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Kyoto University
(1, 1)-BRIDGE SPLITTINGS WITH DISTANCE EXACTLY $n$

AYAKO IDO, YEONHEE JANG AND TSUYOSHI KOBAYASHI

1. INTRODUCTION


On the other hand, the above concept and results have been extended to bridge splittings for links in closed 3-manifolds (for definitions, see subsection 2.3), and have been studied by several authors. For example, Saito [11] showed that for any closed 3-manifold admitting a Heegaard splitting of genus one, there is a knot in the manifold with a (1, 1)-bridge splitting of arbitrary high distance. Recently, Blair, Tomova and Yoshizawa [2] showed that for given integers $b$, $c$, $g$, and $n$, there exists a manifold $M$ containing a $c$-component link $L$ so that $(M, L)$ admits a $(g, b)$-bridge splitting of distance at least $n$. Moreover, Ichihara and Saito [7] showed that for any given closed orientable 3-manifold $M$ with a Heegaard surface of genus $g$, and for any positive integers $b$ and $n$, there exists a knot $K$ in $M$ which admits a $(g, b)$-bridge splitting of distance greater than $n$.

In [6], we showed that there exists a Heegaard splitting of a closed orientable 3-manifold with distance exactly $n$ for each positive integer $n$. To prove this, we gave a method to extend a geodesic in the curve complex of a closed orientable surface to a geodesic with given length, and constructed a concrete example (for details, see [6, Section 4]).

In this paper, we apply the idea of [6, Section 4] to construct a geodesic of any given length in the curve complex of a twice-punctured torus, and show the following.

Theorem 1.1. For any integer $n > 0$, there exists a (1, 1)-bridge splitting with distance exactly $n$.

2. DEFINITIONS AND NOTATIONS

2.1. Curve complexes. Let $S$ be an orientable surface with genus $g$, $b$ boundary components and $p$ punctures. A simple closed curve in $S$ is essential if it does not bound a disk or a once-punctured disk in $S$ and is not parallel to a component of $\partial S$. An arc properly embedded in $S$ is essential if it does not co-bound a disk in $S$ together with an arc on $\partial S$. We say that $S$ is sporadic if $g = 0, b + p \leq 4$ or $g = 1, b + p \leq 1$.

Except in sporadic cases, the curve complex $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on $S$, and a collection of $k + 1$ vertices forms a $k$-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in $S$. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. We assume that $S$ is a torus, a torus with one boundary component, or a sphere with 4 boundary components since, otherwise, there are no essential simple closed curves in
$S$. When $S$ is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of $k + 1$ vertices forms a $k$-simplex of $C(S)$ if they can be realized by curves in $S$ which mutually intersect exactly once (resp. twice). The arc-and-curve complex $AC(S)$ is defined similarly, as follows: each vertex of $AC(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on $S$, and a collection of $k + 1$ vertices forms a $k$-simplex of $AC(S)$ if they can be realized by disjoint arcs or simple closed curves in $S$.

We can define the distance between two vertices in the curve complex $C(S)$ to be the minimal number of 1-simplices of a simplicial path in $C(S)$ joining the two vertices. We denote by $d_{C(S)}(x, y)$, or $d_{S}(x, y)$ in brief, the distance in $C(S)$ between the vertices $x$ and $y$. For subsets $X$ and $Y$ of the vertices of $C(S)$, we define $\text{diam}_{S}(X, Y) = \text{diam}_{S}(X \cup Y)$. Similarly, we can define the distance $d_{AC(S)}(x, y)$ and $\text{diam}_{AC(S)}(X, Y)$. We denote by $[a_0, a_1, \ldots, a_n]$ the path in $C(S)$ with vertices $a_0, a_1, \ldots, a_n$ such that $a_i \cap a_{i+1} = \emptyset$ ($i = 0, 1, \ldots, n - 1$). We call a path $[a_0, a_1, \ldots, a_n]$ a geodesic if $n = d_{s}(a_0, a_n)$.

