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COMBINATORIAL INEQUALITIES OF KAZHDAN-LUSZTIG POLYNOMIALS ON BRUHAT GRAPHS

MASATO KOBAYASHI

ABSTRACT. In terms of Bruhat graphs, we establish two combinatorial inequalities on $q=1$ values of KL polynomials for crystallographic Coxeter systems: (1) We give a lower bound of their $q=1$ values by graph-theoretic distance. (2) We show a sufficient condition for a Bruhat interval to be rationally singular by our new idea, "final intervals" as an application of Deodhar's inequality.

CONTENTS

1. INTRODUCTION

What can we say about $q=1$ specialization of Kazhdan-Lusztig (KL) polynomials from a combinatorial perspective, particularly in terms of Bruhat graphs? This was a motivation of our work. Let us begin with some background.

Kazhdan and Lusztig introduced a family of polynomials in 1979 to study Schubert varieties as well as Verma modules. They conjectured [14] that $q=1$ specialization of these polynomials express multiplicity of composition factors of certain Verma modules (KL conjecture). Soon later, Beilinson-Bernstein [1] and Brylinski-Kashiwara [7] gave proofs from rather geometric and representation-theoretic points of view.

"Combinatorics of KL polynomials" have grown little by little in the 1990s and 2000s. One direction is Dyer's idea, *Bruhat graphs*; This graph encodes crucial

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information of Bruhat order structure as an Eulerian poset together with certain extra edge relations (which do not appear in Hasse diagram). Brenti and Dyer each developed this idea to enumerate label-rising Bruhat paths under reflection orders by R- and $\widetilde{R}$-polynomials; see [5, 6] and [9, 10], for example.

The aim of this article is to establish two kinds of combinatorial inequalities of KL polynomials (Theorems 4.4 and 6.10) in terms of Bruhat graphs for crystallographic Coxeter systems; Our approach is a numerical point of view, somewhat different from Brenti and Dyer. First, Theorem 4.4 gives a lower bound of $P_{uw}(1)$ values in terms of graph-theoretic distance. Second, Theorem 6.10 shows a sufficient condition for a Bruhat interval to be rationally singular with our new idea, "final intervals" as an application of Deodhar's inequality. Proofs are elementary throughout.

Notation. We follow common notation in the context of Coxeter groups as books Björner-Brenti [3] and Humphreys [12]. By (W, S) (or simply W) we mean a Coxeter system with length function ℓ. Unless otherwise specified, u, v, w are elements of W and e is the unit. Let $T = \bigcup_{w \in W} w^{-1}Sw$ denote the set of reflections. Write $u \longrightarrow w$ if $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$. Define *Bruhat order* $u \leq w$ if there exist $v_1, \dots, v_n \in W$ such that $u \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_n = w$. For $u \leq w$, let $[u, w] \overset{\text{def}}{=} \{v \in W \mid u \leq v \leq w\}$ denote a *Bruhat interval.* Often $\ell(u, w) \overset{\text{def}}{=} \ell(w) - \ell(u)$ abbreviates the length of intervals.

Convention. Furthermore, we assume that W is *crystallographic.* This is to ensure the validity of Facts 3.2, 3.3 and 5.3.

2. Bruhat graphs

We begin with Bruhat graphs, our main idea. Recall that $u \longrightarrow w$ means $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$.

Definition 2.1. The *Bruhat graph* of W is a directed graph for vertices $w \in W$ and for edges $u \longrightarrow w$. By a *Bruhat path* we always mean a directed path such as $u \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_n = w$.

Often, we consider also the induced subgraph for each subset $X \subseteq W$.

Figure 1 illustrates an example. Observe that the underlying graph is 3-regular; every vertex is incident to 3 edges. We will come back to this idea later.

