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Kyoto University
Notes on liftable vector fields

Yusuke Mizota*

Abstract

We introduce the author’s research about an estimate for the highest degrees of liftable vector fields and the module of liftable vector fields for non-singular mono-germs and function mono-germs of one variable.

1 Introduction

In this paper, we introduce the author’s research about liftable vector fields. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. In this paper, suppose that all mappings are smooth (that is, of class $C^\infty$ if $\mathbb{K} = \mathbb{R}$ or holomorphic if $\mathbb{K} = \mathbb{C}$).

The notion of liftable vector fields was introduced by Arnol’d [1] for studying bifurcations of wave front singularities. As results and applications of liftable vector fields, Bruce and West [2] obtained diffeomorphisms preserving a crosscap to classify functions on it, and Houston and Littlestone [4] obtained generators for the module of vector fields liftable over the generalized cross cap to find $A_e$-codimension 1 maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$. Houston and Atique [3] classified $\nu \mathcal{K}$-codimension 2 maps on the generalized crosscap to apply to a classification of $A_e$-codimension 2 maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$. Nishimura [8] characterized the minimal number of generators for the module of vector fields liftable over a finitely determined multigerm of corank at most one satisfying a special condition when $n \leq p$.

In previous work [6], the author showed that we can find polynomial vector fields liftable over $f$ if $f$ is a polynomial multigerm and gave an estimate for the highest degrees of liftable vector fields. The highest degree of polynomial vector field $\xi$ means maximum of that of component functions of $\xi$. Let $|x|$ be the greatest integer not exceeding $x$. Lift$(f)$ denotes the module of vector fields liftable over $f$. We proved the following theorem.

**Theorem 1.1** ([6]). Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n < p$) be a polynomial multigerm. Then, there exists a non-zero polynomial vector field in Lift$(f)$ such that

*Research Fellow DC2 of Japan Society for the Promotion of Science
The author was supported by JSPS and CAPES under the Japan-Brazil Research Cooperative Program.
Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, JAPAN.
e-mail: y-mizota@math.kyushu-u.ac.jp
The highest degree is at most
\[ N = \left\lfloor p^{-\sqrt{(\alpha + 1)(p-1)!}} \right\rfloor + 1, \]
where
\[ \alpha = r \left( \frac{p \cdot 2^n - n}{n!} \right) (D + n - 1)^n, \quad r = |S|, \]
\[ D = \max\{D_i | i \in \{1, 2, \ldots, r\} \}, \quad D_i = \max\{\deg(X_j \circ f_i) | j \in \{1, 2, \ldots, p\}\}. \]

The proof of Theorem 1.1 also gives a method to find a non-zero element of \( \text{Lift}(f) \). However, we can usually take values of \( N \) that are much lower than those calculated in Theorem 1.1. Therefore, we needed to improve this estimate. In [7], a better estimate for the highest degrees of liftable vector fields was discovered when \( n = 1 \). It is the following result. Let \( \lceil x \rceil \) be the smallest integer greater than or equal to \( x \).

**Theorem 1.2 ([7])**. Let \( f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^p, 0) \) be a polynomial multigerm which contains no branch of zero map. Then there exist a non-zero polynomial vector field of \( \text{Lift}(f) \) such that the highest degree is at most
\[ N = \max \left\{ \left\lfloor p^{-\sqrt{(A \cdot p - A)(p-1)!}} - \sqrt{p!} \right\rfloor, 1 \right\}, \]
and the highest degree of a corresponding lowerable vector field for \( f_i \) is at most
\[ D_i N - D_i + 1, \]
where
\[ A = \sum_{i=1}^{r} D_i, \quad D_i = \max\{\deg(X_j \circ f_i) | j \in \{1, \ldots, p\}\}. \]

This paper is organized as follows. In Section 2, we explain various definitions, basic facts and examples implying difference of estimates between Theorem 1.1 and Theorem 1.2. In Section 3 the sketch of proof of Theorem 1.2 is described. In Section 4 and 5, topics about the module of liftable vector fields are given. The number of generators for the module of vector fields liftable over a non-singular mono-germ is identified in Section 4. Theorem 1.1 and Theorem 1.2 claims that there exist non-zero polynomial vector fields liftable over \( f \) when \( f \) is a polynomial. It is natural to ask whether there exist non-zero polynomial liftable vector fields when \( f \) is not a polynomial. In Section 5 we investigate the module of vector fields liftable over a function germ of one variable and also consider this problem.

