamma$ is virtually polycyclic. When $\langle a, a \rangle \neq 0$, as $\langle a, a \rangle = \langle \lambda Aa, \lambda Aa \rangle = \lambda^2 \langle a, a \rangle$, it follows $\lambda = 1$. Thus

$$\Gamma \leq \mathbb{R}^{m+2} \rtimes (O(k+1,1) \times O(m-k)) \leq E(m+1,1).$$

$\mathbb{R}^{m+2}/\Gamma$ becomes a compact complete Lorentz flat space form. It is well known that $\Gamma$ is virtually polycyclic. This proves the theorem 5.1.

6. Conformally flat Fefferman-Lorentz geometry

Let $(O(2n+2,2), S^1 \times S^{2n+1})$ be the conformally flat Lorentz geometry (which is a 2-fold cover.) There is the natural embedding $U(n+1,1) \rightarrow O(2n+2,2)$. $U(n+1,1)$ acts transitively on $S^1 \times S^{2n+1}$ so we have a subgeometry $(U(n+1,1), S^1 \times S^{2n+1})$.

**Proposition 6.1.** A manifold locally modelled on $(U(n+1,1), S^1 \times S^{2n+1})$ admits a Lorentz parabolic structure.
Proof. We see that
\[ \hat{U}(n+1,1) \cap O(2n+2,2)_{\infty} = \mathbb{R}^{2n+2} \times (O(2n+1,1) \times \mathbb{R}^*). \]
Then the intersection \( \hat{U}(n+1,1)_{\infty} \triangleleft (O(2n+1,1) \times \mathbb{R}^*) \) belongs to the maximal amenable group \( \mathbb{R}^{2n+2} \times (O(2n+1,1)_{\infty} \times \mathbb{R}^*). \) Thus the structure group of \((U(n+1,1), S^1 \times S^{2n+1})\) belongs to the maximal amenable group \( \mathbb{R}^{2n+2} \times (O(2n+1,1)_{\infty} \times \mathbb{R}^*)\). Thus the structure group of \((U(n+1,1), S^1 \times S^{2n+1})\) belongs to the parabolic group \( O(2n+1,1)_{\infty} \times \mathbb{R}^* \).

\[ \square \]

**Definition 6.2.** A manifold locally modelled on \((U(n+1,1), S^1 \times S^{2n+1})\) is said to be a conformally flat Fefferman-Lorentz parabolic manifold.

To the rest of this section we shall give our recent results concerning compact conformally flat Fefferman-Lorentz parabolic manifolds. The details will be given elsewhere.

Recall that the center \( S^1 \) acts freely on the 2-fold covering \( S^1 \times S^{2n+1} \) of \( S^{2n+1,1} \), there is the equivariant principal bundle:

\[ (S^1, S^1) \rightarrow (U(n+1,1), S^1 \times S^{2n+1}) \xrightarrow{(P,p)} (PU(n+1,1), S^{2n+1}). \]

Let \( X \) be a domain of \( S^1 \times S^{2n+1} \). If \( h \) is an element of the group of conformal Lorentz transformations \( \text{Conf}(X) \), then \( h : X \rightarrow X \) extends uniquely to a conformal diffeomorphism of \( S^1 \times S^{2n+1} \) by Liouville's theorem. We assume that

\[ \text{Conf}(X) \leq U(n+1,1). \]

Suppose that a discrete subgroup \( \Gamma \) of \( U(n+1,1) \) acts properly discontinuously on \( X \) such that the quotient \( X/\Gamma \) is compact. Note that there is a covering group extension:

\[ \begin{array}{cccc}
1 & \rightarrow & \Gamma & \rightarrow & N_{\text{Conf}(X)}(\Gamma) & \xrightarrow{\nu} & \text{Conf}(X/\Gamma) & \rightarrow & 1.
\end{array} \]

We shall determine \( X/\Gamma \) when \( X/\Gamma \) admits a 1-parameter subgroup \( H \) whose lift \( H \) to \( U(n+1,1) \) is not the center \( \mathcal{Z}U(n+1,1) \).

**Theorem 6.3.** Let \( X/\Gamma \) be a 2n + 2-dimensional compact conformally flat Fefferman-Lorentz parabolic manifold. If \( X/\Gamma \) admits a 1-parameter subgroup \( H \) whose lift \( H \) to \( U(n+1,1) \) is not the center \( \mathcal{Z}U(n+1,1) \), then \( X/\Gamma \) is a Seifert fiber space over a spherical CR-orbifold. Moreover \( X/\Gamma \) is either one of (i), . . . , (v). As a consequence, a finite covering of such \( X/\Gamma \) is a Fefferman-Lorentz manifold.

