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(1,1)-BRIDGE SPLITTINGS WITH DISTANCE EXECTLY $n$

AYAKO IDO, YEONHEE JANG AND TSUYOSHI KOBAYASHI

1. Introduction


On the other hand, the above concept and results have been extended to bridge splittings for links in closed 3-manifolds (for definitions, see subsection 2.3), and have been studied by several authors. For example, Saito [11] showed that for any closed 3-manifold admitting a Heegaard splitting of genus one, there is a knot in the manifold with a (1,1)-bridge splitting of arbitrary high distance. Recently, Blair, Tomova and Yoshizawa [2] showed that for given integers $b$, $c$, $g$, and $n$, there exists a manifold $M$ containing a $c$-component link $L$ so that $(M, L)$ admits a $(g, b)$-bridge splitting of distance at least $n$. Moreover, Ichihara and Saito [7] showed that for any given closed orientable 3-manifold $M$ with a Heegaard surface of genus $g$, and for any positive integers $b$ and $n$, there exists a knot $K$ in $M$ which admits a $(g, b)$-bridge splitting of distance greater than $n$.

In [6], we showed that there exists a Heegaard splitting of a closed orientable 3-manifold with distance exactly $n$ for each positive integer $n$. To prove this, we gave a method to extend a geodesic in the curve complex of a closed orientable surface to a geodesic with given length, and constructed a concrete example (for details, see [6, Section 4]).

In this paper, we apply the idea of [6, Section 4] to construct a geodesic of any given length in the curve complex of a twice-punctured torus, and show the following.

Theorem 1.1. For any integer $n > 0$, there exists a (1,1)-bridge splitting with distance exactly $n$.

2. Definitions and notations

2.1. Curve complexes. Let $S$ be an orientable surface with genus $g$, $b$ boundary components and $p$ punctures. A simple closed curve in $S$ is essential if it does not bound a disk or a once-punctured disk in $S$ and is not parallel to a component of $\partial S$. An arc properly embedded in $S$ is essential if it does not co-bound a disk in $S$ together with an arc on $\partial S$. We say that $S$ is sporadic if $g = 0, b + p \leq 4$ or $g = 1, b + p \leq 1$.

Except in sporadic cases, the curve complex $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on $S$, and a collection of $k + 1$ vertices forms a $k$-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in $S$. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. We assume that $S$ is a torus, a torus with one boundary component, or a sphere with 4 boundary components since, otherwise, there are no essential simple closed curves in
S. When S is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of k + 1 vertices forms a k-simplex of $C(S)$ if they can be realized by curves in S which mutually intersect exactly once (resp. twice). The \textit{arc-and-curve complex} $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on S, and a collection of k + 1 vertices forms a k-simplex of $\mathcal{AC}(S)$ if they can be realized by disjoint arcs or simple closed curves in S.

We can define the \textit{distance} between two vertices in the curve complex $C(S)$ to be the minimal number of 1-simplexes of a simplicial path in $C(S)$ joining the two vertices. We denote by $d_{C(S)}(x, y)$, or $d_{S}(x, y)$ in brief, the distance in $C(S)$ between the vertices x and y. For subsets X and Y of the vertices of $C(S)$, we define $\text{diam}_{S}(X, Y) = \text{diam}_{S}(X \cup Y)$. Similarly, we can define the distance $d_{\mathcal{AC}(S)}(x, y)$ and $\text{diam}_{\mathcal{AC}(S)}(X, Y)$. We denote by $[a_{0}, a_{1}, \ldots, a_{n}]$ the path in $C(S)$ with vertices $a_{0}, a_{1}, \ldots, a_{n}$ such that $a_{i} \cap a_{i+1} = \emptyset$ (i = 0, 1, \ldots, n - 1). We call a path $[a_{0}, a_{1}, \ldots, a_{n}]$ a \textit{geodesic} if n = $d_{S}(a_{0}, a_{n})$.