2 Preliminary

Let \( S \) be a subset of \( \mathbb{K}^{p} \). A map germ \( f : (\mathbb{K}^{p}, S) \rightarrow (\mathbb{K}^{p}, 0) \) is called a multigerm. If \( S \) is a singleton, \( f \) is called a mono-germ. Let \( C_{S} \) (resp., \( C_{0} \)) be the set of
function germs \((\mathbb{K}^n, S) \to \mathbb{K}\) (resp., \((\mathbb{K}^p, 0) \to \mathbb{K}\)), and let \(m_S\) (resp., \(m_0\)) be the subset of \(C_S\) (resp., \(C_0\)) consisting of function germs \((\mathbb{K}^n, S) \to (\mathbb{K}, 0)\) (resp., \((\mathbb{K}^p, 0) \to (\mathbb{K}, 0)\)). The sets \(C_S\) and \(C_0\) have natural \(\mathbb{K}\)-algebra structures. A multigerm \(f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)\) can be defined by \((f_1, f_2, \ldots, f_r)\), where \(f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)\). Each \(f_i\) is called a branch. In this paper, for a multigerm \(f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)\) defined by \((f_1, f_2, \ldots, f_r)\) with \(f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)\), we consider \(S\) to be a set consisting of \(r\) distinct points.

For a map germ \(f : (\mathbb{K}^n, S) \to \mathbb{K}^p\), let \(\theta_S(f)\) be the set of germs of vector fields along \(f\). The set \(\theta_S(f)\) has a natural \(C_S\)-module structure and is identified with the direct sum of \(p\) copies of \(C_S\). Put \(\theta_S(n) = \theta_S(id_{(\mathbb{K}^n, S)})\) and \(\theta_S(p) = \theta_{\{0\}}(id_{(\mathbb{K}^p, 0)})\), where \(id_{(\mathbb{K}^n, S)}\) (resp., \(id_{(\mathbb{K}^p, 0)}\)) is the germ of the identity mapping of \((\mathbb{K}^n, S)\) (resp., \((\mathbb{K}^p, 0)\)). For a multigerm \(f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)\), following Mather [5], we define \(tf\) and \(\omega f\) as

\[
\begin{align*}
tf : \theta_S(n) &\to \theta_S(f), \quad tf(\eta) = df \circ \eta,
\omega f : \theta_0(p) &\to \theta_S(f), \quad \omega f(\xi) = \xi \circ f,
\end{align*}
\]

where \(df\) is the differential of \(f\). Following Wall [9], we put \(TR_\epsilon(f) = tf(\theta_S(n))\) and \(TL_\epsilon(f) = \omega f(\theta_0(p))\).

For a multigerm \(f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)\), a vector field \(\xi \in \theta_0(p)\) is said to be \underline{liftable over \(f\)} if \(\xi \circ f \in TR_\epsilon(f) \cap TL_\epsilon(f)\). The set of vector fields liftable over \(f\) is denoted by \(\text{Lift}(f)\). Note that \(\text{Lift}(f)\) has a natural \(C_0\)-module structure. Let \((x_1, x_2, \ldots, x_n)\) (resp., \((X_1, \ldots, X_p)\)) be the standard local coordinates of \(\mathbb{K}^n\) (resp., \(\mathbb{K}^p\)) at the origin. Sometimes \((x_1, x_2)\) (resp., \((X_1, X_2)\)) is denoted by \((x, y)\) (resp., \((X, Y)\)) and \((x_1, x_2, x_3)\) (resp., \((X_1, X_2, X_3)\)) is denoted by \((x, y, z)\) (resp., \((X, Y, Z)\)). We see easily that

\[
\xi = (\psi_1(X_1, X_2, \ldots, X_p), \ldots, \psi_p(X_1, X_2, \ldots, X_p)) \in \text{Lift}(f),
\]

