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Wave fronts with one principal curvature a constant in the hyperbolic three-space

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Abstract

In this note, we prove that weakly complete wave fronts with one principal curvature a constant $c$ in the hyperbolic 3-space is either a totally umbilical sphere or umbilic free, if $|c| > 1$. Moreover, we derive their orientability.

1 Introduction

By the Hartman-Nirenberg theorem, complete flat surfaces in the Euclidean 3-space $R^3$ are cylinders over a complete planar regular curve (cf. [2]). This fact implies that such surfaces are trivial. On the other hand, if we admit some singularities, there exist many nontrivial examples of flat surfaces. Murata-Umehara investigated global properties of flat surfaces with admissible singularities called flat fronts and then proved the following (for precise definitions, see Section 2).

Fact 1.1 ([5]). A complete flat front in the Euclidean 3-space whose singular point set is non-empty has no umbilics, is orientable and co-orientable. Moreover, if its ends are embedded, there exist at least four singular points other than cuspidal edges.

This estimate is sharp (see Figure 1).

![Figure 1: A complete flat front in $R^3$ which has four singular points other than cuspidal edges.](image)

We here remark that a flat surface is considered to be a surface such that one of the principal curvatures is identically zero. In the case of nonzero constant, Shiohama and Takagi [6] showed that a complete surface one of whose principal
curvatures is a nonzero constant is either totally umbilical or umbilic-free. The latter case, such a surface is a tube of a complete regular curve in $R^3$ (i.e., a channel surface). In [4], the author investigated wave fronts such that one of the principal curvatures is a nonzero constant (cf. Definition 3.1) and proved the following.

**Fact 1.2** ([4]). A weakly complete wave front in the Euclidean 3-space such that one of the principal curvatures is a nonzero constant has no umbilics and is orientable.

Although wave fronts with one principal curvature a nonzero constant are co-orientable by definition (cf. Remark 3.2), there exists co-orientable and non-orientable ones (see Figure 2).

![Figure 2: A non-orientable wave front with one principal curvature a nonzero constant in $R^3$.](image)

In the case of non-flat space forms, Aledo-Gálvez [1] investigated (immersed) surfaces with one principal curvature a constant $c$ in the hyperbolic 3-space $H^3$. In particular, they proved that a complete surface one of whose principal curvatures is a constant $c$ is either totally umbilical or umbilic-free, if $|c| > 1$ [1, Theorem 1.1]. Moreover, they showed that, if $|c| \leq 1$, such a result does not hold. That is, if $|c| \leq 1$, they exhibited examples of non-totally-umbilical complete surfaces one of whose principal curvatures is a constant $c$ whose umbilic point set is not empty [1, Example 2.1, Example 2.2]. While their examples are given by the first and second fundamental forms, Izumiya-Saji-Takahashi gave an explicit description of such examples in the case of $|c| = 1$ [3, Example 5.7].

In this paper, we give a generalization of Aledo-Gálvez's Theorem [1, Theorem 1.1] as follows (cf. Theorem 3.7 and Theorem 3.8).

**Theorem 1.3.** A weakly complete wave front in the hyperbolic 3-space such that one of the principal curvatures is a constant $c$ satisfying $|c| > 1$ has no umbilics and is orientable.

This theorem is a direct conclusion of Theorem 3.7 and Theorem 3.8. In the case of $|c| \leq 1$, such a result does not hold (see [1, Example 2.1, Example 2.2]).

This paper is organized as follows. In Section 2, we review fundamental properties of wave fronts in $H^3$. Then, in Section 3, we define wave fronts one of whose principal curvatures is a constant and give a proof of Theorem 1.3.
2 Preliminaries: wave fronts in $H^3$

In this section, we review fundamental properties of wave fronts in the hyperbolic 3-space $H^3$. Here, we regard $H^3$ as

$$H^3 = \{ x = (x_0, x_1, x_2, x_3) \in \mathbb{R}_+^4; \langle x, x \rangle = -1, x_0 > 0 \},$$

where $\mathbb{R}_+^4$ is the Lorentz-Minkowski 4-space with the inner product

$$\langle x, x \rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2, \quad x = (x_0, x_1, x_2, x_3) \in \mathbb{R}_+^4.$$

