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Quantum representation and dual Garside structure

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1 Introduction

It is widely known that the quantum group $U_q(\mathfrak{g})$, the quantum enveloping algebra of a lie algebra $\mathfrak{g}$, gives rise to a representation of the braid group $B_n$ called a quantum representation. For an $U_q(\mathfrak{g})$-module $V$, one gets a linear representation $B_n \to \text{GL}(V^\otimes n)$ using a universal $R$-matrix. Such braid representations, especially for the case $\mathfrak{g}$ is a simple lie algebra such as $\mathfrak{sl}_2$, have gathered much attentions since they produce topological invariants of knots, links and 3-manifolds called quantum invariants.

Although quantum invariants have been actively studied, the quantum representation themselves are still mysterious. In this paper we illustrate a new point of view in the study of quantum braid representations. We show that “generic” quantum representations nicely behaves with respect to the dual Garside structure of the braid groups. This suggests that quantum representations have various nice properties than we first expected.

The dual Garside structure is a combinatorial structure of braid groups which dates back to Garside’s solution of words and conjugacy problem for the braid groups [G]. The dual Garside structure introduces a normal form of braids called a (dual Garside) normal form, and we have a nice length function called the dual Garside length which can be computed quite effectively.

A relationship between a linear representation of the braid groups and dual Garside structure was inspired by author’s previous works [I1, IW], which established a connection between Homological representations of the braid groups and the dual Garside length.

In this paper, we restrict our attention to the simplest case, $\mathfrak{g} = \mathfrak{sl}_2$ and we omit the proof of the main theorem. The proof of our main Theorem, Theorem 4.2, consists of several (tricky) calculations of the action of $B_n$, with a help of dual Garside structures. Details will be included in [I2]. We will treat the case where $\mathfrak{g}$ is a general lie algebra in [I3].

2 Dual Garside structure of the braid groups

In this section we summarize basic facts on the dual Garside structure of the braid groups. For details, see Birman-Ko-Lee [BKL]. [BGG, Section 1] provides a good overview of Garside structures emphasizing the role of normal forms. See [DDKM] for general and categorical treatments of Garside theory.
2.1 Dual Garside structure and normal forms

For $1 \leq i < j \leq n$, let $a_{i,j}$ be the braid

$$a_{i,j} = (\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1})^{-1}\sigma_{i}(\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1}).$$

The generating set $\Sigma^* = \{a_{i,j} \mid 1 \leq i < j \leq n\}$ was introduced in [BKL]. An element of $\Sigma^*$ is called the dual Garside generators, or band generators, or Birman-Ko-Lee generators. The dual braid monoid $B_n^{++}$ is a submonoid of $B_n$ generated by $\Sigma^*$. An element of $B_n^{++}$ is called a dual-positive braid. The braid $\delta = a_{1,2}a_{2,3} \cdots a_{n-1,n}$ is called the dual Garside element.

Let $\preceq$ be the suffix ordering with respect to the dual Garside generators $\Sigma^*$: $\beta_1 \preceq \beta_2$ if and only if $\beta_2\beta_1^{-1} \in B_n^{++}$. This defines a lattice ordering on $B_n$, that is, for $s, t \in B_n$, there exists a unique least common multiple $s \vee t$ and a unique greatest common divisor $s \wedge t$. A dual-positive braid $x$ is called a dual-simple if $x \preceq \delta$. The set of dual-simple elements is denoted by $[1, \delta]$. Instead of $\Sigma^*$, we will often use $[1, \delta]$ as a generator of $B_n$.

A (right-greedy, dual Garside) normal form of a braid $\beta \in B_n$ is a decomposition of $\beta$ as a product of dual simple elements of the form

$$\beta = x_r \cdots x_1\delta^p$$

that is defined by

1. $p$ is the maximal integer that satisfies $\delta^p \preceq \beta$.
2. For $i = 1, \ldots, r$, $x_i = (x_r \cdots x_i) \wedge \delta$.

We will denote the normal form of $\beta$ by $N(\beta)$. The normal form has the following remarkable property.

**Proposition 2.1.** $x_r \cdots x_1\delta^p$ is a normal form if and only if $x_1 \neq \delta$ and $x_{i+1}x_i \wedge \delta = x_i$ for each $i$ (in other words, $x_{i+1}x_i$ is a normal form for each $i$).

This proposition leads to an effective way of computing a normal form. Moreover, the normal forms induces a bi-automatic structure of the braid groups. See [ECHLPT, Deh].