where \(\psi_j : (\mathbb{K}^p, 0) \to \mathbb{K} \quad (j = 1, 2, \ldots, p)\), if and only if for every \(i \in \{1, \ldots, r\}\) there exist a vector field

\[
\eta_i = (\phi_{i,1}(x_1, x_2, \ldots, x_n), \ldots, \phi_{i,n}(x_1, x_2, \ldots, x_n)),
\]

where \(\phi_{i,k} : (\mathbb{K}^n, 0) \to \mathbb{K} \quad (k = 1, 2, \ldots, n)\), such that \(\xi \circ f_i = df_i \circ \eta_i\) i. e.

\[
\begin{pmatrix}
\psi_1(X_1, X_2, \ldots, X_p) \\
\vdots \\
\psi_p(X_1, X_2, \ldots, X_p)
\end{pmatrix}
\circ f_i(x_1, x_2, \ldots, x_n)
\]

\[
= \begin{pmatrix}
\frac{\partial(X_1 \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_1 \circ f_i)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial(X_p \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_p \circ f_i)}{\partial x_n}
\end{pmatrix}
\begin{pmatrix}
\phi_{i,1}(x_1, x_2, \ldots, x_n) \\
\vdots \\
\phi_{i,n}(x_1, x_2, \ldots, x_n)
\end{pmatrix}.
\]

We call this \(\eta_i\) a lowerable vector field for \(f_i\) corresponding to \(\xi\).
Example 2.1. Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be given by $f(x) = (x^2, x^3)$. Then, it can be seen easily that the following vector fields are liftable over $f$:

\[
\begin{pmatrix}
2X \\
3Y
\end{pmatrix}, \begin{pmatrix}
2Y \\
3X^2
\end{pmatrix}.
\]

The forms of vector fields liftable over a polynomial multigerm are complicated generally.

Example 2.2. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (x^3, x^2)$. Then, it is known [8] that the following vector fields are liftable over $f$:

\[
(-6X^2Y^2 + 6XY) \frac{\partial}{\partial X} + (-9XY^3 + 5X^3 + 4Y^2) \frac{\partial}{\partial Y},
\]

In fact,

\[
\begin{pmatrix}
-6X^2Y^2 + 6XY \\
-9XY^3 + 5X^3 + 4Y^2
\end{pmatrix} \circ f_1 = \begin{pmatrix}
-6x^{10} + 6x^5 \\
-9x^{11} + 9x^6
\end{pmatrix} = \begin{pmatrix}
2x \\
3x^2
\end{pmatrix}
\]

\[
\begin{pmatrix}
-6X^2Y^2 + 6XY \\
-9XY^3 + 5X^3 + 4Y^2
\end{pmatrix} \circ f_2 = \begin{pmatrix}
-6x^{10} + 6x^5 \\
-4x^9 + 4x^4
\end{pmatrix} = \begin{pmatrix}
3x^2 \\
2x
\end{pmatrix}
\]

holds.

The following property is very fundamental and important.

Proposition 2.3. We assume $g = t \circ f \circ s$, which $t$ and $s$ are diffeomorphisms (that is, $f$ is $\mathcal{A}$-equivalent to $g$). Then,

\[
\xi \in \text{Lift}(f) \Rightarrow dt \circ \xi \circ t^{-1} \in \text{Lift}(g).
\]

This means that only diffeomorphism of the target of $f$ affects liftable vector fields of $g$. In addition, we will see that only that of the source affects lowerable vector fields in the proof.

Proof. There exists $\eta \in \theta_{S}(n)$ such that

\[
\xi \circ f = df \circ \eta.
\]

Then, the following holds:

\[
(dt \circ \xi \circ t^{-1}) \circ g = dt \circ (\xi \circ f) \circ s
= dt \circ (df \circ \eta) \circ s
= d(t \circ f) \circ \eta \circ s
= d(g \circ s^{-1}) \circ \eta \circ s
= dg \circ (ds^{-1} \circ \eta \circ s)
\]

Thus, $dt \circ \xi \circ t^{-1} \in \text{Lift}(g)$. \qed
We compare estimates of Theorem 1.1 and Theorem 1.2 using examples.