(i) \( X/\Gamma = S^1 \times_{\mathbb{Z}_t} S^{2n+1} \) where \( \mathbb{Z}_t \leq T^{n+1} \).

(ii) \( S^1 \rightarrow X/\Gamma \rightarrow \mathcal{N}/Q \) where \( Q \leq \mathcal{N} \rtimes U(n) \).
The idea of proof is as follows. Let $S^1 = \mathcal{Z}U(n+1,1)$ be the center of $U(n+1,1)$. Then $S^1 \cdot \tilde{H} \leq U(n+1,1)$.

There is an equivariant fibration:

$$(S^1, S^1) \longrightarrow (S^1 \cdot \tilde{H}, \Gamma, X) \longrightarrow^{(p,p)} (G, Q, W)$$

where we put $G = S^1 \cdot \tilde{H}/S^1$, $Q = \Gamma/S^1 \cap \Gamma$ and $W = X/S^1$. As $Q, G \leq PU(n+1,1)$, the quotient $W/Q$ is a spherical $CR$-orbifold with $CR$-action $G$. To determine $X/\Gamma$ reduces to the classification of $CR$-manifolds $(Q, W)$ with the 1-parameter group $G$ of $CR$-transformations. The classification is accomplished by the result in [4].

When $\dim X/\Gamma = 4$, then $Q \leq U(1,1)$ so that $L(Q) \subset S^1$ (for $k = 1$). According to whether $L(Q)$ is a Cantor set in $S^1$ or $L(Q) = S^1$, it is well known that $S^3 - L(Q)/Q = S^1 \times S^2 \# \cdots \# S^1 \times S^2$ or some finite cover of $S^3 - L(Q)/Q = V^3_1/Q$ is a principal $S^1$-bundle with nonzero euler class over a closed surface of genus $g \geq 2$.

**6.1. Non Fefferman-Lorentz manifold.** It is conceivable whether some finite cover of any compact conformally flat Fefferman-Lorentz parabolic manifold is a Fefferman-Lorentz manifold. It is not true in general. It will be shown

**Proposition 6.4.** There exists a compact conformally flat Fefferman-Lorentz parabolic manifold $P$ of dimension $2n+2$ ($n \geq 1$) but no finite covering is a Fefferman-Lorentz manifold.

This manifold $P$ supports a principal fiber space: $T^2 \rightarrow P \rightarrow \mathbb{H}^2_c/Q_0$ where $\mathbb{H}^2_c/Q_0$ is a compact complex hyperbolic manifold.

**7. REPRESENTATION SPACE**

Let $X/\Gamma$ be a compact conformally flat Lorentz manifold with $S^1$-action so that $X \subset \mathbb{R} \times S^{2n+1}$, $\Gamma, \tilde{S}^1 \leq O(m+2,2)$. If $p: O(m+2,2) \rightarrow O(m+2,2)$ is the covering homomorphism, put $G = p(\tilde{S}^1) \leq O(m+2,2)$. We will prove that

- If $G$ is compact, then $m = 2n$ and $G = S^1$, $C_{O(m+2,2)}(S^1) = U(n+1,1)$. $(S^1, X/\Gamma)$ is locally modelled on $(U(n+1,1), S^1 \times S^{2n+1})$ where $S^1 = \mathcal{Z}U(n+1,1)$, i.e. $X/\Gamma$ is a conformally flat Fefferman-Lorentz parabolic manifold.
• If $G$ is noncompact, the either $\Gamma \leq \mathbb{R}^{m+2} \rtimes O(m+1,1)$ and $G = \mathbb{R}$ or $\Gamma \leq O(m+1,1) \times \mathbb{R}^+$ and $G = \mathbb{R}^+$.

Proposition 7.1. Let $X/\Gamma = S^1 \times N^3/\Delta$ which is a conformally flat Lorentz parabolic manifold and $\Gamma = \mathbb{Z} \times \Delta \leq O(4,2)$, $\# \subset \mathbb{R} \times S^3$. There are exactly two distinct faithful representations up to conjugate in $O(4,2)$:

\[
\begin{align*}
\rho_1 : \Gamma &\to \mathbb{R} \times (N \rtimes U(1)), \text{ } S^1 \text{ is lightlike.} \\
\rho_2 : \Gamma &\to \mathbb{R}^3 \times (\mathbb{R}^2 \rtimes O(2)) \leq \mathbb{R}^4 \rtimes O(3,1), \text{ } S^1 \text{ is spacelike.}
\end{align*}
\]  

Then the space of discrete faithful representations $R(\Gamma, O(4,2))$ consists of two components $R(\Gamma, \mathbb{R} \times (N \rtimes U(1)))$, $R(\Gamma, \mathbb{R}^3 \times (\mathbb{R}^2 \rtimes O(2)))$.

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