The supremum $\sup(\beta)$ and the infimum $\inf(\beta)$ of $\beta$ are integers defined by

$$\begin{align*}
\sup(\beta) &= \min\{m \in \mathbb{Z} \mid \beta \preceq_{\Sigma^*} \delta^m\} \\
\inf(\beta) &= \max\{M \in \mathbb{Z} \mid \delta^M \preceq_{\Sigma^*} \beta\}
\end{align*}$$

These values are closely related to the normal form of $\beta$. If $N(\beta) = x_r \cdots x_1\delta^p$ then

$$\sup(\beta) = p + r, \quad \text{and} \quad \inf(\beta) = p.$$
2.2 Diagrammatic expression of dual Garside structures

Here we explain a convenient expression of dual simple elements using convex polytopes.

Let $D_n = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the $n$-punctured disc. We put the puncture points $p_1, \ldots, p_n$ on the circle $|z| = \frac{1}{2}$, as shown in the right of Figure 1. Then there is a one-to-one correspondence between the set of disjoint collections of convex polygons in $D_n$ whose vertices are puncture points, and the set of dual-simple elements.

This correspondence is given as follows: First assume that a convex polygon $P$ is connected. Let $p_{m_1}, \ldots, p_{m_k}$ ($1 \leq m_1 < m_2 < \cdots < m_k \leq n$) be the vertices of $P$. We define a braid $x_P$

$$x_P = a_{m_1,m_2}a_{m_2,m_3}\cdots a_{m_{k-1},m_k}.$$ 

For a disjoint collection of convex polygons $\mathbb{P} = \{P_1, \ldots, P_M\}$, we define

$$x_P = x_{P_1}x_{P_2}\cdots x_{P_M}.$$ 

Then it is seen that $x_P$ is a dual-simple element. Conversely, every dual-simple element can be expressed in such a way. For $x \in [1, \delta]$, we will write the corresponding convex polygons by $P_x$.

This correspondence can be easily understood by using geometric interpretation of the braid groups. As is well-known, the braid group $B_n$ is identified with the mapping class group of $n$-punctured disc $D_n$.

For $1 \leq i < j \leq n$, let $e_{ij}$ be the line segment that connects the $i$-th and the $j$-th punctures. As an element of mapping class group, the band-generator $a_{i,j}$ corresponds to the left-handed half-Dehn twist along $e_{ij}$: $a_{i,j}$ interchanges the position of punctures $p_i$ and $p_j$ by rotating the small disc neighborhood of $e_{ij}$ in a clockwise direction (see Figure 1).

![Figure 1: n-punctured disc $D_n$ and action of $a_{ij}$](image)

By generalizing this move of punctures, for a collection of convex polygons we associate a dance of the puncture points. Each puncture which belongs to some polygon $P$ moves to the position of the adjacent vertex, in the clockwise direction along the boundary of $P$, see Figure 2. In particular, the dual Garside element $\delta$ acts on $D_n$ as rotation of disc by $(2\pi/n)$. 
3 Generic quantum $\mathfrak{sl}_2$ representation

In this section, we review a construction of generic quantum $\mathfrak{sl}_2$-representation following Jackson-Kerler [JK]. For basics of $U_q(\mathfrak{sl}_2)$ we refer [Kas]. (Here we remark that to make correspondence between the dual Garside structure and quantum representations simple, we slightly modified the sign convention: the variable $s$ in this paper corresponds to $s^{-1}$ in [JK, Il].)

We define the $q$-numbers, $q$-fractionals, and $q$-binomial coefficients as

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!}.$$

Let $\mathbb{C}[[\hbar]]$ be the algebra of the complex formal power series in one variable $\hbar$. A quantum enveloping algebra $U_q(\mathfrak{s\ell}_2)$ is a topological Hopf algebra over $\mathbb{C}[[\hbar]]$ generated by $H, E, F$, with relations

$$\begin{cases}
[E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}.
\end{cases} \quad (3.1)$$

The coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ (here $\otimes$ denotes the topological tensor product, the $\hbar$-adic completion of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$), and the antipode $S$ are given by

$$\begin{cases}
\Delta(E) = E \otimes e^{\hbar H} + 1 \otimes E, \\
\Delta(F) = F \otimes 1 + e^{-\hbar H} \otimes F, \\
\Delta(H) = H \otimes 1 + 1 \otimes H, \\
S(E) = -e^{-\hbar H} F, & S(F) = -e^{\hbar H} E, & S(H) = -H.
\end{cases} \quad (3.2)$$

