<table>
<thead>
<tr>
<th>Title</th>
<th>Notes on liftable vector fields (Pursuit of the Essence of Singularity Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Mizota, Yusuke</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1868: 63-73</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195420">http://hdl.handle.net/2433/195420</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Notes on liftable vector fields

Yusuke Mizota*

Abstract

We introduce the author’s research about an estimate for the highest degrees of liftable vector fields and the module of liftable vector fields for non-singular mono-germs and function mono-germs of one variable.

1 Introduction

In this paper, we introduce the author’s research about liftable vector fields. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. In this paper, suppose that all mappings are smooth (that is, of class $C^\infty$ if $\mathbb{K} = \mathbb{R}$ or holomorphic if $\mathbb{K} = \mathbb{C}$).

The notion of liftable vector fields was introduced by Arnol’d [1] for studying bifurcations of wave front singularities. As results and applications of liftable vector fields, Bruce and West [2] obtained diffeomorphisms preserving a crosscap to classify functions on it, and Houston and Littlestone [4] obtained generators for the module of vector fields liftable over the generalized cross cap to find $A_\epsilon$-codimension 1 maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$. Houston and Atique [3] classified $\nu K$-codimension 2 maps on the generalized crosscap to apply to a classification of $A_\epsilon$-codimension 2 maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$. Nishimura [8] characterized the minimal number of generators for the module of vector fields liftable over a finitely determined multigerm of corank at most one satisfying a special condition when $n \leq p$.

In previous work [6], the author showed that we can find polynomial vector fields liftable over $f$ if $f$ is a polynomial multigerm and gave an estimate for the highest degrees of liftable vector fields. The highest degree of polynomial vector field $\xi$ means maximum of that of component functions of $\xi$. Let $[x]$ be the greatest integer not exceeding $x$. Lift($f$) denotes the module of vector fields liftable over $f$. We proved the following theorem.

Theorem 1.1 ([6]). Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0) \ (n < p)$ be a polynomial multigerm. Then, there exists a non-zero polynomial vector field in Lift($f$) such that...

*Research Fellow DC2 of Japan Society for the Promotion of Science

The author was supported by JSPS and CAPES under the Japan-Brazil Research Cooperative Program.

Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, JAPAN.

e-mail: y-mizota@math.kyushu-u.ac.jp
the highest degree is at most

\[ N = \left\lfloor p^{-\frac{1}{2}}(\alpha + 1)(p - 1)! \right\rfloor + 1, \]

where

\[ \alpha = r \left( \frac{p \cdot 2^n - n}{n!} \right)(D + n - 1)^n, \quad r = |S|, \]

\[ D = \max\{D_i | i \in \{1, 2, \ldots, r\}\}, \quad D_i = \max\{\deg(X_j \circ f_i) | j \in \{1, 2, \ldots, p\}\}. \]

The proof of Theorem 1.1 also gives a method to find a non-zero element of \( \text{Lift}(f) \). However, we can usually take values of \( N \) that are much lower than those calculated in Theorem 1.1. Therefore, we needed to improve this estimate. In [7], a better estimate for the highest degrees of liftable vector fields was discovered when \( n = 1 \). It is the following result. Let \( \lceil x \rceil \) be the smallest integer greater than or equal to \( x \).

**Theorem 1.2 ([7]).** Let \( f : (K, S) \to (K^p, 0) \) \((p \geq 2)\) be a polynomial multigerm which contains no branch of zero map. Then there exist a non-zero polynomial vector field of \( \text{Lift}(f) \) such that the highest degree is at most

\[ N = \max \left\{ \left\lfloor p^{-\frac{1}{2}}(Ap - A)(p - 1)! - \sqrt[p]{p!} \right\rfloor, 1 \right\}, \]

and the highest degree of a corresponding lowerable vector field for \( f_i \) is at most

\[ D_iN - D_i + 1, \]

where

\[ A = \sum_{i=1}^{r} D_i, \quad D_i = \max\{\deg(X_j \circ f_i) | j \in \{1, \ldots, p\}\}. \]

This paper is organized as follows. In Section 2, we explain various definitions, basic facts and examples implying difference of estimates between Theorem 1.1 and Theorem 1.2. In Section 3 the sketch of proof of Theorem 1.2 is described. In Section 4 and 5, topics about the module of liftable vector fields are given. The number of generators for the module of vector fields liftable over a non-singular mono-germ is identified in Section 4. Theorem 1.1 and Theorem 1.2 claims that there exist non-zero polynomial vector fields liftable over \( f \) when \( f \) is a polynomial. It is natural to ask whether there exist non-zero polynomial liftable vector fields when \( f \) is not a polynomial. In Section 5 we investigate the module of vector fields liftable over a function germ of one variable and also consider this problem.