If we denote by $S_1^3$ the de Sitter 3-space $S_1^3 = \{ x \in \mathbb{R}_+^4; \langle x, x \rangle = 1 \}$, the unit tangent bundle $T_1H^3$ of $H^3$ is given by

$$T_1H^3 = \{ (p, v) \in H^3 \times S_1^3; \langle p, v \rangle = 0 \}.$$

Let $M^2$ be a smooth 2-manifold and $f : M^2 \to H^3$ be a smooth map. We call $f$ a frontal, if for any point $p \in M^2$, there exists a neighborhood $U$ of $p$ and a smooth map $\nu : U \to S_1^3$ such that

$$\langle df_p(v), \nu(p) \rangle = 0$$

holds for all $v \in T_pM^2$. Then, $\nu$ is said to be the unit normal vector field of the frontal $f$. If $\nu$ is well-defined on $M^2$, $f$ is called co-orientable. Moreover, $f$ is orientable if $M^2$ is orientable. A point $p \in M^2$ is said to be a singular (resp. regular) point if $\text{rank}(df)_p < 2$ (resp. $\text{rank}(df)_p = 2$). As in the introduction, we call the frontal $f$ wave front, if the map

$$L := (f, \nu) : U \to T_1H^3$$

is an immersion. The map $L$ is called the Legendrian lift of $f$.

**Lemma 2.1** ([5, Lemma 1.1]). Let $M^2$ be a smooth 2-manifold and $f : M^2 \to H^3$ be a co-orientable wave front. If $p \in M^2$ is a singular point of $f$, then there exist a real number $\delta > 0$ such that $p$ is a regular point of the parallel front $f_\delta := (\cosh \delta)f + (\sinh \delta)\nu$.

For a co-orientable wave front $f : M^2 \to H^3$, take $p \in M^2$ arbitrary. By Lemma 2.1, there exist a neighborhood $U$ and a real number $\delta$ such that $f_\delta$ is immersion on $U$. Then, a point $p \in M^2$ is called umbilic of $f$ if $p$ is umbilic point of $f_\delta$. By definition, umbilic points are common in its parallel family.

**Lemma 2.2.** Let $M^2$ be a smooth 2-manifold, $f : M^2 \to H^3$ be a co-orientable wave front and $p \in M^2$ be a singular point of $f$. Then, $p$ is umbilic if and only if $\text{rank}(df)_p = 0$ holds.

Lemma 2.2 is an analogue of [4, Lemma 2.2].

**Lemma 2.3** ([5, Lemma 1.3]). Let $M^2$ be a smooth 2-manifold, $f : M^2 \to H^3$ be a co-orientable wave front and $\nu$ be a unit normal vector field of $f$. For a non-umbilic point $p \in M^2$, there exist a local coordinate system $(U; u, v)$ centered at $p$ such that
$f_u$ and $\nu_u$ (resp. $f_v$ and $\nu_v$) are linearly independent on $U$. In particular, the pair 
$\{f_u, \nu_u\}$ (resp. $\{f_v, \nu_v\}$) does not vanishes at the same time and 
$$\langle f_u, f_v \rangle = \langle f_u, \nu_v \rangle = \langle f_v, \nu_u \rangle = 0$$
holds.

Such a coordinate system is called principal curvature line.

**Definition 2.4** (cf. [5, Definition 1.5]). Let $M^2$ be a smooth 2-manifold and $f : M^2 \to H^3$ be a co-orientable wave front. A direction $\nu \in T_p M^2$ is called a principal direction of $f$ if $df(\nu)$ and $d\nu(\nu)$ are linearly dependent. Moreover, for an open interval $I \subseteq \mathbb{R}$, a curve $\sigma(t) : I \to M^2$ is called a principal curvature line if $\sigma'(t)$ gives a principal direction for all $t \in I$.

On a principal curvature line coordinate neighborhood, every coordinate curve gives a principal curvature line.

For $j = 1, 2$, let $\Lambda_j : M^2 \to P^1(\mathbb{R})$ be the principal curvature map of a wave front $f$ (for a precise definition, see [5, Section 1]). In particular, if $(U; u, v)$ is a principal curvature line coordinate system, $\Lambda_j|_U : U \to P^1(\mathbb{R}) (j = 1, 2)$ coincide with the smooth maps 
$$\Lambda_1 = [-\nu_u : f_u], \quad \Lambda_2 = [-\nu_v : f_v],$$
respectively. Here, where $[-\nu_u : f_u]$ and $[-\nu_v : f_v]$ mean the proportional ratio of 
$\{-\nu_u, f_u\}$ and $\{-\nu_v, f_v\}$ respectively as elements of the real projective line $P^1(\mathbb{R})$.