$U_q(\mathfrak{sl}_2)$ is a quasi-triangular topological Hopf algebra. Namely, there exists an element $R \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ called a universal $R$-matrix that satisfies the properties

$$\begin{cases}
R \Delta(x) = \Delta^\text{op}(x) R, \\
(\Delta \otimes \text{id}) R = R_{12} R_{23}, \\
(\text{id} \otimes \Delta) R = R_{13} R_{12}.
\end{cases}$$
where $\Delta^{op}$ denotes the opposite of $\Delta$. These properties show that $\mathcal{R}$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. $$

The universal $R$-matrix $\mathcal{R}$ for $U_{\hbar}(\mathfrak{sl}_2)$ is given by

$$\mathcal{R} = e^{\frac{\hbar}{2}(H\otimes H)}\left( \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n \otimes F^n \right),$$  

(3.3)

where we put $q = e^\hbar$.

For $\lambda \in \mathbb{C}^*$, let $V_\lambda$ be the Verma module with highest weight $\lambda$, which is a topologically free $U_{\hbar}(\mathfrak{sl}_2)$-module generated by a highest weight vector $v_0$ that satisfies

$$Hv_0 = \lambda v_0, \quad Ev_0 = 0.$$

If $\lambda$ is not an integer ("generic"), then as a $\mathbb{C}\llbracket \hbar \rrbracket$-module, the Verma module $V_\lambda$ is freely generated by $\{v_i\}_{i=0,1,\ldots}$ and the action of $U_{\hbar}(\mathfrak{sl}_2)$ is given by

$$\begin{align*}
Hv_i &= (\lambda - 2i)v_i \\
Ev_i &= v_{i-1} \\
Fv_i &= [i+1]_q \frac{e^{\hbar(\lambda-i)} - e^{-\hbar(\lambda-i)}}{e^\hbar - e^{-\hbar}} v_{i+1}.
\end{align*}$$  

(3.4)

Now we treat all generic Verma modules at once by regarding a weight $\lambda$ as a variable instead of treating as a complex parameter. To this end, we regard $U_{\hbar}(\mathfrak{sl}_2)$ as a topological Hopf algebra over the coefficient ring $\mathbb{C}[\lambda][[\hbar]]$, the polynomial ring with coefficients in $\mathbb{C}[[\hbar]]$. We regard the formula (3.4) as a definition of a $U_{\hbar}(\mathfrak{sl}_2)$-module $V_\hbar$: namely, as a topological $\mathbb{C}[\lambda][[\hbar]]$-module, $V_\hbar$ is a $\mathbb{C}[\lambda][[\hbar]]$-module freely generated by $\{v_0, v_1, \ldots\}$ and $U_{\hbar}(\mathfrak{sl}_2)$ acts on $V_\hbar$ by the formula (3.4). We call $V_\hbar generic Verma module$.

Let $L = \mathbb{C}[q^{\pm 1}, s^{\pm 1}]$ be the ring of two-variable Laurent polynomial, and we regard $L$ as a subring of $\mathbb{C}[\lambda][[\hbar]]$ via the injective homomorphism $i_h : L \rightarrow \mathbb{C}[\lambda][[\hbar]]$ defined by $i_h(q) = e^\hbar$, $i_h(s) = e^{\hbar\lambda}$.

Let $V_L \subset V_\hbar$ be the free $L$-module generated by basis vectors $\{v_0, v_1, \ldots\}$ of $V_\hbar$, and let $R = e^{-\frac{\hbar}{2}\lambda^2} o \mathcal{R} o T : V_L^\otimes 2 \rightarrow V_L^\otimes 2$.

Then by (3.3), the action of $R$ is written as

$$R(v_i \otimes v_j) = s^{(i+j)} \sum_{n=0}^{n} F_{i,j,n}(q) \prod_{k=0}^{n-1}(s^{-1}q^{-k-j} - sq^{k+j})v_{j+n} \otimes v_{i-n},$$  

(3.5)

where $F_{i,j,n}(q) = q^{2(i-n)(j+n) \frac{n(n-1)}{2}} \binom{n+j}{j}$. 