## 2 Preliminary

Let \( S \) be a subset of \( K^n \). A map germ \( f : (K^n, S) \to (K^p, 0) \) is called a multigerm. If \( S \) is a singleton, \( f \) is called a mono-germ. Let \( C_S \) (resp., \( C_0 \)) be the set of
function germs \( (\mathbb{K}^n, S) \to \mathbb{K} \) (resp., \( (\mathbb{K}^p, 0) \to \mathbb{K} \)), and let \( m_S \) (resp., \( m_0 \)) be the subset of \( C_S \) (resp., \( C_0 \)) consisting of function germs \( (\mathbb{K}^n, S) \to (\mathbb{K}, 0) \) (resp., \( (\mathbb{K}^p, 0) \to (\mathbb{K}, 0) \)). The sets \( C_S \) and \( C_0 \) have natural \( \mathbb{K} \)-algebra structures. A multigerm \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) can be defined by \( (f_1, f_2, \ldots, f_r) \), where \( f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0) \). Each \( f_i \) is called a branch. In this paper, for a multigerm \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) defined by \( (f_1, f_2, \ldots, f_r) \) with \( f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0) \), we consider \( S \) to be a set consisting of \( r \) distinct points.

For a map germ \( f : (\mathbb{K}^n, S) \to \mathbb{K}^p \), let \( \theta_S(f) \) be the set of germs of vector fields along \( f \). The set \( \theta_S(f) \) has a natural \( C_S \)-module structure and is identified with the direct sum of \( p \) copies of \( C_S \). Put \( \theta_S(n) = \theta_S(\text{id}_{(\mathbb{K}^n, S)}) \) and \( \theta_S(p) = \theta_S(\text{id}_{(\mathbb{K}^p, 0)}) \), where \( \text{id}_{(\mathbb{K}^n, S)} \) (resp., \( \text{id}_{(\mathbb{K}^p, 0)} \)) is the germ of the identity mapping of \( (\mathbb{K}^n, S) \) (resp., \( (\mathbb{K}^p, 0) \)). For a multigerm \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \), following Mather [5], we define \( tf \) and \( \omega f \) as

\[
    tf : \theta_S(n) \to \theta_S(f), \quad tf(\eta) = df \circ \eta,
\]

\[
    \omega f : \theta_0(p) \to \theta_S(f), \quad \omega f(\xi) = \xi \circ f,
\]

where \( df \) is the differential of \( f \). Following Wall [9], we put \( TR_e(f) = tf(\theta_S(n)) \) and \( TL_e(f) = \omega f(\theta_0(p)) \).

For a multigerm \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \), a vector field \( \xi \in \theta_0(p) \) is said to be liftable over \( f \) if \( \xi \circ f \in \mathcal{R}_e(f) \cap TL_e(f) \). The set of vector fields liftable over \( f \) is denoted by \( \text{Lift}(f) \). Note that \( \text{Lift}(f) \) has a natural \( C_0 \)-module structure. Let \( (x_1, x_2, \ldots, x_n) \) (resp., \( (X_1, X_2, X_3) \)) be the standard local coordinates of \( \mathbb{K}^n \) (resp., \( \mathbb{K}^p \)) at the origin. Sometimes \( (x_1, x_2) \) (resp., \( (X_1, X_2) \)) is denoted by \( (x, y) \) (resp., \( (X, Y) \)) and \( (x_1, x_2, x_3) \) (resp., \( (X_1, X_2, X_3) \)) is denoted by \( (x, y, z) \) (resp., \( (X, Y, Z) \)). We see easily that

