<table>
<thead>
<tr>
<th>Title</th>
<th>(1,1)-BRIDGE SPLITTINGS WITH DISTANCE EXACTLY $n$ (Pursuit of the Essence of Singularity Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>IDO, AYAKO; JANG, YEONHEE; KOBAYASHI, TSUYOSHI</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1868: 32-37</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195424">http://hdl.handle.net/2433/195424</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
</tbody>
</table>

Kyoto University
(1,1)-BRIDGE SPLITTINGS WITH DISTANCE EXECTLY $n$

AYAKO IDO, YEONHEE JANG AND TSUYOSHI KOBAYASHI

1. Introduction


On the other hand, the above concept and results have been extended to bridge splittings for links in closed 3-manifolds (for definitions, see subsection 2.3), and have been studied by several authors. For example, Saito [11] showed that for any closed 3-manifold admitting a Heegaard splitting of genus one, there is a knot in the manifold with a (1,1)-bridge splitting of arbitrary high distance. Recently, Blair, Tomova and Yoshizawa [2] showed that for given integers $b$, $c$, $g$, and $n$, there exists a manifold $M$ containing a $c$-component link $L$ so that $(M, L)$ admits a $(g, b)$-bridge splitting of distance at least $n$. Moreover, Ichihara and Saito [7] showed that for any given closed orientable 3-manifold $M$ with a Heegaard surface of genus $g$, and for any positive integers $b$ and $n$, there exists a knot $K$ in $M$ which admits a $(g, b)$-bridge splitting of distance greater than $n$.

In [6], we showed that there exists a Heegaard splitting of a closed orientable 3-manifold with distance exactly $n$ for each positive integer $n$. To prove this, we gave a method to extend a geodesic in the curve complex of a closed orientable surface to a geodesic with given length, and constructed a concrete example (for details, see [6, Section 4]).

In this paper, we apply the idea of [6, Section 4] to construct a geodesic of any given length in the curve complex of a twice-punctured torus, and show the following.

Theorem 1.1. For any integer $n > 0$, there exists a (1,1)-bridge splitting with distance exactly $n$.

2. Definitions and notations

2.1. Curve complexes. Let $S$ be an orientable surface with genus $g$, $b$ boundary components and $p$ punctures. A simple closed curve in $S$ is essential if it does not bound a disk or a once-punctured disk in $S$ and is not parallel to a component of $\partial S$. An arc properly embedded in $S$ is essential if it does not co-bound a disk in $S$ together with an arc on $\partial S$. We say that $S$ is sporadic if $g = 0, b + p \leq 4$ or $g = 1, b + p \leq 1$.

Except in sporadic cases, the curve complex $C(S)$ is defined as follows: each vertex of $C(S)$ is the isotopy class of an essential simple closed curve on $S$, and a collection of $k + 1$ vertices forms a $k$-simplex of $C(S)$ if they can be realized by disjoint curves in $S$. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. We assume that $S$ is a torus, a torus with one boundary component, or a sphere with 4 boundary components since, otherwise, there are no essential simple closed curves in
Suppose $X$ is a closed connected surface of genus $g$. Then $S$ is a handlebody if and only if $S$ is isotopic to a surface obtained from $X$ by a union of two or more essential simple closed curves. If $S$ is of genus $g$, then $S$ is a closed orientable 3-manifold with boundary $X$.

We define an arc-and-curve complex $\mathcal{AC}(S)$ of $S$ as follows: every arc $\alpha$ in $S$ is an element of $\mathcal{AC}(S)$ if it is essential and intersects $X$ in at least two points. For any $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of arcs in $\mathcal{AC}(S)$, we define $d_{\mathcal{AC}(S)}(\alpha, \alpha')$ to be the minimal number of intersections between $\alpha$ and $\alpha'$, and we define $\mathcal{A}C(S)$ to be the metric space on $\mathcal{AC}(S)$. We define the distance between two vertices of $\mathcal{AC}(S)$ to be the minimal number of 1-simplexes of a simplicial path in $\mathcal{AC}(S)$ joining the two vertices. We denote by $d_{C}(x, y)$, or $d_{S}(x, y)$ in brief, the distance in $C(S)$ between the vertices $x$ and $y$. For subsets $X$ and $Y$ of the vertices of $C(S)$, we define $diam_{S}(X, Y) = diam_{S}(X \cup Y)$. Similarly, we can define the distance $d_{\mathcal{AC}(S)}(x, y)$ and $diam_{\mathcal{AC}(S)}(X, Y)$.