Similarly, the action of $R^{-1}$ is written as

$$R^{-1}(v_i \otimes v_j) = s^{-(i+j)} \sum_{n=0}^{n} (-1)^n F'_{i,j,n}(q) \prod_{k=0}^{n-1}(sq^{k+i} - s^{-1}q^{-k-i})v_{j-n} \otimes v_{i+n},$$
where $F_{i,j,n}(q) = q^{-2ji}q^{-\binom{n-1}{2}}\left[\begin{array}{c}n+i \\ i \end{array}\right]_q$.

Thus $R(V_{L}^\otimes 2) = V_{L}^\otimes 2$, so we get an (infinite dimensional) linear representation

$$\rho : B_n \to \text{GL}(V_{L}^\otimes m), \quad \rho(\sigma_i) = \text{id} \otimes (i-1) \otimes R \otimes \text{id}^\otimes (n-i-1)$$

which we call a generic quantum $\mathfrak{sl}_2$-representation.

To deduce finite dimensional representation, we take a weight decomposition of $\rho$. For $m \geq 0$, let $V_{n,m} = \{v \in V_{L}^\otimes n | e^H v = s^{-n}q^{-2m}v\}$ be the weight space corresponding to the weight $s^{-n}q^{-2m}$. (Here $e^H$ is often denoted by $K$ in a literature). It is directly checked that $V_{L}^\otimes n$, as a $\mathbb{C}B_n$-module, decomposes as $V_{L}^\otimes n = \bigoplus_{m=0}^{\infty} V_{n,m}$.

Then the set $\{v_{k_1} \otimes \cdots \otimes v_{k_n} | k_i \geq 0, k_1 + \cdots + k_n = m\}$ forms a basis of $V_{n,m}$. To relate the representation $V_{n,m}$ and the dual Garside structure, we use the following slightly modified basis of $V_{n,m}$, obtained by shifting the degree of the variable $s$. For $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^n$, we define $|k| = \sum_{i=1}^{n} k_i$, and $w_k = s^{-\sum_{i=1}^{n} ik_i}v_{k_1} \otimes v_{k_2} \otimes \cdots \otimes v_{k_n} \in V^\otimes n$.

Then the set $B = B(m) = \{w_k | |k| = m\}$ form a basis of $V_{n,m}$. The cardinal of $B(m)$ is $\binom{n+m-1}{m}$. By using this basis $B(m)$, we express the braid group representation $V_{n,m}$ as an explicit matrix

$$\rho_{m,n} : B_n \to \text{GL}\left(\begin{array}{c}n+m-1 \\ m \end{array}\right) ; L \right).$$

We call this representation a generic quantum $\mathfrak{sl}_2$ representation.

4 Main Theorem

From now on, we fix $m > 1$, and we put $V = V_{n,m}$ and $B = B(m)$. By abuse of notation, we may often identify the basis vector $w_k \in B$ and its corresponding sequence of integers $k = (k_1, \ldots, k_n)$. To make notation simple, for $\beta \in B_n$ and $w \in V$, we will write $\beta(w)$ to imply $\rho_{m,n}(\beta)(w)$.

4.1 Statement of Main theorem

For monomials $s^iq^j$ and $s^{i'}q^{j'}$ of $L = \mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$, we define the lexicographical ordering $\leq_{s,q}$ by

$$s^iq^j \leq_{s,q} s^{i'}q^{j'} \text{ if } i < i', \text{ or if } i = i' \text{ and } j \leq j'.$$

For $a \in L$, we will concentrate our attention to the $\leq_{s,q}$-maximal monomial. We denote the maximal and the minimum degree of the variable $s$ in $a$ by $M_s(a)$ and $m_s(a)$, respectively, and we write the $\leq_{s,q}$-maximal monomial in $a$ by $s^{M_s(a)}q^{N_s(a)} = s^m q^N$. The
\( \text{sign } \epsilon(a) \in \{ \pm 1 \} \) is defined as the sign of the coefficient of the \( <_{s,q} \)-maximal monomial \( s^M q^N \) in \( a \).

For \( i = 1, \ldots, n \), let \( k_i \in \mathcal{B} \) be a basis vector

\[
k(i) = (0, \ldots, 0, \check{m}, 0, \ldots, 0).
\]

and define \( w \in V \) by

\[
w = \sum_{i=1}^{n} w_{k(i)}.
\]

and \( w_{k(i)} \) plays an important role in computations in quantum representations.