\[
    \xi = (\psi_1(X_1, X_2, \ldots, X_3), \ldots, \psi_p(X_1, X_2, \ldots, X_3)) \in \text{Lift}(f),
\]

where \( \psi_j : (\mathbb{K}^p, 0) \to \mathbb{K} \) (\( j = 1, 2, \ldots, p \)), if and only if for every \( i \in \{1, \ldots, r\} \) there exist a vector field

\[
    \eta_i = (\phi_{i,1}(x_1, x_2, \ldots, x_n), \ldots, \phi_{i,n}(x_1, x_2, \ldots, x_n)),
\]

where \( \phi_{i,k} : (\mathbb{K}^n, 0) \to \mathbb{K} \) (\( k = 1, 2, \ldots, n \)), such that \( \xi \circ f = df \circ \eta_i \) i. e.

\[
    \begin{pmatrix}
        \psi_1(X_1, X_2, \ldots, X_3) \\
        \vdots \\
        \psi_p(X_1, X_2, \ldots, X_3)
    \end{pmatrix}
    \circ f_i(x_1, x_2, \ldots, x_n)
    =
    \begin{pmatrix}
        \frac{\partial(X_1 \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_1 \circ f_i)}{\partial x_n} \\
        \vdots & \ddots & \vdots \\
        \frac{\partial(X_p \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_p \circ f_i)}{\partial x_n}
    \end{pmatrix}
    \begin{pmatrix}
        \phi_{i,1}(x_1, x_2, \ldots, x_n) \\
        \vdots \\
        \phi_{i,n}(x_1, x_2, \ldots, x_n)
    \end{pmatrix}.
\]

We call this \( \eta_i \) a lowerable vector field for \( f_i \) corresponding to \( \xi \).
Example 2.1. Let $f : (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be given by $f(x) = (x^2, x^3)$. Then, it can be seen easily that the following vector fields are liftable over $f$:

$$
\begin{pmatrix}
2X \\
3Y
\end{pmatrix}, \begin{pmatrix}
2Y \\
3X^2
\end{pmatrix}.
$$

The forms of vector fields liftable over a polynomial multigerm are complicated generally.

Example 2.2. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (x^3, x^2)$. Then, it is known [8] that the following vector fields are liftable over $f$:

$$
(-6X^2Y^2 + 6XY) \frac{\partial}{\partial X} + (-9XY^3 + 5X^3 + 4Y^2) \frac{\partial}{\partial Y},
$$

In fact,

$$
\begin{pmatrix}
-6X^2Y^2 + 6XY \\
-9XY^3 + 5X^3 + 4Y^2
\end{pmatrix} \circ f_1 = \begin{pmatrix}
-6x^{10} + 6x^5 \\
-9x^{11} + 9x^6
\end{pmatrix} = \begin{pmatrix}
2x \\
3x^2
\end{pmatrix} \begin{pmatrix}
-3x^9 + 3x^4
\end{pmatrix},
$$

$$
\begin{pmatrix}
-6X^2Y^2 + 6XY \\
-9XY^3 + 5X^3 + 4Y^2
\end{pmatrix} \circ f_2 = \begin{pmatrix}
-6x^{10} + 6x^5 \\
-4x^9 + 4x^4
\end{pmatrix} = \begin{pmatrix}
3x^2 \\
2x
\end{pmatrix} \begin{pmatrix}
-2x^8 + 2x^3
\end{pmatrix}
$$

holds.

The following property is very fundamental and important.

Proposition 2.3. We assume $g = t \circ f \circ s$, which $t$ and $s$ are diffeomorphisms (that is, $f$ is $A$-equivalent to $g$). Then,

$$
\xi \in \operatorname{Lift}(f) \Rightarrow dt \circ \xi \circ t^{-1} \in \operatorname{Lift}(g).
$$

This means that only diffeomorphism of the target of $f$ affects liftable vector fields of $g$. In addition, we will see that only that of the source affects lowerable vector fields in the proof.