2.2. **Subsurface projections.** Let $P(Y)$ denote the power set of a set $Y$. Suppose that $X$ is an essential subsurface of $S$ that contains an essential simple closed curve. We call the composition $\pi_0 \circ \pi_A$ of maps $\pi_A : C^0(S) \to P(\mathcal{AC}^0(X))$ and $\pi_0 : P(\mathcal{AC}^0(X)) \to P(C^0(X))$ a subsurface projection if they satisfy the following (see Figure 1): for a vertex $\alpha$, take a representative $\alpha$ so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_A(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union for all $i = 1, \ldots, n$ of the set of all isotopy classes of the components of $\partial N(\alpha_i \cup \partial X)$ which are essential in $X$, where $N(\alpha_i \cup \partial X)$ is a regular neighborhood of $\alpha_i \cup \partial X$ in $X$.

![Figure 1](image)

2.3. **$(g, b)$-bridge splittings.** Let $H$ be a genus-$g(\geq 0)$ handlebody. We say that a set of $n$ arcs $\{t_1, \ldots, t_n\}$ properly embedded in $H$ is a set of trivial $n$ arcs if $t_1 \cup \cdots \cup t_n$ is parallel to $\partial H$. Let $H$ be a handlebody and $\tau = \{t_1, \ldots, t_n\}$ a set of trivial $n$ arcs in $H$. Then $\tau$ can be isotoped in $H$ so that the projection from $\partial H \times [0, 1)$ to $[0, 1)$ has exactly one critical point in each $t_i$.

It is well known that every closed orientable 3-manifold $M$ has a genus-$g$ Heegaard splitting for some $g(\geq 0)$, i.e., $M = V_1 \cup_F V_2$, where $V_1$ and $V_2$ are genus-$g$ handlebodies...
such that $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = P$. Let $L$ be a link in $M$. We say that $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is a $(g, b)$-bridge splitting (or $(g, b)$-splitting for short) for the pair $(M, L)$ if $P$ separates $(M, L)$ into two components $(V_1, \tau_1)$ and $(V_2, \tau_2)$ where $\tau_1 = L \cap V_1$ (resp. $\tau_2 = L \cap V_2$) is a set of trivial $b$ arcs in $A$ (resp. $B$). Then we say that $P$ is a $(g, b)$-bridge surface (or a bridge surface for short). It is known that each $(M, L)$ has a $(g, b)$-bridge splitting for some $g$ and $b$. (For a detailed discussion, see [5, Lemma 2.1]).

For $i = 1$ or 2, $D(V_i)$ denotes the subset of $C^0(\partial V_i - \tau_i)$ consisting of the vertices with representatives bounding disks in $V_i - \tau_i$. Then the (Hempel) distance of $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is defined by $d_P(D(V_1), D(V_2))$, where $P' = \partial V_1 - \tau_1 = \partial V_2 - \tau_2$.

3. Extending geodesics

Let $F$ be a twice-punctured torus. The following two propositions can be shown by using arguments in the proof of [6, Propositions 4.1 and 4.4].

**Proposition 3.1** (cf. [6, Proposition 4.1]). For an integer $n(\geq 4)$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $C(F)$ satisfying the following.

(H1) $[\alpha_0, \ldots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $C(F)$,

(H2) $diam_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \geq 4n$, where $X_{n-2}$ is the component of $Cl(F \backslash N(\alpha_{n-2}))$ that contains an essential simple closed curve.

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $C(F)$.

**Proposition 3.2** (cf. [6, Proposition 4.4]). For an integer $n(\geq 3)$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $C(F)$ satisfying the following.

(H1) $[\alpha_0, \ldots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $C(F)$,

(H2) $diam_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_n)) > 2n$, where $X_{n-2}$ is the component of $Cl(F \backslash N(\alpha_{n-2}))$ that contains an essential simple closed curve.

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $C(F)$.

By using Propositions 3.1 and 3.2, we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ in $C(F)$, i.e., $d_F(\alpha_0, \alpha_n) = n$, for a positive integer $n$.