**Proposition 2.5** ([5, Lemma 1.7]). Let $f : M^2 \to H^3$ be a co-orientable wave front and $\Lambda_1, \Lambda_2$ be the principal curvature maps of $f$. Then, a point $p \in M^2$ is umbilic if and only if $\Lambda_1(p) = \Lambda_2(p)$ holds. On the other hand, $p \in M^2$ is a singular point if and only if either $\Lambda_1(p) = [1 : 0]$ or $\Lambda_2(p) = [1 : 0]$ holds.

At the end of this section, we recall the weakly completeness of wave fronts as follows. Let $f : M^2 \to H^3$ be a wave front and $\nu$ be a (locally defined) unit normal vector field of $f$. Then the symmetric covariant 2-tensor
$$ds^2_\# := \langle df, df \rangle + \langle d\nu, d\nu \rangle$$
gives a Riemannian metric on $M^2$ which is called a lift metric of $f$. The lift metric is a pull-back metric of the Sasakian metric of the unit tangent bundle $T'_1 H^3$ of $H^3$ through the Legendrian lift $L = (f, \nu)$ of $f$. The lift metric $ds^2_\#$ is independent of a choice of $\nu$.

**Definition 2.6.** A wave front is called weakly complete if its lift metric gives a complete Riemannian metric.

### 3 Wave fronts one of whose principal curvatures is a nonzero constant

In this section, we give a definition of wave fronts one of whose principal curvatures is a nonzero constant. Then, we give a proof of Theorem 1.3 by showing Theorem 3.7 and Theorem 3.8.
3.1 Definitions

Let $M^2$ be a smooth 2-manifold. Consider a co-orientable front $f : M^2 \to H^3$ such that for some real numbers $a, b \in \mathbb{R}$ $(a^2 + b^2 \neq 0)$, $f$ satisfies

\[(3.1) \quad \text{rank}(a(\partial \nu)_p + b(\partial f)_p) < 2\]

for any $p \in M^2$, where $\nu : M^2 \to S^3_1$ is the unit normal vector field of $f$. If $a \neq 0, b = 0$, then $f$ is called an extrinsically flat front, and if $a = 0, b \neq 0$, then all the points of $M^2$ are singular.

From now on, we consider the case $a \neq 0, b \neq 0$. Setting $c = b/a$, $(3.1)$ turns out to be

\[(3.2) \quad \text{rank}((\partial \nu)_p + c(\partial f)_p) < 2\]

for any $p \in M^2$.

Definition 3.1. Let $c$ be a real number, $f : M^2 \to H^3$ be a co-orientable front and $\nu : M^2 \to S^3_1$ the unit normal vector field of $f$. Then, $f$ is called one of whose principal curvatures is a constant $c$ if $f$ satisfies $(3.2)$.

Remark 3.2 (Non-co-orientable case). Consider a non-co-orientable wave front satisfying $(3.2)$. Changing $\nu$ to $-\nu$, we have that such a wave front satisfies both of

\[(3.3) \quad \text{rank}((\partial \nu)_p + c(\partial f)_p) < 2 \quad \text{and} \quad \text{rank}((\partial \nu)_p - c(\partial f)_p) < 2,\]

for any $p \in M^2$. (3.3) implies that such a wave front must be isoparametric (i.e., both of the principal curvatures are constant), and hence has no singular points. Since isoparametric surfaces must be orientable, a wave front satisfying $(3.3)$ must be co-orientable. This is a contradiction. Therefore, we have that wave fronts satisfying $(3.2)$ must be co-orientable.

3.2 Proof of Theorem 1.3

From now on, we denote by $\mathcal{U}_f$ the umbilic point set of a wave front $f : M^2 \to H^3$. Lemma 3.3 and Lemma 3.4 can be proved in the similar way as [4, Lemma 3.5] and [4, Lemma 3.6], respectively.