2.2. Subsurface projections. Let $\mathcal{P}(Y)$ denote the power set of a set Y. Suppose that X is an essential subsurface of S that contains an essential simple closed curve. We call the composition $\pi_{0} \circ \pi_{A}$ of maps $\pi_{A} : C^{0}(S) \rightarrow \mathcal{P}(\mathcal{AC}^{0}(X))$ and $\pi_{0} : \mathcal{P}(\mathcal{AC}^{0}(X)) \rightarrow \mathcal{P}(C^{0}(X))$ a \textit{subsurface projection} if they satisfy the following (see Figure 1): for a vertex $\alpha$, take a representative $\alpha$ so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_{A}(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_{0}(\{\alpha_{1}, \ldots, \alpha_{n}\})$ is the union for all $i = 1, \ldots, n$ of the set of all isotopy classes of the components of $\partial N(\alpha_{i} \cup \partial X)$ which are essential in X, where $N(\alpha_{i} \cup \partial X)$ is a regular neighborhood of $\alpha_{i} \cup \partial X$ in X.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

2.3. $(g, b)$-bridge splittings. Let H be a genus-g($\geq 0$) handlebody. We say that a set of n arcs $\{t_{1}, \ldots, t_{n}\}$ properly embedded in H is a \textit{set of trivial n arcs} if $t_{1} \cup \ldots \cup t_{n}$ is parallel to $\partial H$. Let H be a handlebody and $\tau = \{t_{1}, \ldots, t_{n}\}$ a set of trivial n arcs in H. Then $\tau$ can be isotoped in H so that the projection from $\partial H \times [0, 1]$ to [0, 1] has exactly one critical point in each $t_{i}$.

It is well known that every closed orientable 3-manifold M has a genus-g Heegaard splitting for some g($\geq 0$), i.e., $M = V_{1} \cup_{F} V_{2}$, where $V_{1}$ and $V_{2}$ are genus-g handlebodies
such that $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = P$. Let $L$ be a link in $M$. We say that $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is a $(g, b)$-bridge splitting (or $(g, b)$-splitting for short) for the pair $(M, L)$ if $P$ separates $(M, L)$ into two components $(V_1, \tau_1)$ and $(V_2, \tau_2)$ where $\tau_1 = L \cap V_1$ (resp. $\tau_2 = L \cap V_2$) is a set of trivial $b$ arcs in $A$ (resp. $B$). Then we say that $P$ is a $(g, b)$-bridge surface (or a bridge surface for short). It is known that each $(M, L)$ has a $(g, b)$-bridge splitting for some $g$ and $b$. (For a detailed discussion, see [5, Lemma 2.1]).

For $i = 1$ or 2, $\mathcal{D}(V_i)$ denotes the subset of $C^0(\partial V_i, \tau_i)$ consisting of the vertices with representatives bounding disks in $V_i - \tau_i$. Then the (Hempel) distance of $(V_1, \tau_1) \cup_P (V_2, \tau_2)$ is defined by $d_P(\mathcal{D}(V_1), \mathcal{D}(V_2))$, where $P = \partial V_1 - \tau_1 = \partial V_2 - \tau_2$.

3. Extending Geodesics

Let $F$ be a twice-punctured torus. The following two propositions can be shown by using arguments in the proof of [6, Propositions 4.1 and 4.4].

**Proposition 3.1** (cf. [6, Proposition 4.1]). For an integer $n \geq 4$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $C(F)$ satisfying the following.

(H1) $[\alpha_0, \ldots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $C(F)$,

(H2) $\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \geq 4n$, where $X_{n-2}$ is the component of $C(F \setminus N(\alpha_{n-2}))$ that contains an essential simple closed curve.

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $C(F)$.

**Proposition 3.2** (cf. [6, Proposition 4.4]). For an integer $n \geq 3$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $C(F)$ satisfying the following.

(H1) $[\alpha_0, \ldots, \alpha_{n-2}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $C(F)$,

(H2') $\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_n)) > 2n$, where $X_{n-2}$ is the component of $C(F \setminus N(\alpha_{n-2}))$ that contains an essential simple closed curve.

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $C(F)$.

By using Propositions 3.1 and 3.2, we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ in $C(F)$, i.e., $d_F(\alpha_0, \alpha_n) = n$, for a positive integer $n$.

3.1. A construction of a concrete example: the case when $n$ is even. We first assume that $n$ is even. Let $\alpha_0, \alpha_2$ be essential non-separating simple closed curves on $F$ which intersect transversely in one point, and let $\alpha_1$ be an essential simple closed curve on $S$ which is disjoint from $\alpha_0 \cup \alpha_2$. Let $X_2 = C(F \setminus N(\alpha_2))$. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $C(F)$. Choose a homeomorphism $f_2 : F \to F$ such that $f_2(N(\alpha_2)) = N(\alpha_2)$ and that $\text{diam}_{X_2}((\pi_{X_2}(\alpha_0), \pi_{X_2}(f_2(\alpha_0))) \geq 4n$. This is possible by [9, Proposition 4.6]. Let $\alpha_3 = f_2(\alpha_1)$ and $\alpha_4 = f_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $C(F)$ and $\alpha_2$ intersects $\alpha_4$ transversely in one point.