For \( v = \sum_{w_{k} \in \mathcal{B}} a_{k}(s, q) w_{k} \in V \), we define

\[
M_{s}(v) = \max\{M_{s}(a_{k}) | k \in \mathcal{B}\}.
\]

By looking at the \( <_{s,q} \)-maximal monomials of \( a_{k} \), we assign a graph \( \Gamma(v) \) in \( D_n \) in the following manner:

The vertices of \( \Gamma(v) \) is a subset of the puncture points of \( D_n \). The \( i \)-th puncture \( p_i \) is a vertex of \( \Gamma(v) \) if and only if

(V) \( M_{s}(a_{k(i)}) = M_{s}(v) \)

holds.

Now assume that for \( 1 \leq i < j \leq n \), both \( p_i \) and \( p_j \) are vertices of \( \Gamma(v) \). For \( e = 0, \ldots, m \), let us put

\[
k(e; i, j) = (0, \ldots, 0, \check{e}, 0, \ldots, 0, m - e, 0, \ldots, 0) \in \mathcal{B}.
\]

We connect two vertices \( p_i \) and \( p_j \) by an edge if and only if

(E) The \( <_{s,q} \)-maximal monomial part of \( a_{k(e,i,j)} \) is

\[
(-1)^{e} \epsilon(a_{k(0)}) \cdot c \cdot s^{M_{s}(v)} q^{N_{q}(a_{k(0)})} q^{2em-e^{2}-e}.
\]

where \( c > 0 \) is the absolute value of the coefficient of the \( <_{s,q} \)-maximal monomial.

Finally we assign a graph \( \Gamma(x) \) for each dual simple element \( x \).

**Definition 4.1.** For a dual simple element \( x \in [1, \delta] \) we define the graph \( \Gamma(x) \) by \( \Gamma(x) = \Gamma(x(w)) \).

At first glance, the definition of the graph \( \Gamma \) seems to be artificial. Here we explain the background motivation of the definition of \( \Gamma \).

Let us rewrite a formula of the \( R \)-action on \( V \otimes V \) in terms of our modified (\( s \)-degree shifted) basis \( \{ w_{i,j} = s^{i+2j} v_{i} \otimes v_{j} \} \) of \( V \otimes V \), and concentrate our attention to the \( <_{s,q} \)-maximal monomials. Then the \( <_{s,q} \)-maximal monomial is given by

\[
R(w_{i,j}) = \sum_{n=0}^{i} s^{2i-n} q^{2(i-n)(j+n)} q^{\frac{n(n-1)}{2}} \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{k+j}) w_{j+n,i-n}
\]

\[
= \sum_{n=0}^{i} ((-1)^{n} s^{2i} q^{2ij+2ni-n^{2}-n} + \cdots) w_{j+n,i-n}
\]
This formula says that $M_s(R(w_{i,j})) = 2i \leq 2m$. Since we are interested in the case $s$-degree maximal part, let us consider the case $i = m$ and $j = 0$. Then we get

$$R(w_{m,0}) = \sum_{n=0}^{m}((-1)^m s^{2m} q^{2nm-n^2-n} + \ldots)w_{n,m-n}$$

This shows that the graph $\Gamma(a_{i+1}(w_{k(i)}))$ coincides with the convex polygon $e_{i+1}$ (an edge connecting $p_i$ and $p_{i+1}$).

More generally by using the above formula of $R$, one can check that for $i \leq k < j$, $\Gamma(a_{i,j}(w_{k(k)}))$ coincides with the convex polygon $e_{i,j}$ (an edge connecting $p_i$ and $p_j$). Thus, the graph $\Gamma$ was defined so that it captures the behaviour of the $<_{s,q}$-maximal part of $a_{i,j}(w)$ or $a_{i,j}(w_{k(k)})$.

For a general dual simple element $x$, like $x = a_{i,j}$ case, its graph $\Gamma(x)$ is closely related to the corresponding convex polygon $P_x$ although the relations are more complicated (especially when $P_x$ is not connected): Figure 3 shows several examples of the graph $\Gamma(x)$. As $\Gamma(a_{1,4}a_{2,3})$ suggests, not all edges of $\Gamma(x)$ is contained in the corresponding convex polygon $P_x$. It is checked that $\Gamma(x)$ does not depend on $m$, and for $x, y \in [1, \delta]$, $\Gamma(x) \neq \Gamma(y)$ if $x \neq y$.