**Example 2.4.** When $n = 1$ and $p = 2$, by Theorem 1.1

$$N = 3rD + 2.$$  

For $f(x) = (x^2, x^3)$, since $r = 1$ and $D = 3$, we get $N = 11$. On the other hand, when $p = 2$ by Theorem 1.2

$$N = \left\lfloor A - \sqrt{2} \right\rfloor,$$

Since $A = 3$, we get $N = 2$. In fact, the highest degree of the following liftable vector fields are 1 and 2 respectively;

$$\left( \begin{array}{l} 2X \\ 3Y \end{array} \right), \left( \begin{array}{l} 2Y \\ 3X^2 \end{array} \right).$$

**Example 2.5.** For $f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^2, 0)$ given by $f_1(x) = (x^2, x^3), f_2(x) = (-x^3, x^2), f_3(x) = (x^2 - x^3, x^2 + x^3)$, since $r = 3$ and $D = 3$, by Theorem 1.1 $N = 29$. On the other hand, by Theorem 1.2 since $A = 9$, we get $N = 8$. In fact, the following vector field are liftable over $f$ and the highest degree is 6;

$$(-15X^6 - 45X^5Y - 45X^4Y^2 + 19X^3Y^3 + 4X^2Y^4 \\
-4X^5 - 64X^4Y + 45X^3Y^2 + 41X^2Y^3 + 57XY^4 \\
-7Y^5 + 4X^4 - 12X^3Y - 8X^2Y^2 + 52XY^3 \\
-14Y^4 + 8X^3 - 16X^2Y) \frac{\partial}{\partial X} \\
+( -10X^5Y - 30X^4Y^2 - 38X^3Y^3 + 18X^2Y^4 + 6XY^5 \\
+8X^5 - 8X^4Y - 46X^3Y^2 + 34X^2Y^3 + 24XY^4 \\
+56Y^5 - 4X^4 + 6X^3Y - 26X^2Y^2 - 10XY^3 \\
+28Y^4 + 12X^2Y - 20XY^2) \frac{\partial}{\partial Y}.$$

**Example 2.6.** When $n = 1$ and $p = 3$, by Theorem 1.1

$$N = \left\lfloor \sqrt{2(5rD + 1)} \right\rfloor.$$  

For $f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^3, 0)$ given by $f_1(x) = (x^3, x^4, x^5), f_2(x) = (x^4, x^5, x^3), f_3(x) = (x^5, x^3, x^4)$, we get $r = 3$ and $D = 5$. Therefore, $N = 13$. On the other hand, when $p = 3$ by Theorem 1.2

$$N = \left\lfloor 2\sqrt{A} - \frac{\sqrt{6}}{2} \right\rfloor.$$  

Since $A = 15$, we get $N = 6$. In fact, the following vector field are liftable over $f$ and the highest degree is 5;
\[(12X^3 + 18X^4 - 12X^2Y + 12X^3Y + 18X^4Y + 6X^2Y^2 - 6X^3Y^2 + 18XY^3 - 18X^2Y^3 + 6Y^4 + 18X^3Z - 6XYZ - 12X^2YZ - 18XY^2Z - 18X^2Y^2Z + 12Y^3Z + 6XY^3Z + 18Y^4Z - 6X^2Z^2 - 18X^3Z^2 - 18XYZ^2 - 6X^2YZ^2 - 18Y^2Z^2 + 12XZ^3 + 6X^2Z^3 + 6YZ^3 - 12Y^2Z^3 - 6Z^4 + 12XZ^4) \frac{\partial}{\partial X} + \]
\[(-8X^4 + 16X^2Y + 16X^3Y + 24X^4Y - 8XY^2 + 8X^2Y^2 + 16X^3Y^2 + 16XY^3 - 8Y^4 - 8X^2Z + 8XYZ + 16X^2YZ + 8X^3YZ - 8XY^2Z - 24X^2Y^2Z - 16Y^3Z - 16X^3Z - 24X^3Z^2 + 8YZ^2 - 16X^2Y^2Z - 8Y^2Z^2 - 16X^3Z^2 + 8Y^4Z^2 + 32YZ^3 - 8Z^4) \frac{\partial}{\partial Y} + \]
\[(10X^4Y + 10XY^2 + 10X^2Y^2 + 30X^3Y^2 + 10X^3Y^3 + 10X^2YZ + 10X^3YZ - 20X^2Y^2Z - 10X^3Z - 10Y^4Z - 10X^3Z^2 + 10Y^2Z^2 - 20XY^2Z^2 - 40X^2Y^2Z^2 - 40XY^2Z^2 - 10Y^3Z^2 - 10X^2Z^3 + 10Y^4Z^2 + 30YZ^3 + 20Z^4 + 10X^4 Z^4) \frac{\partial}{\partial Z}.\]