3.1. A construction of a concrete example: the case when $n$ is even. We first assume that $n$ is even. Let $\alpha_0, \alpha_2$ be essential non-separating simple closed curves on $F$ which intersect transversely in one point, and let $\alpha_1$ be an essential simple closed curve on $S$ which is disjoint from $\alpha_0 \cup \alpha_2$. Let $X_2 = Cl(F \backslash N(\alpha_2))$. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $C(F)$. Choose a homeomorphism $f_2 : F \rightarrow F$ such that $f_2(N(\alpha_2)) = N(\alpha_2)$ and that $diam_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(f_2(\alpha_0))) \geq 4n$. This is possible by [9, Proposition 4.6]. Let $\alpha_3 = f_2(\alpha_1)$ and $\alpha_4 = f_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $C(F)$ and $\alpha_2$ intersects $\alpha_4$ transversely in one point.

![Diagram](image2)
We repeat this process to construct a path \([a_0, a_1, \ldots, a_n]\) inductively as follows. Suppose that we have constructed a path \([a_0, a_1, \ldots, a_i]\) with \(|a_{i-2} \cap a_i| = 1\) for each even \(i(< n)\). Then let \(X_i = \text{Cl}(F \setminus N(a_i))\). Choose a homeomorphism \(f_i : F \to F\) such that \(f_i(N(a_i)) = N(a_i)\) and that

\[
\text{diam}_{X_i} (\pi_{X_i}(a_{i-2}), \pi_{X_i}(f_i(a_{i-2}))) \geq 4n.
\]

Then we let \(a_{i+1} = f_i(a_{i-1})\) and \(a_{i+2} = f_i(a_{i-2})\). Note that \([a_i, a_{i+1}, a_{i+2}]\) is a geodesic of length two in \(C(F)\), and we have obtained a path \([a_0, a_1, \ldots, a_{i+2}]\) with \(|a_i \cap a_{i+2}| = 1\).

**Claim 3.3.** For each \(k \in \{2, 4, \ldots, n\}\), the path \([a_0, a_1, \ldots, a_k]\) constructed above is a geodesic in \(C(F)\).

**Proof.** We prove the claim by mathematical induction on \(k\). It is clear that \([a_0, a_1, a_2]\) is a geodesic in \(C(F)\). Hence, Claim 3.1 holds for \(k = 2\). Assume that \([a_0, a_1, \ldots, a_k]\) is a geodesic in \(C(F)\) for some \(k \in \{2, 4, \ldots, n-2\}\). We note that \([a_k, a_{k+1}, a_{k+2}]\) is a geodesic in \(C(F)\). Furthermore, by the inequality (1), we have \(\text{diam}_{X_k} (\pi_{X_k}(a_{k-2}), \pi_{X_k}(a_{k+2})) \geq 4n > 4k\). Hence, by Proposition 3.1, the path \([a_0, a_1, \ldots, a_{k+2}]\) is a geodesic in \(C(F)\), which shows that Claim 3.3 holds for \(k + 2\). This completes the proof of Claim 3.3.

---

3.2. A construction of a concrete example: the case when \(n\) is odd. Suppose that \(n\) is odd. Let \(a_2, a_3\) be essential non-separating simple closed curves on \(F\) which are mutually disjoint. Let \(x\) be an essential simple closed curve which intersects \(a_2\) and \(a_3\) transversely in one point, respectively. Choose an essential simple closed curve \(y_1\) on \(F\) that is disjoint from \(a_2\) and \(x\). Let \(X_2 = \text{Cl}(F \setminus N(a_2))\). By [10, Proposition 4.6], there exists homeomorphism \(f_2 : F \to F\) such that \(f_2(N(a_2)) = N(a_2)\) and that

\[
\text{diam}_{X_2} (\pi_{X_2}(a_3), \pi_{X_2}(f_2(x))) > 2n.
\]