2.2. Subsurface projections. Let $P(Y)$ denote the power set of a set $Y$. Suppose that $X$ is an essential subsurface of $S$ that contains an essential simple closed curve. We call the composition $\pi_{0} \circ \pi_{A}$ of maps $\pi_{A}: C^{0}(S) \rightarrow P(AC^{0}(X))$ and $\pi_{0}: P(AC^{0}(X)) \rightarrow P(C^{0}(X))$ a subsurface projection if they satisfy the following (see Figure 1): for a vertex $\alpha$, take a representative $\alpha$ so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_{A}(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,

- $\pi_{0}(\{\alpha_{1}, \ldots, \alpha_{n}\})$ is the union for all $i = 1, \ldots, n$ of the set of all isotopy classes of the components of $\partial N(\alpha_{i} \cup \partial X)$ which are essential in $X$, where $N(\alpha_{i} \cup \partial X)$ is a regular neighborhood of $\alpha_{i} \cup \partial X$ in $X$.

2.3. $(g, b)$-bridge splittings. Let $H$ be a genus-$g$ $(\geq 0)$ handlebody. We say that a set of $n$ arcs $\{t_{1}, \ldots, t_{n}\}$ properly embedded in $H$ is a set of trivial $n$ arcs if $t_{1} \cup \cdots \cup t_{n}$ is parallel to $\partial H$. Let $H$ be a handlebody and $\tau = \{t_{1}, \ldots, t_{n}\}$ a set of trivial $n$ arcs in $H$. Then $\tau$ can be isotoped in $H$ so that the projection from $\partial H \times [0, 1]$ to $[0, 1]$ has exactly one critical point in each $t_{i}$.

It is well known that every closed orientable 3-manifold $M$ has a genus-$g$ Heegaard splitting for some $g \geq 0$, i.e., $M = V_{1} \cup_{F} V_{2}$, where $V_{1}$ and $V_{2}$ are genus-$g$ handlebodies.
such that $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = P$. Let $L$ be a link in $M$. We say that $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is a $(g, b)$-bridge splitting (or $(g, b)$-splitting for short) for the pair $(M, L)$ if $P$ separates $(M, L)$ into two components $(V_1, \tau_1)$ and $(V_2, \tau_2)$ where $\tau_1 = L \cap V_1$ (resp. $\tau_2 = L \cap V_2$) is a set of trivial $b$ arcs in $A$ (resp. $B$). Then we say that $P$ is a $(g, b)$-bridge surface (or a bridge surface for short). It is known that each $(M, L)$ has a $(g, b)$-bridge splitting for some $g$ and $b$. (For a detailed discussion, see [5, Lemma 2.1]).

For $i = 1$ or $2$, $D(V_i)$ denotes the subset of $C^0(\partial V_i - \tau_i)$ consisting of the vertices with representatives bounding disks in $V_i - \tau_i$. Then the (Hempel) distance of $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is defined by $d_P(D(V_1), D(V_2))$, where $P' = \partial V_1 - \tau_1 = \partial V_2 - \tau_2$.

3. Extending Geodesics

Let $F$ be a twice-punctured torus. The following two propositions can be shown by using arguments in the proof of [6, Propositions 4.1 and 4.4].

**Proposition 3.1** (cf. [6, Proposition 4.1]). For an integer $n(\geq 4)$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $C(F)$ satisfying the following.

(H1) $[\alpha_0, \ldots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $C(F)$,
(H2) $\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \geq 4n$, where $X_{n-2}$ is the component of $\text{Cl}(F \setminus N(\alpha_{n-2}))$ that contains an essential simple closed curve.

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $C(F)$.

**Proposition 3.2** (cf. [6, Proposition 4.4]). For an integer $n(\geq 3)$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $C(F)$ satisfying the following.

(H1) $[\alpha_0, \ldots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $C(F)$,
(H2') $\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_n)) > 2n$, where $X_{n-2}$ is the component of $\text{Cl}(F \setminus N(\alpha_{n-2}))$ that contains an essential simple closed curve.

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $C(F)$.