3 The sketch of proof of Theorem 1.2

We want non-negative integers \(N\) and \(N'_i (i = 1, 2, \ldots, r)\) such that we can find a coefficient vector
\[(a_1^{(0,0,\ldots,0)}, a_1^{(1,0,\ldots,0)}, \ldots, a_p^{(0,0,\ldots,N)}, a_{1,0}, a_{1,1}, \ldots, a_{r,N'}(i = 1, 2, \ldots, r)) \neq 0\]
such that for every \(i \in \{1, 2, \ldots, r\}\), the following polynomial equation with respect to the variable \(x_1\) holds:
\[
\left( \sum_{d=0}^{N} \left( \sum_{i_1 + \cdots + i_p = d} a_1^{(i_1, i_2, \ldots, i_p)} \prod_{h=1}^{p} X_i^{i_h} \right) \right) \circ f_i = \left( \sum_{d=0}^{N} \left( \sum_{i_1 + \cdots + i_p = d} a_p^{(i_1, i_2, \ldots, i_p)} \prod_{h=1}^{p} X_i^{i_h} \right) \right) \frac{\partial (X_1 \circ f_i)}{\partial x_1} = \left( \sum_{d=0}^{N} \left( \sum_{i_1 + \cdots + i_p = d} a_r^{(i_1, i_2, \ldots, i_p)} \prod_{h=1}^{p} X_i^{i_h} \right) \right) \frac{\partial (X_p \circ f_i)}{\partial x_1} = (a_{i_0} + a_{i_1} x_1 + \cdots + a_{i,N'} x_1^{N'_i}),
\]
where \(i_1, i_2, \ldots, i_p\) are non-negative integers. Note that for every \(i \in \{1, 2, \ldots, r\}\), the highest degree of the left-hand side is at most \(N \cdot D_i\) and that of the right-hand side is at most \(N'_i + D_i - 1\). By comparing the coefficients of the terms on the both sides, a system of linear equations with respect to
\[a_1^{(0,0,\ldots,0)}, a_1^{(1,0,\ldots,0)}, \ldots, a_p^{(0,0,\ldots,N)}, a_{1,0}, a_{1,1}, \ldots, a_{r,N'}(i = 1, 2, \ldots, r)\]
is obtained. Let the number of unknowns of the system of linear equations be denoted by \( U \) and the number of equations by \( E \). We assume that \( N \geq 1 \) to put
\[
N'_i = ND_i - D_i + 1.
\]
The number of combinations of non-negative integers \( i_1, \ldots, i_p \) such that \( i_1 + \cdots + i_p = d \) is
\[
\binom{d + p - 1}{d} = \frac{(d + p - 1)!}{d!(p - 1)!} = \frac{(d + p - 1) \cdots (d + 1)}{(p - 1)!}.
\]
Thus, we get
\[
U = p \sum_{d=0}^{N} \frac{(d + p - 1) \cdots (d + 1)}{(p - 1)!} + \sum_{i=1}^{r} (ND_i - D_i + 2)
\]
and
\[
E \leq p \sum_{i=1}^{r} (ND_i + 1).
\]
Here, the following formula is known.

**Proposition 3.1.** For a non-negative integer \( k \),
\[
\sum_{d=1}^{n} d(d + 1) \cdots (d + k) = \frac{n(n + 1) \cdots (n + k + 1)}{k + 2}.
\]

In addition, the following lemma holds.