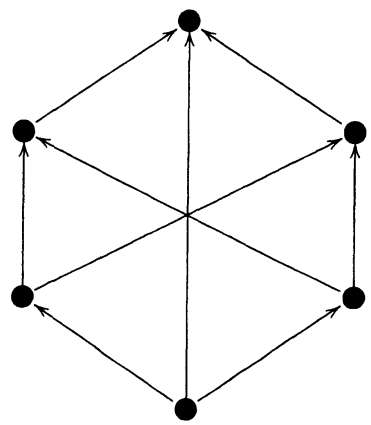
3. KL polynomials

We now introduce KL polynomials following [3, Section 5.1]; See *loc.cit.* for R-polynomials which we do not define here.

Fact 3.1. There exists a unique family of polynomials $\{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q]$ (*Kazhdan-Lusztig polynomials*) such that

(1) $P_{uw}(q) = 0$ if $u \not\leq w$,

FIGURE 1. Bruhat graph of a dihedral interval



(2) $P_{uw}(q) = 1$ if $u = w$,
(3) $\deg P_{uw}(q) \leq (\ell(u,w) - 1)/2$ if $u < w$,
(4) if $u \leq w$, then

$$q^{\ell(u,w)} P_{uw}(q^{-1}) = \sum_{u \leq v \leq w} R_{uv}(q) P_{vw}(q),$$

(5) $P_{uw}(0) = 1$ if $u \leq w$,
(6) if $u \leq w$, $s \in S$ and $ws \rightarrow w$, then $P_{uw}(q) = P_{us,w}(q)$.

In what follows, our discussion goes with a fixed element $w \in W$ in mind. Then we investigate behavior of $P_{uw}(q)$'s with u running over the lower interval $[e, w]$. To emphasize this context, we use notation $X(w) \stackrel{\text{def}}{=} [e, w]$. By slight abuse of language, we refer to $X(w)$ even as a Bruhat graph.

Now recall two key facts under the assumption W to be crystallographic:

Fact 3.2 (Nonnegativity [13, Corollary 4]). All coefficients of KL polynomials in W are nonnegative.

To state another fact, we need this notation: For $f = a_0 + a_1 q + \cdots + a_c q^c, g = b_0 + b_1 q + \cdots + b_d q^d \in \mathbb{N}[q]$ $(c = \deg f, d = \deg g)$, define a partial order $f \leq g$ if $a_i \leq b_i$ for all i (hence $c \leq d$).

Fact 3.3 (Monotonicity [4, Corollary 3.7]). Suppose $u \leq v \leq w$ in W. Then $P_{uw}(q) \geq P_{vw}(q)$.

In other words, fixing w as the second index, the function $P_{-,w}(q) : X(w) \rightarrow \mathbb{N}[q]$ is *weakly* monotonically decreasing. Actually, there is a convenient criterion for strict monotonicity:

Proposition 3.4. *Let $u < v \leq w$. Then $P_{uw}(q) > P_{vw}(q) \iff P_{uw}(1) > P_{vw}(1)$.*

Proof. Suppose $u < v \leq w$. Then we have the inequality $P_{uw}(q) \geq P_{vw}(q)$ as assumed above. Say $P_{uw}(q) = 1 + b_1 q + \cdots + b_d q^d$, $P_{vw}(q) = 1 + a_1 q + \cdots + a_d q^d$ with $0 \leq a_i \leq b_i$ for all i. If $P_{uw}(q) > P_{vw}(q)$, then $a_j < b_j$ for some j $(1 \leq j \leq d)$. Then

$$P_{uw}(1) - P_{vw}(1) = (b_1 - a_1) + \cdots + (b_j - a_j) + \cdots + (b_d - a_d) > 0$$

and vice versa. □

Consequently, $P_{-,w}(1) : X(w) \to \mathbb{N}$ is also *weakly* monotonically decreasing.

Definition 3.5. Let $u \in X(w)$. Say $[u, w]$ is *rationally singular* if $P_{uw}(1) > 1$. Say it is *rationally smooth* if $P_{uw}(1) = 1$.

Remark 3.6. We borrowed such terminology from geometry of Schubert varieties; see Billey-Lakshmibai [2] for this theory.