Proof. There exists $\eta \in \theta_S(n)$ such that

$$
\xi \circ f = df \circ \eta.
$$

Then, the following holds:

$$
(dt \circ \xi \circ t^{-1}) \circ g = dt \circ \xi \circ t^{-1} \circ (t \circ f \circ s) = dt \circ (\xi \circ f) \circ s = dt \circ (df \circ \eta) \circ s = d(t \circ f) \circ \eta \circ s = d(g \circ s^{-1}) \circ \eta \circ s = dg \circ (ds^{-1} \circ \eta \circ s)
$$

Thus, $dt \circ \xi \circ t^{-1} \in \operatorname{Lift}(g)$. 

$\square$
We compare estimates of Theorem 1.1 and Theorem 1.2 using examples.

**Example 2.4.** When $n = 1$ and $p = 2$, by Theorem 1.1

$$N = 3rD + 2.$$  

For $f(x) = (x^2, x^3)$, since $r = 1$ and $D = 3$, we get $N = 11$. On the other hand, when $p = 2$ by Theorem 1.2

$$N = \left\lfloor A - \sqrt{2} \right\rfloor,$$

Since $A = 3$, we get $N = 2$. In fact, the highest degree of the following liftable vector fields are 1 and 2 respectively;

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix}, \begin{pmatrix} 2Y \\ 3X^2 \end{pmatrix}.$$

**Example 2.5.** For $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ given by $f_1(x) = (x^2, x^3), f_2(x) = (-x^3, x^2), f_3(x) = (x^2 - x^3, x^2 + x^3)$, since $r = 3$ and $D = 3$, by Theorem 1.1 $N = 29$. On the other hand, by Theorem 1.2 since $A = 9$, we get $N = 8$. In fact, the following vector field are liftable over $f$ and the highest degree is 6;

$$(-15X^6 - 45X^5Y - 45X^4Y^2 + 19X^3Y^3 + 4X^2Y^4$$
$$ -4X^5 - 64X^4Y + 45X^3Y^2 + 41X^2Y^3 + 57XY^4$$
$$ -7Y^5 + 4X^4 - 12X^3Y - 8X^2Y^2 + 52XY^3$$
$$ -14Y^4 + 8X^3 - 16X^2Y) \frac{\partial}{\partial X}$$
$$ +(-10X^5Y - 30X^4Y^2 - 38X^3Y^3 + 18X^2Y^4 + 6XY^5$$
$$ + 8X^5 - 8X^4Y - 46X^3Y^2 + 34X^2Y^3 + 24XY^4$$
$$ + 56Y^5 - 4X^4 + 6X^3Y - 26X^2Y^2 - 10XY^3$$
$$ + 28Y^4 + 12X^2Y - 20XY^2) \frac{\partial}{\partial Y}.$$

**Example 2.6.** When $n = 1$ and $p = 3$, by Theorem 1.1

$$N = \left\lfloor \sqrt{2(5rD + 1)} \right\rfloor.$$

For $f : (\mathbb{K}, S) \to (\mathbb{K}^3, 0)$ given by $f_1(x) = (x^3, x^4, x^5), f_2(x) = (x^4, x^5, x^3), f_3(x) = (x^5, x^3, x^4)$, we get $r = 3$ and $D = 5$. Therefore, $N = 13$. On the other hand, when $p = 3$ by Theorem 1.2