Figure 3: (1) $\Gamma(a_{1,2}a_{2,3}a_{3,4}), P_{a_{1,2}a_{2,3}a_{3,4}}$ and (2) $\Gamma(a_{1,4}a_{2,3}), P_{a_{1,4}a_{2,3}}$

Now we are ready to state the main theorem.

**Theorem 4.2** (Dual Garside normal form and generic quantum $sl_2$-representation). Let $N(\beta) = x_r \cdots x_1 \delta^p$ be the normal form of $\beta \in B_n$. Then

1. $M_s(\beta w) = 2m \text{sup}(\beta)$.
2. $m_s(\beta w) = 2m \text{inf}(\beta)$.
3. $\Gamma(\beta w) = \Gamma(x_r)$.

This theorem shows that, the maximal $<_{s,q}$-maximal part of a generic quantum $sl_2$ representation nicely reflects the dual Garside normal form. In particular, one can compute the normal form of the braid $\beta$ by looking at the single vector $\beta(w)$.

Recall that the variable $s$ in a generic quantum representation $\rho_{m,n}$ comes from the weight of the Verma module. Thus we may view the maximal $s$-degree part of $\beta(w)$ as "highest weight" parts. Thus or main theorem suggests that there is an unexpected relationship between representation theory of lie algebras and quantum groups (highest weight vectors), and the dual Garside structures.
4.2 Several consequences of main theorem

We close the paper by presenting several consequences of our main theorem.

First we observe that as a corollary of our main theorem, we provide an alternative, algebraic proof of the main results in [11]. For an $N \times N$-matrix of $L$ coefficient $A = (a_{ij})$, we denote the maximal and the minimal degree of $s$ in $A$, $\max_{i,j} M_s(a_{ij})$ and $\min_{i,j} m_s(a_{ij})$, by $M_s(A)$ and $m_s(A)$, respectively.

**Corollary 4.3** (Dual Garside length formula [11]). Let $\beta \in B_n$.

1. $M_s(\rho_{m,n}(\beta)) = 2m \sup(\beta)$.
2. $m_s(\rho_{m,n}(\beta)) = -2m \inf(\beta)$.
3. $l(\beta) = 2m (\max\{0, M_s(\rho_{m,n}(\beta))\} - \min\{0, m_s(\rho_{m,n}(\beta))\})$.

Our argument provides a remarkable restriction for an image of generic quantum representation $\rho_{n,m}$.

**Theorem 4.4** (Image of quantum representation). Let $A \in \text{GL}(\binom{n+m-1}{m}; \mathbb{L})$.

1. If $A$ lies in the image of the generic quantum representation $\rho_{n,m}$, then for $1 \leq i \leq n$, $\Gamma(A_i) = \Gamma(x)$ for some $x \in [1, \delta]$. Here $A_i$ denotes the low of $A$ that corresponds to the basis vector $k(i)$.
2. There is an effective algorithm to determine whether $A$ lies in the image of the generic quantum representation $\rho_{n,m}$ or not.

We also remark that Theorem 4.2 gives a new, quantum-group theoretical proof of the faithfulness of the Lawrence-Krammer-Bigelow representation and its natural generalizations called Lawrence's representation $L_{n,m}$,

$$L_{n,m} : B_n \to \text{GL} \left( \binom{n+m-2}{m} ; \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \right).$$

Lawrence's representations are obtained by considering the action of the braid groups on the homology of the configuration space of $m$-points in $n$-punctured disc $D_n$. For details, see [11, Law]. $L_{n,1}$ is identical with the reduced Burau representation, and $L_{n,2}$ is called the Lawrence-Krammer-Bigelow representation. It is known that generic quantum representation decomposes as $\rho_{n,m} = \bigoplus_{i=0}^{m} L_{n,i}$.

**Theorem 4.5** (Dual Garside length formula [11]). For $\beta \in B_n$,

1. $M_s(L_{n,m}(\beta)) = m \sup(\beta)$.
2. $m_s(L_{n,m}(\beta)) = -m \inf(\beta)$.
3. $l(\beta) = m (\max\{0, M_s(L_{n,m}(\beta))\} - \min\{0, m_s(L_{n,m}(\beta))\})$.

In particular, $L_{n,m}$ is faithful.

It is already known that $L_{n,m}$ is faithful ([12] for details). However, the known proof of the faithfulness for $m > 2$ is based on a topological argument due to Bigelow [Big]. Our quantum-representation proof gives a purely algebraic proof.
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