By using Propositions 3.1 and 3.2, we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ in $C(F)$, i.e., $d_F(\alpha_0, \alpha_n) = n$, for a positive integer $n$.

3.1. A construction of a concrete example: the case when $n$ is even. We first assume that $n$ is even. Let $\alpha_0, \alpha_2$ be essential non-separating simple closed curves on $F$ which intersect transversely in one point, and let $\alpha_1$ be an essential simple closed curve on $S$ which is disjoint from $\alpha_0 \cup \alpha_2$. Let $X_2 = \text{Cl}(F \setminus N(\alpha_2))$. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $C(F)$. Choose a homeomorphism $f_2 : F \to F$ such that $f_2(N(\alpha_2)) = N(\alpha_2)$ and that $\text{diam}_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(f_2(\alpha_0))) \geq 4n$. This is possible by [9, Proposition 4.6]. Let $\alpha_3 = f_2(\alpha_1)$ and $\alpha_4 = f_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $C(F)$ and $\alpha_2$ intersects $\alpha_4$ transversely in one point.

![Figure 2](image-url)
We repeat this process to construct a path $[a_0, a_1, \ldots, a_n]$ inductively as follows. Suppose that we have constructed a path $[a_0, a_1, \ldots, a_i]$ with $|a_{i-2} \cap a_i| = 1$ for each even $i(< n)$. Then let $X_i = \text{Cl}(F \setminus N(a_i))$. Choose a homeomorphism $f_i : F \to F$ such that $f_i(N(a_i)) = N(a_i)$ and that

$$\text{diam}_{X_i}(\pi_{X_i}(a_{i-2}), \pi_{X_i}(f_i(a_{i-2}))) \geq 4n.$$ 

Then we let $a_{i+1} = f_i(a_{i-1})$ and $a_{i+2} = f_i(a_{i-2})$. Note that $[a_{i}, a_{i+1}, a_{i+2}]$ is a geodesic of length two in $C(F)$, and we have obtained a path $[a_0, a_1, \ldots, a_{i+2}]$ with $|a_i \cap a_{i+2}| = 1$.

Claim 3.3. For each $k \in \{2, 4, \ldots, n\}$, the path $[a_0, a_1, \ldots, a_k]$ constructed above is a geodesic in $C(F)$.

Proof. We prove the claim by mathematical induction on $k$. It is clear that $[a_0, a_1, a_2]$ is a geodesic in $C(F)$. Hence, Claim 3.1 holds for $k = 2$. Assume that $[a_0, a_1, \ldots, a_k]$ is a geodesic in $C(F)$ for some $k \in \{2, 4, \ldots, n-2\}$. We note that $[a_k, a_{k+1}, a_{k+2}]$ is a geodesic in $C(F)$. Furthermore, by the inequality (1), we have $\text{diam}_{X_k}(\pi_{X_k}(a_{k-2}), \pi_{X_k}(a_{k+2})) \geq 4n > 4k$. Hence, by Proposition 3.1, the path $[a_0, a_1, \ldots, a_{k+2}]$ is a geodesic in $C(F)$, which shows that Claim 3.3 holds for $k + 2$. This completes the proof of Claim 3.3.

3.2. A construction of a concrete example: the case when $n$ is odd. Suppose that $n$ is odd. Let $\alpha_2, \alpha_3$ be essential non-separating simple closed curves on $F$ which are mutually disjoint. Let $x$ be an essential simple closed curve which intersects $\alpha_2$ and $\alpha_3$ transversely in one point, respectively. Choose an essential simple closed curve $y_1$ on $F$ that is disjoint from $\alpha_2$ and $x$. Let $X_2 = \text{Cl}(F \setminus N(\alpha_2))$. By [10, Proposition 4.6], there exists homeomorphism $f_2 : F \to F$ such that $f_2(N(\alpha_2)) = N(\alpha_2)$ and that