$$N = \left\lfloor 2\sqrt{A} - \sqrt[3]{6} \right\rfloor.$$

Since $A = 15$, we get $N = 6$. In fact, the following vector field are liftable over $f$ and the highest degree is 5;
\[(12X^3 + 18X^4 - 12X^2Y + 12X^3Y + 18X^4Y + 6X^2Y^2 - 6X^3Y^2 + 18X^4Y^2 - 6X^2Y^3 - 18X^3Y^3 + 6Y^4 + 18X^3Z - 6XYZ - 12X^2YZ - 6X^3YZ - 18YZ^2 + 12XZ^3 + 6X^2Z^3 + 6YZ^3 - 12Y^2Z^3 - 6Z^4 + 12XZ^4) \frac{\partial}{\partial X} +
\cdot
(-8X^4 + 16X^2Y + 16X^3Y + 24X^4Y - 8XY^2 + 8X^2Y^2 + 16X^3Y^2 + 8XYZ + 16X^2YZ + 8X^3YZ - 8XY^2Z - 24X^2Y^2Z - 16X^3Z - 16XY^3Z - 24X^3Z^2 + 8YZ^2 - 16X^2YZ^2 - 16XZ^3 + 8YZ^3 + 32X^2Y^3 - 8Y^2Z^2 - 16XZ^3 + 6X^2Z^3 + 6YZ^3 - 12Y^2Z^3 - 6Z^4 + 12XZ^4) \frac{\partial}{\partial Y} +
(10X^4Y + 10XY^2 + 10X^2Y^2 + 30X^3Y^2 + 10XY^3 + 10X^2Y^3 + 20X^3Z - 20XY^2Z + 10X^2YZ + 10X^3YZ - 20XY^2Z - 10X^2YZ - 20Y^2Z^2 - 30XY^2Z^2 - 10Y^3Z^2 - 30X^2Y^3Z^2 + 10XZ^4 + 10X^2Z^4 + 10X^4Y) \frac{\partial}{\partial Z}.\]

\section{The sketch of proof of Theorem 1.2}

We want non-negative integers \(N\) and \(N'_i\) \((i = 1, 2, \ldots, r)\) such that we can find a coefficient vector

\[
(a_1^{(0,0,\ldots,0)}, a_1^{(1,0,\ldots,0)}, \ldots, a_p^{(0,0,\ldots,N)}, a_{1,0}, a_{1,1}, \ldots, a_{r,N'_r}) \neq 0
\]

such that for every \(i \in \{1, 2, \ldots, r\}\), the following polynomial equation with respect to the variable \(x_1\) holds:

\[
\begin{pmatrix}
\sum_{d=0}^{N} \left( \sum_{i_1+i_2+\cdots+i_p=d} a_1^{(i_1,i_2,\ldots,i_p)} \prod_{h=1}^{p} X_h^{i_h} \right) f_i \\
\vdots \\
\sum_{d=0}^{N} \left( \sum_{i_1+i_2+\cdots+i_p=d} a_p^{(i_1,i_2,\ldots,i_p)} \prod_{h=1}^{p} X_h^{i_h} \right) f_i \\
\frac{\partial(X_1 \circ f_i)}{\partial x_1} \\
\vdots \\
\frac{\partial(X_p \circ f_i)}{\partial x_1}
\end{pmatrix}
= (a_{i,0} + a_{i,1}x_1 + \cdots + a_{i,N'_i}x_1^{N'_i}),
\]

where \(i_1, i_2, \ldots, i_p\) are non-negative integers. Note that for every \(i \in \{1, 2, \ldots, r\}\), the highest degree of the left-hand side is at most \(N \cdot D_i\) and that of the right-hand side is at most \(N'_i + D_i - 1\). By comparing the coefficients of the terms on the both sides, a system of linear equations with respect to

\[
a_1^{(0,0,\ldots,0)}, a_1^{(1,0,\ldots,0)}, \ldots, a_p^{(0,0,\ldots,N)}, a_{1,0}, a_{1,1}, \ldots, a_{r,N'_r}
\]
is obtained. Let the number of unknowns of the system of linear equations be denoted by $U$ and the number of equations by $E$. We assume that $N \geq 1$ to put

$$N'_i = ND_i - D_i + 1.$$ 

The number of combinations of non-negative integers $i_1, \ldots, i_p$ such that $i_1 + \cdots + i_p = d$ is

$$\binom{d + p - 1}{d} = \frac{(d + p - 1)!}{d!(p - 1)!} = \frac{(d + p - 1) \cdots (d + 1)}{(p - 1)!}.$$ 

Thus, we get

$$U = p \sum_{d=0}^{N} \frac{(d + p - 1) \cdots (d + 1)}{(p - 1)!} + \sum_{i=1}^{r} (ND_i - D_i + 2)$$ 

and

$$E \leq p \sum_{i=1}^{r} (ND_i + 1).$$ 

Here, the following formula is known.

**Proposition 3.1.** For a non-negative integer $k$,

$$\sum_{d=1}^{n} d(d+1) \cdots (d+k) = \frac{n(n+1) \cdots (n+k+1)}{k+2}.$$ 

In addition, the following lemma holds.