**Lemma 3.2.** For \( p \in \mathbb{N}_{\geq 2} \) and \( x \in \mathbb{R}_{>0} \),
\[
\left( x + \sqrt[p]{p!} \right)^p < (x + 1)(x + 2) \cdots (x + p)
\]

Put
\[
A = \sum_{i=1}^{r} D_i.
\]
By Proposition 3.1 and Lemma 3.2, we get
\[
U - E > \frac{1}{(p - 1)!} \left( N + \sqrt[p]{p!} \right) \left\{ \left( N + \sqrt[p]{p!} \right)^{p-1} - A (p - 1) (p - 1)! \right\}.
\]
Thus, if we put
\[
N = \max \left\{ \left\lfloor \frac{p}{\sqrt{(Ap - A) (p - 1)!}} \right\rfloor, 1 \right\}
\]
and
\[
N'_i = ND_i - D_i + 1,
\]
then we can obtain \( U - E > 0 \). This completes the proof.
4 Liftable vector fields for non-singular mono-germs

A mono-germ \( f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0) \) is singular if \( \text{rank} Jf(0) < \min\{n, p\} \) holds, where \( Jf(0) \) is the Jacobian matrix of \( f \) at \( 0 \). A mono-germ \( f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0) \) is non-singular if \( f \) is not singular.

We identify the number of generators for \( \text{Lift}(f) \) for a non-singular mono-germ \( f \).

**Proposition 4.1.** Let \( f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0) \) be non-singular. Then, the number of generators for \( \text{Lift}(f) \) is

\[
\begin{cases}
  n + (p - n)(p - n) & (n < p) \\
  p & (n \geq p)
\end{cases}
\]

**Proof.** When \( n \geq p \) we know that \( f \) is \( \mathcal{A} \)-equivalent to the following form:

\[
g(x_1, \ldots, x_n) = (x_1, \ldots, x_p).
\]

Then, we can easily check that \( \text{Lift}(g) = C_0 \). Therefore, the number of generators for \( \text{Lift}(f) \) is \( p \).

When \( n < p \) we know that \( f \) is \( \mathcal{A} \)-equivalent to the following form:

\[
h(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).
\]

Then, we can check that the following vector fields belong to \( \text{Lift}(h) \):

\[
\frac{\partial}{\partial X_i} \partial (1 \leq i \leq n), \quad X_{n+i} \frac{\partial}{\partial X_{n+j}} (1 \leq i, j \leq p-n).
\]

Therefore, we know

\[
\left< \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, X_{n+1} \frac{\partial}{\partial X_{n+1}}, X_{n+2} \frac{\partial}{\partial X_{n+1}}, \ldots, X_p \frac{\partial}{\partial X_p} \right>_{C_0}
\]

is contained in \( \text{Lift}(h) \).

Conversely, for \( \xi = (\psi_1(X_1, X_2, \ldots, X_p), \ldots, \psi_p(X_1, X_2, \ldots, X_p)) \in \text{Lift}(h) \), since there exist smooth function germs \( \phi_i(x_1, x_2, \ldots, x_n)(i = 1, 2, \ldots, n) \) such that

\[
\psi_i(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) = \begin{cases}
  \phi_i(x_1, x_2, \ldots, x_n) & (1 \leq i \leq n) \\
  0 & (n + 1 \leq i \leq p)
\end{cases}
\]

when \( n + 1 \leq i \leq p \) there exist smooth function germs \( \tilde{\psi}_i(X_1, X_2, \ldots, X_p)(i = 1, 2, \ldots, p - n) \) such that

\[
\psi_i(X_1, X_2, \ldots, X_p) = \psi_i(X_1, X_2, \ldots, X_n, 0, 0, \ldots, 0) + \tilde{\psi}_1 X_{n+1} + \cdots + \tilde{\psi}_{p-n} X_p
\]

\[
= \tilde{\psi}_1 X_{n+1} + \cdots + \tilde{\psi}_{p-n} X_p.
\]
Therefore, $\xi = (\psi_1(X_1, X_2, \ldots, X_p), \cdots, \psi_p(X_1, X_2, \ldots, X_p)) \in \text{Lift}(h)$ belongs to
\[
\left\langle \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, X_{n+1} \frac{\partial}{\partial X_{n+1}}, X_{n+2} \frac{\partial}{\partial X_{n+2}}, \ldots, X_p \frac{\partial}{\partial X_p} \right\rangle_{C_0}.
\]
Thus, the number of generators for \text{Lift}(f) is $n + (p - n)(p - n)$. This completes the proof. $\square$

When $f$ is singular, characterization of the number of generators for \text{Lift}(f) is essentially difficult (see [8]).