Lemma 3.3. Let $f : M^2 \to H^3$ be a wave front one of whose principal curvatures is a constant $c$. If $p \in M^2$ is a umbilic point of $f$, the $f$ is regular at $p$.

Lemma 3.4. Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant $c$ and $q \in M^2 \setminus \mathcal{U}_f$ be a non-umbilic point of $f$. Then there exists a curvature line coordinate system $(U; u, v)$ around $q$ such that

- $u$-curves are curvature line of $\Lambda_1$, $v$-curves are curvature line of $\Lambda_2 \equiv [c : 1]$,
- $|f_v| \equiv 1$.
- $\nu_u + cf_u \neq 0$, $\nu_v + cf_v = 0$, $f_{vv} = f + cv$

hold on $U$, where $0 = (0, 0, 0, 0)$. 


A regular curve in $H^3$ is called a planar circle, if its curvature function is a constant greater than 1 and its torsion function is identically zero. For a planar circle $\hat{\sigma} = \hat{\sigma}(t)$, there exist a point $p \in H^3$ such that $\text{dist}_{H^3}(p, \hat{\sigma}(t))$ is a constant for all $t$, where $\text{dist}_{H^3}(\cdot, \cdot)$ is the distance function of $H^3$. We call $p$ the center of $\hat{\sigma}$. Lemma 3.5 and Lemma 3.6 can be proved in the similar way as [4, Lemma 3.7] and [4, Lemma 3.8], respectively.

**Lemma 3.5.** Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant $c$ and $\sigma(t) : R \supseteq I \to M^2$ be a principal curvilinear of $\Lambda_2 \equiv c$ parametrized by arc-length passing through a non-umbilic point $q \in M^2 \backslash U_f$. If $|c| > 1$, $\hat{\sigma}(t) := f \circ \sigma(t)$ is a planar circle in $H^3$ whose curvature is $c$ and there exist real constants $a, b \in R$ such that $\Lambda_1$ is given by

$$
(3.4) \quad \Lambda_1(\sigma(t)) = \left[ 1 + c(c^2 - 1) \left( a \cos \left( \sqrt{c^2 - 1} t \right) + b \sin \left( \sqrt{c^2 - 1} t \right) \right) : 
\right. 
\left. c + (c^2 - 1) \left( a \cos \left( \sqrt{c^2 - 1} t \right) + b \sin \left( \sqrt{c^2 - 1} t \right) \right) \right]
$$
on $\sigma(t)$. Furthermore, $\sigma(1)$ and $U_f$ has no intersection.

**Lemma 3.6.** Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant $c$ with $|c| > 1$ and $(U; u, v)$ be a curvilinear coordinate system as in Lemma 3.4 around a non-umbilic point $q \in M^2 \backslash U_f$. Then, the map $C : U \to H^3$ defined by

$$
C(u, v) = \frac{1}{\sqrt{c^2 - 1}} (c f(u, v) + \nu(u, v))
$$
is independent of $v$ and is a regular curve $C = \gamma(u)$ in $H^3$. Moreover, if we set $\sigma_{u_0,v_0}(t) : R \supseteq J \to M^2$ as the curvilinear of $\Lambda_2$ such that $\sigma_{u_0,v_0}(0) = (u_0, v_0) \in U$, the center of the planar circle $\hat{\sigma}_{u_0,v_0} := f \circ \sigma_{u_0,v_0}$ is $\gamma(u_0)$ and the image of $\hat{\sigma}_{u_0,v_0}$ is included in the normal plane $\gamma(u_0)^\perp$.

**Theorem 3.7.** Let $c$ be a constant satisfying $|c| > 1$ and $f : M^2 \to H^3$ be a wave front one of whose principal curvature is $c$. If $f$ is weakly complete, $f$ is totally umbilic or umbilic-free. In the latter case, $f$ is described as

$$
(3.5) \quad f(u, v) = \frac{1}{\sqrt{c^2 - 1}} \left( -c \gamma(u) + \cos \left( \sqrt{c^2 - 1} t \right) e_1(u) + \sin \left( \sqrt{c^2 - 1} t \right) e_2(u) \right),
$$

where $(u, v) \in R \times S^1$, $S^1 = R/2\pi Z$, $\gamma(u)$ is a complete regular curve in $H^3$ and $\{e_1, e_2\}$ is an orthonormal frame of the normal bundle of $\gamma$.