**Lemma 3.2.** For $p \in \mathbb{N}_{\geq 2}$ and $x \in \mathbb{R}_{>0}$,

$$(x + \sqrt[p]{p!})^p < (x + 1)(x + 2) \cdots (x + p)$$

Put

$$A = \sum_{i=1}^{r} D_i.$$ 

By Proposition 3.1 and Lemma 3.2, we get

$$U - E > \frac{1}{(p - 1)!} \left( N + \sqrt[p]{p!} \right) \left\{ \left( N + \sqrt[p]{p!} \right)^{p-1} - A(p - 1)(p - 1)! \right\}.$$ 

Thus, if we put

$$N = \max \left\{ \left\lfloor \sqrt[p]{(Ap - A)(p - 1)!} - \sqrt[p]{p!} \right\rfloor, 1 \right\}$$

and

$$N'_i = ND_i - D_i + 1,$$

then we can obtain $U - E > 0$. This completes the proof.
4 Liftable vector fields for non-singular monogerms

A mono-germ $f: (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ is singular if rank$Jf(0) < \min\{n, p\}$ holds, where $Jf(0)$ is the Jacobian matrix of $f$ at $0$. A mono-germ $f: (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ is non-singular if $f$ is not singular.

We identify the number of generators for Lift($f$) for a non-singular mono-germ $f$.

**Proposition 4.1.** Let $f: (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ be non-singular. Then, the number of generators for Lift($f$) is

$$\begin{cases} n + (p - n)(p - n) & (n < p) \\
p & (n \geq p) \end{cases}$$

**Proof.** When $n \geq p$ we know that $f$ is $\mathcal{A}$-equivalent to the following form:

$$g(x_1, \ldots, x_n) = (x_1, \ldots, x_p).$$

Then, we can easily check that Lift($g$) = $C_0$. Therefore, the number of generators for Lift($f$) is $p$.

When $n < p$ we know that $f$ is $\mathcal{A}$-equivalent to the following form:

$$h(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

Then, we can check that the following vector fields belong to Lift($h$):

$$\frac{\partial}{\partial X_i} (1 \leq i \leq n)$$

$$X_{n+i} \frac{\partial}{\partial X_{n+j}} (1 \leq i, j \leq p - n).$$

Therefore, we know

$$\left\langle \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, X_{n+1} \frac{\partial}{\partial X_{n+1}}, X_{n+2} \frac{\partial}{\partial X_{n+1}}, \ldots, X_p \frac{\partial}{\partial X_p} \right\rangle_{C_0}$$

is contained in Lift($h$).

Conversely, for $\xi = (\psi_1(X_1, X_2, \ldots, X_p), \cdots, \psi_p(X_1, X_2, \ldots, X_p)) \in$ Lift($h$), since there exist smooth function germs $\phi_i(x_1, x_2, \ldots, x_n)(i = 1, 2, \ldots, n)$ such that

$$\psi_i(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) = \begin{cases} \phi_i(x_1, x_2, \ldots, x_n) & (1 \leq i \leq n) \\
0 & (n + 1 \leq i \leq p) \end{cases},$$

when $n + 1 \leq i \leq p$ there exist smooth function germs $\tilde{\psi}_i(X_1, X_2, \ldots, X_p)(i = 1, 2, \ldots, p - n)$ such that

$$\psi_i(X_1, X_2, \ldots, X_p) = \psi_i(X_1, X_2, \ldots, X_n, 0, 0, \ldots, 0) + \tilde{\psi}_1 X_{n+1} + \cdots + \tilde{\psi}_{p-n} X_p$$

$$= \tilde{\psi}_1 X_{n+1} + \cdots + \tilde{\psi}_{p-n} X_p.$$
Therefore, \( \xi = (\psi_1(X_1, X_2, \ldots, X_p), \cdots, \psi_p(X_1, X_2, \ldots, X_p)) \in \text{Lift}(h) \) belongs to 
\[
\left< \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, X_{n+1}\frac{\partial}{\partial X_{n+1}}, X_{n}2\frac{\partial}{\partial X_{n+1}}, \ldots X_{p}\frac{\partial}{\partial X_{p}} \right>_{C_0}.
\]
Thus, the number of generators for \( \text{Lift}(f) \) is \( n + (p-n)(p-n) \). This completes the proof.