5 Liftable vector fields for function mono-germs of one variable

We investigate \text{Lift}(f) for $n = p = 1$.

**Proposition 5.1.** Let a smooth function $f : (K, 0) \rightarrow (K, 0)$ be denoted by $f(x) = \tilde{f}(x)x^n$, where $\tilde{f} : (K, 0) \rightarrow K$ satisfies $\tilde{f}(0) \neq 0$ and $n$ is an integer greater than 1. Then, \text{Lift}(f) = \langle X \rangle_{C_0}.

**Proof.** Since $f(x) = \tilde{f}(x)x^n$, we can see easily
\[
f'(x) = \tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1}.
\]
At first we show $(\subset)$. Since $f(0) = f'(0) = 0$, for $\xi \in \text{Lift}(f)$, $\xi(0) = 0$ holds. Therefore, there exists a function $\psi(X)$ such that $\xi(X) = \psi(X)X$. This means \text{Lift}(f) $\subset$ $\langle X \rangle_{C_0}$.

Next, we show $(\supset)$. It is sufficient to show $X \in \text{Lift}(f)$. In fact,
\[
\tilde{f}(x)x^n = (\tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1}) \left( \frac{\tilde{f}(x)x}{\tilde{f}'(x)x + n\tilde{f}(x)} \right)
\]
holds. Thus, \text{Lift}(f) $\supset$ $\langle X \rangle_{C_0}$. This completes the proof. $\square$

This implies that there exist some cases that we can take non-zero polynomial vector fields liftable over $f$ even though $f$ is not a polynomial. Proposition 5.1 does not hold generally for a flat function $f$. For example, if
\[
f(x) = \begin{cases} 
    x^2e^{-1/x} & (x > 0) \\
    0 & (x = 0) \\
    x^2e^{1/x} & (x < 0)
\end{cases}
\]
than $X$ is not liftable. We prove this statement. we show that
\[
X \notin \text{Lift}(f).
\]
At first, since
\[
g(x) = \begin{cases} 
    e^{-1/x} & (x > 0) \\
    0 & (x \leq 0)
\end{cases}
\]
is a $C^\infty$ function, so is $g(-x)$. Therefore,

$$h(x) = \begin{cases} e^{-1/x} & (x > 0) \\ 0 & (x = 0) \\ e^{1/x} & (x < 0) \end{cases}$$

is a $C^\infty$ function. Thus, $f(x) = x^2 h(x)$ is a $C^\infty$ function.

We assume that $X \in \text{Lift}(f)$. Then, a lowerable vector field $\phi(x)$ for a liftable vector field $X$ must be

$$\phi(x) = \begin{cases} \frac{x^2}{2x+1} & (x > 0) \\ a & (x = 0) \\ \frac{x^2}{2x-1} & (x < 0) \end{cases}$$

($a \in \mathbb{R}$). However, $\phi(x)$ is not class of $C^\infty$. Thus, $X \not\in \text{Lift}(f)$.

On the other hand, we can show that $X^2$ is liftable. In fact, we can give a lowerable vector field $\phi(x)$ as follows:

$$\phi(x) = \begin{cases} \frac{x^4e^{-1/x}}{2x+1} & (x > 0) \\ 0 & (x = 0) \\ \frac{x^4e^{1/x}}{2x-1} & (x < 0) \end{cases}$$

It can be seen easily that $\phi(x)$ is class of $C^\infty$. Thus, $X^2 \in \text{Lift}(f)$.

The author does not know examples of $f$ such that there exist no polynomial vector fields liftable over $f$.

References