**Proof.** Assume that $f$ is not totally umbilic. First of all, we shall prove that the curvilinear of $\Lambda_2 \equiv c$ passing through the non-umbilic point $p \in M^2 \backslash U_f$ is defined on $S^1$. Let $(U; u, v)$ be a curvilinear coordinate system around $p$ as in Lemma 3.4. Then each curvilinear of $\Lambda_2$ is given by the $v$-curves on $U$. The lift metric $ds_#^2$ of $f$ is given by

$$
ds_#^2 = \langle df, df \rangle + \langle d\nu, d\nu \rangle = (\langle f_u, f_u \rangle + \langle \nu_u, \nu_u \rangle) du^2 + (1 + c^2) dv^2
$$
on $U$. In particular, each $v$-curve is a geodesic of $ds^2_{\#}$, and hence it is defined on $\mathbb{R}$ since $f$ is weakly complete. Since the image of each curvatureline of $\Lambda_2$ is a planar circle, the domain of each curvatureline is $S^1$.

Suppose that the umbilic point set $\mathcal{U}_f$ of $f$ is not empty. Take an umbilic point $q \in \partial \mathcal{U}_f$. Then there exists a sequence $\{p_n\} \subseteq M^2 \setminus \mathcal{U}_f$ such that $\lim_{n \to \infty} p_n = q$. For each $p_n$, let $\sigma_n$ be the curvatureline of $\Lambda_2$ passing through $p_n$. By Lemma 3.5, $\hat{\sigma}_n := f \circ \sigma_n$ is a planar circle of a constant curvature $c$. Therefore, there exists a subsequence $\{n_k\}$ such that $\hat{\sigma}_q = \lim_{k \to \infty} \hat{\sigma}_{n_k}$ is also a planar circle of a constant curvature $c$. Every point on the inverse image $\hat{\sigma}_q$ of $\hat{\sigma}_q$ through $f$ is umbilic by Lemma 3.5.

On the other hand, by Lemma 3.5, For each $\sigma_{nk} = \sigma_{nk}(v)$, there exist $v_k$ such that $\Lambda_1(\sigma_{nk}(v_k)) = [1 : c]$. If we take the limit as $k \to \infty$, we have $\hat{\sigma}_q = \lim_{k \to \infty} \sigma_{nk}$. Therefore, by the continuity of the principal curvature map $\Lambda_1$, there exists a point on $\hat{\sigma}_q$ such that $\Lambda_1 = [1 : c] \neq [c : 1] = \Lambda_2$, which is a contradiction. Thus we have $\mathcal{U}_f = \emptyset$.

**Theorem 3.8.** Let $f : M^2 \to H^3$ be a wave front one of whose principal curvature is a constant $c$ with $|c| > 1$. If $f$ is weakly complete, $f$ is orientable.

**Proof.** If $f$ is totally umbilic, $f$ is orientable. Thus we assume that $f$ is not totally umbilic. Then, by Theorem 3.7, $f$ is represented as in (3.5). Take an orthonormal frame $e_1, e_2$ of $\gamma$ such that $\{\gamma'(u), e_1(u), e_2(u)\}$ is a positively oriented orthogonal frame. Setting $e_0 := e_1 \times e_2$, we have

$$\gamma'(u) = \varphi(u) e_0(u), \quad \left(\varphi(u) = \sqrt{\langle \gamma'(u), \gamma'(u) \rangle}\right).$$

If $f$ is not orientable, there exist real numbers $u_0, L$ such that $\gamma(u + L) = \gamma(u)$ holds for each $u \in \mathbb{R}$ and

$$e_1(u_0 + L) \times e_2(u_0 + L) = -e_1(u_0) \times e_2(u_0)$$

holds. Since $e_0(u_0 + L) = -e_0(u_0)$, we have

$$\gamma'(u_0 + L) = \varphi(u_0 + L) e_0(u_0 + L) = -\varphi(u_0) e_0(u_0) = -\gamma'(u_0),$$

which contradicts to $\gamma'(u_0 + L) = \gamma'(u_0)$. \hfill $\Box$

Theorem 3.7 and Theorem 3.8 imply Theorem 1.3.

**References**


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