$$\text{diam}_{X_2}(\pi_{X_2}(\alpha_3), \pi_{X_2}(f_2(x))) > 2n.$$ 

Let $\alpha_0 = f_2(x)$ and $\alpha_1 = f_2(y_1)$. Note that $\alpha_0 \cap \alpha_1 = \emptyset$, $\alpha_1 \cap \alpha_2 = \emptyset$ and $\alpha_0$ intersects $\alpha_2$ transversely in one point, which implies that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $C(F)$. On the other hand, choose an essential simple closed curve $y_3$ on $F$ that is disjoint from $\alpha_3$ and $x$. Let $X_3 = \text{Cl}(F \setminus N(\alpha_3))$. By [10, Proposition 4.6], there exists a homeomorphism $f_3 : F \to F$ such that $f_3(N(\alpha_3)) = N(\alpha_3)$ and that

$$\text{diam}_{X_3}(\pi_{X_3}(\alpha_0), \pi_{X_3}(f_3(x))) > 2n.$$ 

Let $\alpha_4 = f_3(y_2)$ and $\alpha_5 = f_3(x)$. Note that $\alpha_3 \cap \alpha_4 = \emptyset$, $\alpha_4 \cap \alpha_5 = \emptyset$ and $\alpha_3$ intersects $\alpha_5$ transversely in one point, which implies that $[\alpha_3, \alpha_4, \alpha_5]$ is a geodesic in $C(F)$.

![Figure 3](image-url)
Proof. By Proposition 3.2 together with the inequality (2), the path $[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ is a geodesic. By Proposition 3.2 again together with the inequality (3), we see that the path $[\alpha_0, \alpha_1, \ldots, \alpha_5]$ is also a geodesic in $C(F)$.

We extend the above geodesic $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$ as follows. Suppose that we have constructed a path $[\alpha_0, \alpha_1, \ldots, \alpha_i]$ for an odd integer $i$ with $5 \leq i < n$. Let $X_i = Cl(F \setminus N(\alpha_i))$. Then there exists a homeomorphism $f_i : F \to F$ such that $f_i(N(\alpha_i)) = N(\alpha_i)$ and that $\text{diam} \pi X_i(\alpha_{i-2}), \pi X_i(f_i(\alpha_{i-2})) \geq 4n$. Let $\alpha_{i+1} = f_i(\alpha_{i-1})$ and $\alpha_{i+2} = f_i(\alpha_{i-2})$. Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic in $C(F)$ and that $\alpha_i$ intersects $\alpha_{i+2}$ transversely in one point. By Proposition 3.1, the path $[\alpha_1, \alpha_2, \ldots, \alpha_{i+2}]$ is a geodesic in $C(F)$. We repeat this process until we obtain a geodesic $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ of length $n$. Note that $\alpha_{n-2}$ intersects $\alpha_n$ transversely in one point.

4. Proof of Theorem 1.1

Basically we mimic the proof of [6, Theorem 1.1]. For $i = 1, 2$, let $V_i$ be a solid torus and $t_i$ a trivial arc properly embedded in $V_i$. The following assertion is proved by Saito [11, Proposition 3.8].

Assertion 4.1. Let $D_i$ be an essential disk in $V_i - t_i$ as in Figure 4. Then any non-separating essential disk in $V_i - t_i$ is isotopic to $D_i$ and any separating essential disk in $V_i - t_i$ can be isotoped to be disjoint from $D_i$.

![Figure 4](image)

Let $P = \partial V_1$. Then starting with a geodesic $[\alpha_0(= \partial D_1), \alpha_1, \alpha_2]$ in $C(P - t_1)$ with $|\alpha_0 \cap \alpha_2| = 1$, we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]$ with $|\alpha_n \cap \alpha_{n+2}| = 1$ as in Subsections 3.1 and 3.2. We glue $\partial V_1$ and $\partial V_2$ by a homeomorphism $h : \partial V_1 \to \partial V_2$ such that $h(\partial t_1) = \partial t_2$ and $h(\alpha_{n+2}) = \partial D_2$. Then the argument in the proof of [6, Theorem 1.1], together with Assertion 4.1, enables us to show that the distance of the $(1,1)$-bridge splitting $(V_1, t_1) \cup_P (V_2, t_2)$ is exactly $n$.

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