When \( f \) is singular, characterization of the number of generators for \( \text{Lift}(f) \) is essentially difficult (see [8]).

5 \ Liftable vector fields for function mono-germs of one variable

We investigate \( \text{Lift}(f) \) for \( n = p = 1 \).

**Proposition 5.1.** Let a smooth function \( f : (\mathbb{K}, 0) \to (\mathbb{K}, 0) \) be denoted by 
\( f(x) = \tilde{f}(x)x^n \), where \( \tilde{f} : (\mathbb{K}, 0) \to \mathbb{K} \) satisfies \( \tilde{f}(0) \neq 0 \) and \( n \) is an integer greater than 1. Then, \( \text{Lift}(f) = \langle X \rangle_{C_0} \).

**Proof.** Since \( f(x) = \tilde{f}(x)x^n \), we can see easily 
\[
f'(x) = \tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1}.
\]
At first we show \((\subseteq)\). Since \( f(0) = f'(0) = 0 \), for \( \xi \in \text{Lift}(f) \) \( \xi(0) = 0 \) holds. Therefore, there exists a function \( \psi(X) \) such that \( \xi(X) = \psi(X)X \). This means \( \text{Lift}(f) \subseteq \langle X \rangle_{C_0} \).

Next, we show \((\supseteq)\). It is sufficient to show \( X \in \text{Lift}(f) \). In fact,
\[
\tilde{f}(x)x^n = (\tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1}) \left( \frac{\tilde{f}(x)x}{\tilde{f}'(x)x + n\tilde{f}(x)} \right)
\]
holds. Thus, \( \text{Lift}(f) \supset \langle X \rangle_{C_0} \). This completes the proof. \( \square \)

This implies that there exist some cases that we can take non-zero polynomial vector fields liftable over \( f \) even though \( f \) is not a polynomial. Proposition 5.1 does not hold generally for a flat function \( f \). For example, if
\[
f(x) = \begin{cases} 
x^2e^{-1/x} & (x > 0) 
0 & (x = 0) 
x^2e^{1/x} & (x < 0)
\end{cases}
\]
then \( X \) is not liftable. We prove this statement. we show that
\[
X \notin \text{Lift}(f).
\]

At first, since
\[
g(x) = \begin{cases} 
e^{-1/x} & (x > 0) 
0 & (x \leq 0)
\end{cases}
\]
is a \( C^\infty \) function, so is \( g(-x) \). Therefore,

\[
h(x) = \begin{cases} 
  e^{-1/x} & (x > 0) \\
  0 & (x = 0) \\
  e^{1/x} & (x < 0)
\end{cases}
\]

is a \( C^\infty \) function. Thus, \( f(x) = x^2 h(x) \) is a \( C^\infty \) function.

We assume that \( X \in \text{Lift}(f) \). Then, a lowerable vector field \( \phi(x) \) for a liftable vector field \( X \) must be

\[
\phi(x) = \begin{cases} 
  \frac{x^2}{2x+1} & (x > 0) \\
  a & (x = 0) \\
  \frac{x^2}{2x-1} & (x < 0)
\end{cases}
(a \in \mathbb{R}).
\]

However, \( \phi(x) \) is not class of \( C^\infty \). Thus, \( X \not\in \text{Lift}(f) \).

On the other hand, we can show that \( X^2 \) is liftable. In fact, we can give a lowerable vector field \( \phi(x) \) as follows:

\[
\phi(x) = \begin{cases} 
  \frac{x^4e^{-1/x}}{2x+1} & (x > 0) \\
  0 & (x = 0) \\
  \frac{x^4e^{1/x}}{2x-1} & (x < 0)
\end{cases}
\]

It can be seen easily that \( \phi(x) \) is class of \( C^\infty \). Thus, \( X^2 \in \text{Lift}(f) \).

The author does not know examples of \( f \) such that there exist no polynomial vector fields liftable over \( f \).

References


