数理解析研究所講究録
第 1871 巻 2013 年 164-176

THE LIMIT MULTIPLICITY PROBLEM FOR CONGRUENCE SUBGROUPS OF ARITHMETIC LATTICES AND THE TRACE FORMULA

TOBIAS FINIS

ABSTRACT. We review some recent joint work of the author with E. Lapid and W. Müller on the limit multiplicity problem for congruence subgroups of arithmetic lattices in the case of non-compact quotients.

1. INTRODUCTION

The limit multiplicity problem, which goes back to DeGeorge and Wallach [12, 13, 30], concerns the asymptotic behavior of the spectra of lattices Γ (discrete subgroups of finite covolume $\mathrm{vol}(\Gamma\backslash G)$) in a fixed semisimple Lie group G in the situation where $\mathrm{vol}(\Gamma\backslash G)\rightarrow\infty$. In a great number of cases, the normalized discrete spectra μ_{Γ} coverge then to the Plancherel measure μ_{pl} of the group G, which is defined purely in terms of the decomposition of the space $L^{2}(G)$, i.e., without any reference to discrete subgroups.

For uniform lattices Γ (lattices for which the quotient $\Gamma\backslash G$ is compact), general results on this problem have been known for some time. The first results in this direction were proved by DeGeorge-Wallach [loc. cit.] for normal towers, i.e., descending sequences of finite index normal subgroups of a given uniform lattice with trivial intersection. Subsequently, Delorme [14] completely resolved this case of the limit multiplicity problem. Recently, there has been big progress in proving limit multiplicity for much more general sequences of uniform lattices [1, 2].

In the case of non-compact quotients $\Gamma\backslash G$, where the spectrum also contains a continuous part, much less is known. In a recent joint preprint of the author with E. Lapid and W. Müller [20], this case has been analyzed in a rather general setup. An extension of these results will appear in [17]. (See [20, §1] for previous results in the literature.) The main problem is to show that the contribution of the continuous spectrum is negligible in the limit. This was known up to now only in the case of $\mathrm{GL}(2)$ (or for the discrete series). The new approach is based on a careful study of the spectral side of Arthur's trace formula in the recent form given in [16, 18]. The results are unconditional only for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, but in the general case a substantial reduction of the problem is obtained.

Date: July 8, 2013.

Author supported by DFG Heisenberg grant # FI 1795/1-1 and partially supported by grant # 964-107.6/2007 from the German-Israeli Foundation for Scientific Research and Development.

The purpose of this note is to give an introduction to the problem and an overview of the proof. I would like to thank the organizers of the RIMS symposium "Automorphic Representations and Related Topics" (Jan. 21 – 25, 2013), Atsushi Ichino and Taku Ishii, for the opportunity to present this material there.

2. The limit multiplicity problem

We first need to define the basic objects already mentioned above. Let G be a connected linear semisimple Lie group with a fixed choice of a Haar measure. Since the group G is of type I, we can write unitary representations of G on separable Hilbert spaces as direct integrals (with multiplicities) over the unitary dual $\Pi(G)$, the set of isomorphism classes of irreducible unitary representations of G with the Fell topology (cf. [15]). The regular representation of $G \times G$ on $L^2(G)$ decomposes as the direct integral of the tensor products $\pi \otimes \pi^*$ against the *Plancherel measure* μ_{pl} on $\Pi(G)$. The support of the Plancherel measure is called the *tempered dual* $\Pi(G)_{\mathrm{temp}} \subset \Pi(G)$. The Plancherel measure and the tempered dual are well understood, mainly by the work of Harish-Chandra.

Here we recall only that up to a closed subset of Plancherel measure zero, the topological space $\Pi(G)_{\mathrm{temp}}$ is homeomorphic to a countable union of Euclidean spaces of bounded dimensions, and that under this homeomorphism the Plancherel density is given by a continuous function. One can speak of bounded subsets of $\Pi(G)$. By definition, a Jordan measurable subset of $\Pi(G)_{\mathrm{temp}}$ is a bounded set A such that $\mu_{\mathrm{pl}}(\bar{A} - A^\circ) = 0$.

We say that a collection $\mathfrak{M}$ of Borel measures μ on $\Pi(G)$ has the *limit multiplicity property* (property (LM)) if the following two conditions are satisfied:

(1) For any Jordan measurable set $A \subset \Pi(G)_{\mathrm{temp}}$ we have[1]

$$\mu(A) \to \mu_{\mathrm{pl}}(A), \quad \mu \in \mathfrak{M}.$$

(2) For any bounded set $A \subset \Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$ we have

$$\mu(A) \to 0, \quad \mu \in \mathfrak{M}.$$

We will apply this setup to the regular representations R_Γ of G on $L^2(\Gamma\backslash G)$ for lattices Γ in G. Consider the discrete part $L^2_{\mathrm{disc}}(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$, namely the sum of all irreducible subrepresentations, and denote by $R_{\Gamma,\mathrm{disc}}$ the corresponding restriction of R_Γ. For any $\pi \in \Pi(G)$ let $m_\Gamma(\pi)$ be the multiplicity of π in $L^2(\Gamma\backslash G)$. Thus,

$$m_\Gamma(\pi) = \dim \operatorname{Hom}_G(\pi, R_\Gamma) = \dim \operatorname{Hom}_G(\pi, R_{\Gamma,\mathrm{disc}}).$$