Definition 3.7. *Rational smooth* and *singular vertices* of $X(w)$ are

$$X_1(w) = \{u \in X(w) \mid P_{uw}(1) = 1\} \text{ and } X_2(w) = \{u \in X(w) \mid P_{uw}(1) > 1\}.$$

4. Main Theorem 1

In this section, we prove Theorem 4.4. Before that, we need several definitions and facts.

Definition 4.1. An edge $u \to v$ in $X(w)$ is *strict* if $P_{uw}(1) > P_{vw}(1)$.

The following is the key idea for the proof of Theorem 4.4 (author's recent result [15, Theorem 8.2]).

Lemma 4.2. *If $P_{uw}(1) > 1$, then there exists a strict edge $u \to v$ in $X(w)$.*

Since Bruhat order is the transitive closure of edge relations, this result is useful to give a lower bound of $P_{uw}(1)$ in terms of graph-theoretic distance as we recall now.

Definition 4.3. Let G be a finite directed graph. For a vertex u and a nonempty subset A of vertices of G, define a directed-graph-theoretic distance between the vertex and the subset as

$$\text{dist}(u, A) = \min\{d \geq 0 \mid u \to v_1 \to v_2 \to \cdots \to v_d \in A\}.$$

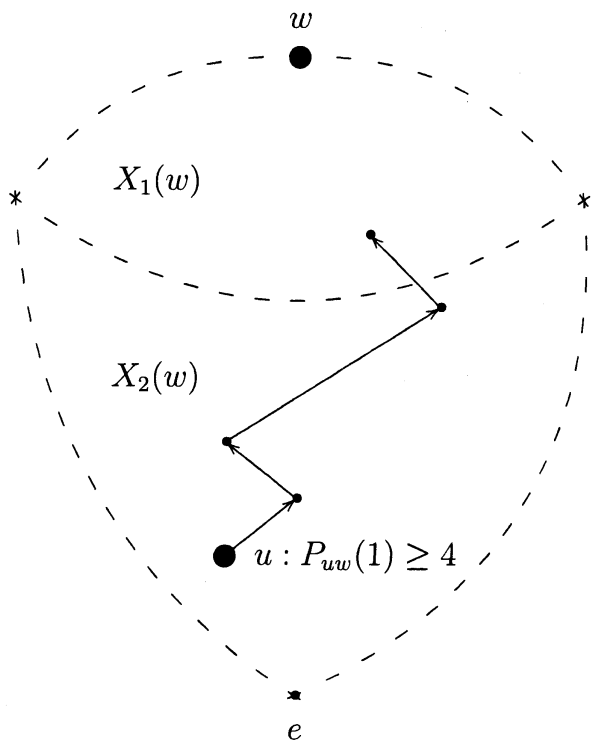
In particular, $\text{dist}(u, A) = 0 \iff u \in A$.

Now consider the case for $G = X(w)$ and $A = X_1(w)$. Then such distance gives a lower bound of $P_{uw}(1)$:

Theorem 4.4. *Let $u \in X(w)$. Then we have*

$$P_{uw}(1) \geq \text{dist}(u, X_1(w)) + 1.$$

FIGURE 2. distance between a singular vertex u and rationally smooth vertices



Proof. For convenience, let $d = \text{dist}(u, X_1(w))$. If u is rationally smooth, then we have $d = 0$; So the assertion is obvious. Suppose u is singular. Lemma 4.2 implies that there exists a strict edge $u \to v_1$ in $X(w)$. If $v_1 \in X_1(w)$, then $d = 1$ so that $P_{uw}(1) \geq 2 = d + 1$. If not, find another strict edge from v_1 in $X(w)$, say $v_1 \to v_2$. We can repeat this procedure at least $d = \text{dist}(u, X_1(w))$ times by definition. Thus the path $u \to v_1 \to v_2 \to \cdots \to v_d$ in $X(w)$ (with all strict edges) induces d strict inequalities of positive integers

$$P_{uw}(1) > P_{v_1 w}(1) > \cdots > P_{v_d w}(1) \geq 1.$$

Conclude that $P_{uw}(1) \geq d + 1$. □

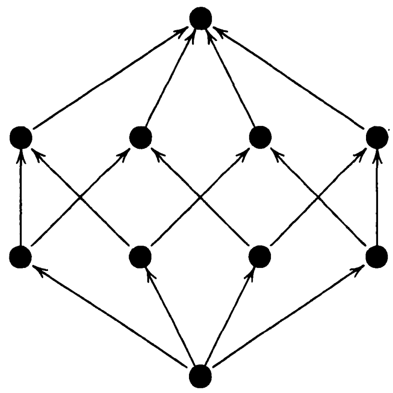
5. Rational singularities and Deodhar's inequality