![Figure 2](image-url)
We repeat this process to construct a path \([a_0, a_1, \ldots, a_n]\) inductively as follows. Suppose that we have constructed a path \([a_0, a_1, \ldots, a_i]\) with \(|\alpha_{i-2} \cap \alpha_i| = 1\) for each even \(i(< n)\). Then let \(X_i = \text{Cl}(F \setminus N(\alpha_i))\). Choose a homeomorphism \(f_i : F \to F\) such that \(f_i(N(\alpha_i)) = N(\alpha_i)\) and that

\[
(1) \quad \text{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(f_i(\alpha_{i-2}))) \geq 4n.
\]

Then we let \(\alpha_{i+1} = f_i(\alpha_{i-1})\) and \(\alpha_{i+2} = f_i(\alpha_{i-2})\). Note that \([\alpha_i, \alpha_{i+1}, \alpha_{i+2}]\) is a geodesic of length two in \(\mathcal{C}(F)\), and we have obtained a path \([a_0, a_1, \ldots, a_{i+2}]\) with \(|\alpha_i \cap \alpha_{i+2}| = 1\).

**Claim 3.3.** For each \(k \in \{2, 4, \ldots, n\}\), the path \([\alpha_0, \alpha_1, \ldots, \alpha_k]\) constructed above is a geodesic in \(\mathcal{C}(F)\).

**Proof.** We prove the claim by mathematical induction on \(k\). It is clear that \([\alpha_0, \alpha_1, \alpha_2]\) is a geodesic in \(\mathcal{C}(F)\). Hence, Claim 3.1 holds for \(k = 2\). Assume that \([\alpha_0, \alpha_1, \ldots, \alpha_k]\) is a geodesic in \(\mathcal{C}(F)\) for some \(k \in \{2, 4, \ldots, n-2\}\). We note that \([\alpha_k, \alpha_{k+1}, \alpha_{k+2}]\) is a geodesic in \(\mathcal{C}(F)\). Furthermore, by the inequality (1), we have \(\text{diam}_{X_1}(\pi_{X_1}(\alpha_{k-2}), \pi_{X_1}(\alpha_{k+2})) \geq 4n > 4k\). Hence, by Proposition 3.1, the path \([\alpha_0, \alpha_1, \ldots, \alpha_{k+2}]\) is a geodesic in \(\mathcal{C}(F)\), which shows that Claim 3.3 holds for \(k + 2\). This completes the proof of Claim 3.3. \(\square\)

### 3.2. A construction of a concrete example: the case when \(n\) is odd

Suppose that \(n\) is odd. Let \(\alpha_2, \alpha_3\) be essential non-separating simple closed curves on \(F\) which are mutually disjoint. Let \(x\) be an essential simple closed curve which intersects \(\alpha_2\) and \(\alpha_3\) transversely in one point, respectively. Choose an essential simple closed curve \(y_1\) on \(F\) that is disjoint from \(\alpha_2\) and \(x\). Let \(X_2 = \text{Cl}(F \setminus N(\alpha_2))\). By [10, Proposition 4.6], there exists homeomorphism \(f_2 : F \to F\) such that \(f_2(N(\alpha_2)) = N(\alpha_2)\) and that

\[
(2) \quad \text{diam}_{X_2}(\pi_{X_2}(\alpha_2), \pi_{X_2}(f_2(x))) > 2n.
\]

Let \(\alpha_0 = f_2(x)\) and \(\alpha_1 = f_2(y_1)\). Note that \(\alpha_0 \cap \alpha_1 = \emptyset\), \(\alpha_1 \cap \alpha_2 = \emptyset\) and \(\alpha_0\) intersects \(\alpha_2\) transversely in one point, which implies that \([\alpha_0, \alpha_1, \alpha_2]\) is a geodesic in \(\mathcal{C}(F)\). On the other hand, choose an essential simple closed curve \(y_2\) on \(F\) that is disjoint from \(\alpha_3\) and \(x\). Let \(X_3 = \text{Cl}(F \setminus N(\alpha_3))\). By [10, Proposition 4.6], there exists a homeomorphism \(f_3 : F \to F\) such that \(f_3(N(\alpha_3)) = N(\alpha_3)\) and that

\[
(3) \quad \text{diam}_{X_3}(\pi_{X_3}(\alpha_3), \pi_{X_3}(f_3(x))) > 2n.
\]