Let \(a_0 = f_2(x)\) and \(a_1 = f_2(y_1)\). Note that \(a_0 \cap a_1 = \emptyset\), \(a_1 \cap a_2 = \emptyset\) and \(a_0\) intersects \(a_2\) transversely in one point, which implies that \([a_0, a_1, a_2]\) is a geodesic in \(C(F)\). On the other hand, choose an essential simple closed curve \(y_2\) on \(F\) that is disjoint from \(a_3\) and \(x\). Let \(X_3 = \text{Cl}(F \setminus N(a_3))\). By [10, Proposition 4.6], there exists a homeomorphism \(f_3 : F \to F\) such that \(f_3(N(a_3)) = N(a_3)\) and that

\[
\text{diam}_{X_3} (\pi_{X_3}(a_0), \pi_{X_3}(f_3(x))) > 2n.
\]

Let \(a_4 = f_3(y_2)\) and \(a_5 = f_3(x)\). Note that \(a_3 \cap a_4 = \emptyset\), \(a_4 \cap a_5 = \emptyset\) and \(a_3\) intersects \(a_5\) transversely in one point, which implies that \([a_3, a_4, a_5]\) is a geodesic in \(C(F)\).

---

**Figure 3**

---

**Claim 3.4.** The path \([a_0, a_1, \ldots, a_5]\) constructed above is a geodesic in \(C(F)\).
Proof. By Proposition 3.2 together with the inequality (2), the path $[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ is a geodesic. By Proposition 3.2 again together with the inequality (3), we see that the path $[\alpha_0, \alpha_1, \ldots, \alpha_6]$ is also a geodesic in $C(F)$. \hfill \square

We extend the above geodesic $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$ as follows. Suppose that we have constructed a path $[\alpha_0, \alpha_1, \ldots, \alpha_i]$ for an odd integer $i$ with $5 \leq i < n$. Let $X_i = Cl(F \setminus N(\alpha_i))$. Then there exists a homeomorphism $f_i : F \to F$ such that $f_i(N(\alpha_i)) = N(\alpha_i)$ and that $\text{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(f_i(\alpha_{i-2}))) \geq 4n$. Let $\alpha_{i+1} = f_i(\alpha_{i-1})$ and $\alpha_{i+2} = f_i(\alpha_{i-2})$.

Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic in $C(F)$ and that $\alpha_i$ intersects $\alpha_{i+2}$ transversely in one point. By Proposition 3.1, the path $[\alpha_1, \alpha_2, \ldots, \alpha_{i+2}]$ is a geodesic in $C(F)$. We repeat this process until we obtain a geodesic $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ of length $n$. Note that $\alpha_{n-2}$ intersects $\alpha_n$ transversely in one point.

4. PROOF OF THEOREM 1.1

Basically we mimic the proof of [6, Theorem 1.1]. For $i = 1, 2$, let $V_i$ be a solid torus and $t_i$ a trivial arc properly embedded in $V_i$. The following assertion is proved by Saito [11, Proposition 3.8].

Assertion 4.1. Let $D_i$ be an essential disk in $V_i - t_i$ as in Figure 4. Then any non-separating essential disk in $V_i - t_i$ is isotopic to $D_i$ and any separating essential disk in $V_i - t_i$ can be isotoped to be disjoint from $D_i$.

![Figure 4](image_url)

Let $P = \partial V_1$. Then starting with a geodesic $[\alpha_0(= \partial D_1), \alpha_1, \alpha_2]$ in $C(P - t_1)$ with $|\alpha_0 \cap \alpha_2| = 1$, we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]$ with $|\alpha_n \cap \alpha_{n+2}| = 1$ as in Subsections 3.1 and 3.2. We glue $\partial V_1$ and $\partial V_2$ by a homeomorphism $h : \partial V_1 \to \partial V_2$ such that $h(\partial t_1) = \partial t_2$ and $h(\alpha_{n+2}) = \partial D_2$. Then the argument in the proof of [6, Theorem 1.1], together with Assertion 4.1, enables us to show that the distance of the (1,1)-bridge splitting $(V_1, t_1) \cup_P (V_2, t_2)$ is exactly $n$.

REFERENCES


DEPARTMENT OF MATHEMATICS, NARA WOMEN’S UNIVERSITY  
*E-mail address*: eaa.ido@cc.nara-wu.ac.jp

DEPARTMENT OF MATHEMATICS, NARA WOMEN’S UNIVERSITY  
*E-mail address*: yeonheejang@cc.nara-wu.ac.jp

DEPARTMENT OF MATHEMATICS, NARA WOMEN’S UNIVERSITY  
*E-mail address*: tsuyoshi@cc.nara-wu.ac.jp