This section recalls some definitions and facts on Deodhar's inequality for the discussion in the next section.

Definition 5.1. Let $u \leq w$. Set

$$\overline{N}(u, w) = \{v \in W \mid u \to v \leq w\} \text{ and } \overline{\ell}(u, w) = |\overline{N}(u, w)|.$$

FIGURE 3. an irregular Bruhat graph



That is, $\overline{N}(u,w)$ is the neighborhood of the bottom vertex on the Bruhat graph of $[u,w]$; $\overline{\ell}(u,w)$ is the number of those outgoing edges.

Definition 5.2. The *defect* of $[u,w]$ is $\mathrm{df}(u,w) = \overline{\ell}(u,w) - \ell(u,w)$.

The defect turns out to be always nonnegative:

Fact 5.3 (Deodhar's inequality [11])**.** $\mathrm{df}(u,w) \geq 0$.

Definition 5.4. Say $[u,w]$ is *rationally singular* if $\mathrm{df}(x,w) > 0$ for some $x \in [u,w]$. Say it is *rationally smooth* if $\mathrm{df}(x,w) = 0$ for all $x \in [u,w]$. Also, we say "u is rationally singular (smooth) under w" for convenience.

This definition is indeed equivalent to Definition 3.5 (in crystallographic cases); see [2, Section 13.2].

Figure 3 shows the Bruhat graph of $[1324, 3412]$ in the symmetric group S_4. It has the positive defect: $\mathrm{df}(1324, 3412) = 1 = 4 - 3$. Observe that this graph is irregular because the bottom vertex is incident to four edges while middle vertices are incident to only three; About regularity of Bruhat graphs, here is a significant result of Carrell-Peterson [8]:

Fact 5.5. Let $[u,w]$ be a Bruhat interval. Then the following are equivalent:

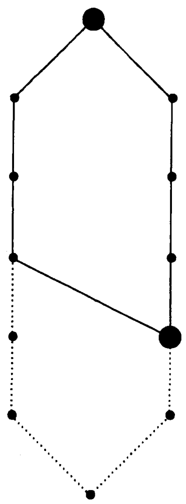
(1) It is rationally smooth.
(2) Its Bruhat graph is $\ell(u,w)$-regular.

Consequently, if we find some vertex incident to more than $\ell(u,w)$ edges, then immediately $[u,w]$ is rationally singular. We apply this idea for the proof of Theorem 6.10.

6. MAIN THEOREM 2

In this section, we prove Theorem 6.10 with some new idea, "final intervals" assuming W is *finite*.

FIGURE 4. a final interval



Definition 6.1. For $I \subseteq S$, let W_I be the standard parabolic subgroup generated by I and uW_I the right I-coset containing u.

Fact 6.2 (Distinguished coset representative of maximal length). Each uW_I has a unique element x such that for all $v \in uW_I$, we have $v \leq x$.

Denote this element by $x = \max uW_I$.

Definition 6.3. An interval $[u, w]$ is *(right) final* if there exists $I \subseteq S$ such that $w = \max uW_I$.

Example 6.4. All right weak edges $u \to us$ are final by definition. More interesting cases are rank 2 cosets (dihedral intervals); Figure 4 shows an example of a final interval in such a coset (we omitted some edges and heads for simplicity). Observe that some final intervals might share the bottom element as in Figure 5.