Let \(\alpha_4 = f_3(y_2)\) and \(\alpha_5 = f_3(x)\). Note that \(\alpha_3 \cap \alpha_4 = \emptyset\), \(\alpha_4 \cap \alpha_5 = \emptyset\) and \(\alpha_3\) intersects \(\alpha_5\) transversely in one point, which implies that \([\alpha_3, \alpha_4, \alpha_5]\) is a geodesic in \(\mathcal{C}(F)\).

**Figure 3**

**Claim 3.4.** The path \([\alpha_0, \alpha_1, \ldots, \alpha_5]\) constructed above is a geodesic in \(\mathcal{C}(F)\).
Proof. By Proposition 3.2 together with the inequality (2), the path \([\alpha_0, \alpha_1, \alpha_2, \alpha_3]\) is a geodesic. By Proposition 3.2 again together with the inequality (3), we see that the path \([\alpha_0, \alpha_1, \ldots, \alpha_5]\) is also a geodesic in \(C(F)\).

We extend the above geodesic \([\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]\) as follows. Suppose that we have constructed a path \([\alpha_0, \alpha_1, \ldots, \alpha_i]\) for an odd integer \(i\) with \(5 \leq i < n\). Let \(X_i = Cl(F \setminus N(\alpha_i))\). Then there exists a homeomorphism \(f_i : F \to F\) such that \(f_i(N(\alpha_i)) = N(\alpha_i)\) and that \(\text{diam}_X(\pi_X(\alpha_{i-2}), \pi_X(f_i(\alpha_{i-2}))) \geq 4n\). Let \(\alpha_{i+1} = f_i(\alpha_{i-1})\) and \(\alpha_{i+2} = f_i(\alpha_{i-2})\). Note that \([\alpha_i, \alpha_{i+1}, \alpha_{i+2}]\) is a geodesic in \(C(F)\) and that \(\alpha_i\) intersects \(\alpha_{i+2}\) transversely in one point. By Proposition 3.1, the path \([\alpha_1, \alpha_2, \ldots, \alpha_{i+2}]\) is a geodesic in \(C(F)\). We repeat this process until we obtain a geodesic \([\alpha_1, \alpha_2, \ldots, \alpha_n]\) of length \(n\). Note that \(\alpha_{n-2}\) intersects \(\alpha_n\) transversely in one point.

4. PROOF OF THEOREM 1.1

Basically we mimic the proof of [6, Theorem 1.1]. For \(i = 1, 2\), let \(V_i\) be a solid torus and \(t_i\) a trivial arc properly embedded in \(V_i\). The following assertion is proved by Saito [11, Proposition 3.8].

Assertion 4.1. Let \(D_i\) be an essential disk in \(V_i - t_i\) as in Figure 4. Then any non-separating essential disk in \(V_i - t_i\) is isotopic to \(D_i\) and any separating essential disk in \(V_i - t_i\) can be isotoped to be disjoint from \(D_i\).

![Figure 4](image-url)

Let \(P = \partial V_1\). Then starting with a geodesic \([\alpha_0 = \partial D_1, \alpha_1, \alpha_2]\) in \(C(P - t_1)\) with \(|\alpha_0 \cap \alpha_2| = 1\), we construct a geodesic \([\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]\) with \(|\alpha_n \cap \alpha_{n+2}| = 1\) as in Subsections 3.1 and 3.2. We glue \(\partial V_1\) and \(\partial V_2\) by a homeomorphism \(h : \partial V_1 \to \partial V_2\) such that \(h(\partial t_1) = \partial t_2\) and \(h(\alpha_{n+2}) = \partial D_2\). Then the argument in the proof of [6, Theorem 1.1], together with Assertion 4.1, enables us to show that the distance of the (1,1)-bridge splitting \((V_1, t_1) \cup_F (V_2, t_2)\) is exactly \(n\).

REFERENCES


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