Proposition 6.5. *Let $[u, w]$ be a final Bruhat interval, say $w = \max uW_I$ with $I \subseteq S$ (necessarily $I \subseteq \{s \in S \mid ws \to w\}$). Then there exists a directed path $u \to us_1 \to us_1s_2 \to \cdots \to us_1s_2 \cdots s_n = w$ such that $s_i \in I$ for all i.*

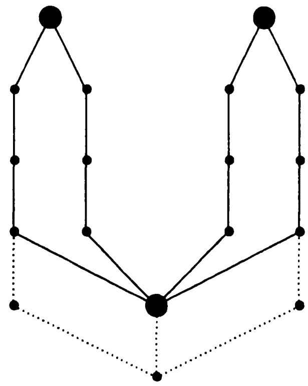
Proof. By definition of a final interval, there exists $x \in W_I$ such that $ux = w$. Choose a reduced expression $x = s_1s_2 \cdots s_n$ with $s_i \in I$ for all i. □

Lemma 6.6. *A final interval $[u, w]$ is rationally smooth.*

Proof. Choose a directed path from u to w as in the previous proposition. Then $P_{uw}(q) = P_{us_1,w}(q) = P_{us_1s_2,w}(q) = \cdots = P_{us_1s_2\cdots s_n,w}(q) = P_{ww}(q) = 1$ since $ws_i \to w$ for all i. □

It follows that $\bar{\ell}(u, w) = \ell(u, w)$ thanks to Deodhar's inequality.

FIGURE 5. final intervals sharing the bottom



Definition 6.7. Let $v \in [u,w]$. Say v is a *final vertex* of $[u,w]$ if $[u,v]$ is final.

Definition 6.8. Let $[u,w]$ be an interval of odd length ≥ 3. A final vertex v of $[u,w]$ is *half* if $\ell(u,v) = (\ell(u,w)+1)/2$.

Definition 6.9. A pair (v_1, v_2) of final vertices of $[u,w]$ is *disjoint* if

$$\overline{N}(u,v_1) \cap \overline{N}(u,v_2) = \varnothing.$$

Theorem 6.10. *If there exists a pair (v_1,v_2) of half and disjoint final vertices in $[u,w]$, then $[u,w]$ is rationally singular.*

The idea is to show the existence of more than $\ell(u,w)$ edges incident to u in $[u,w]$. Then, Deodhar's inequality guarantees that $[u,w]$ is rationally singular.

Proof. It is enough to show that $\overline{\ell}(u,w) > \ell(u,w)$. Let (v_1,v_2) be as above. By definition of the set $\overline{N}(x,y)$, we have $\overline{N}(u,w) \supseteq \overline{N}(u,v_1) \cup \overline{N}(u,v_2)$; this union is disjoint since (v_1,v_2) is disjoint. Hence

$$\begin{aligned}
\overline{\ell}(u,w) &= |\overline{N}(u,w)| \geq |\overline{N}(u,v_1)| + |\overline{N}(u,v_2)| \\
&= \overline{\ell}(u,v_1) + \overline{\ell}(u,v_2) \\
&= \ell(u,v_1) + \ell(u,v_2) \quad \text{(finality of } [u,v_i]) \\
&= \frac{\ell(u,w)+1}{2} + \frac{\ell(u,w)+1}{2} \\
&= \ell(u,w) + 1.
\end{aligned}$$

□

Example 6.11. Let $u = 187654329$, $w = 897654312$, $v_1 = 876543219$ and $v_2 = 198765432$ in $W = A_7 = S_8$. Then we can show that the pair (v_1,v_2) is half and disjoint final vertices of $[u,w]$ with $v_1 = uW_I$, $v_2 = uW_J$, $I = \{s_1,\dots,s_7\}$, $J = \{s_2,\dots,s_8\}$, s_i adjacent transpositions, $\ell(u,w) = 34 - 21 = 13$ (length=the number of inversions) and $\ell(u, v_i) = 28 - 21 = 7$. Theorem 6.10 guarantees $[u, w]$ is rationally singular (in fact, $P_{uw}(q) = 1 + q^6$). We obtained permutations u and w from the construction of arbitrary KL polynomials by Polo [16].

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