These multiplicities are known to be finite, at least under a weak reduction-theoretic assumption on G and Γ [26, p. 62], which is satisfied if either G has no compact factors or if Γ is arithmetic (cf. [ibid., Theorem 3.3]). We define the

---

[1] Here convergence means that for any $\varepsilon > 0$ the set of $\mu \in \mathfrak{M}$ with $|\mu(A) - \mu_{\mathrm{pl}}(A)| \geq \varepsilon$ is finite.

discrete spectral measure on $\Pi(G)$ with respect to Γ by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\delta_\pi,$$

where δ_π is the Dirac measure at π.

The limit multiplicity problem can be formulated as follows: under which conditions does the set of measures μ_Γ, where Γ ranges over a collection of lattices in G, satisfy property (LM)?

There is an obvious obstruction to this property, namely the possibility that the lattices Γ (or just an infinite subset) all contain a non-trivial subgroup Z of the center of G, which forces the corresponding representations R_Γ to be Z-invariant. By passing to the quotient G/Z, we can assume that this is not the case. If we exclude this possibility, we expect the limit multiplicity property to hold at least for congruence subgroups (and even more generally, although some caution is necessary, as we will see in §3 below).

3. Density principle and trace formula

A basic approach to the limit multiplicity problem is to use integration against test functions on G and the trace formula. Let K be a maximal compact subgroup of G. For a test function $f \in C^\infty_{\mathrm{c,fin}}(G)$, the space of smooth, compactly supported bi-K-finite functions on G, we define its "Fourier transform" on the unitary dual by taking traces: $\hat{f}(\pi) = \mathrm{tr}\,\pi(f)$, $\pi \in \Pi(G)$. This defines $\mu(\hat{f})$ for Borel measures μ on $\Pi(G)$ (of course $\mu(\hat{f})$ might in general be divergent). In particular we have $\mu_{\mathrm{pl}}(\hat{f}) = f(1)$ by Plancherel inversion and

$$\mu_\Gamma(\hat{f}) = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \mathrm{tr}\, R_{\mathrm{disc},\Gamma}(f),$$

which converges by the work of W. Müller [22].

Sauvageot's density principle [29], a refinement of the work of Delorme, amounts to the following:

Theorem 1 (Sauvageot). *Let $\mathfrak{M}$ be a collection of Borel measures on G and assume that for all $f \in C^\infty_{\mathrm{c,fin}}(G)$ we have*

$$\mu(\hat{f}) \to \mu_{\mathrm{pl}}(\hat{f}) = f(1), \quad \mu \in \mathfrak{M}.$$

Then $\mathfrak{M}$ satisfies (LM).

For the purpose of illustration let now Γ be a cocompact lattice in G. We have then the trace formula

$$\mathrm{vol}(\Gamma\backslash G)\mu_\Gamma(\hat{f}) = \mathrm{tr}\, R_\Gamma(f) = \sum_{\{\gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx,$$

where γ ranges over a system of representatives for the conjugacy classes of Γ and G_γ and Γ_γ denote the centralizer of γ in G and Γ, respectively.

If we consider a finite index subgroup Δ of Γ, then we may rewrite this as

$$\mu_\Delta(\hat{f}) = \frac{1}{\operatorname{vol}(\Gamma\backslash G)} \sum_{\{\gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \frac{c_\Delta(\gamma)}{[\Gamma:\Delta]} \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx,$$

where

$$c_\Delta(\gamma) = |\{\delta \in \Delta\backslash\Gamma : \delta\gamma\delta^{-1} \in \Delta\}|.$$

Note that for central elements γ (in particular for $\gamma = 1$), we have obviously $c_\Delta(\gamma) = [\Gamma : \Delta]$. By the density principle, we have therefore reduced the limit multiplicity problem for collections $\mathfrak{D}$ of finite index subgroups Δ of Γ to the group-theoretical question whether for any $\gamma \in \Gamma$, $\gamma \neq 1$, we have

$$\frac{c_\Delta(\gamma)}{[\Gamma:\Delta]} \to 0, \quad \Delta \in \mathfrak{D}.$$

For a normal subgroup Δ of Γ this quotient is just the characteristic function of Δ. Therefore the condition is trivially satisfied for normal towers, which implies Delorme's result that they have the limit multiplicity property.

Can we expect that for irreducible arithmetic lattices the limit multiplicity property holds for any collection of subgroups not containing non-trivial central elements? For congruence subgroups (or arbitrary finite index subgroups in the higher rank case) this follows from [1, 2]. An independent alternative proof of the congruence subgroup case will be contained in work in preparation of E. Lapid and the author [17].

In general however, there might be a further obstruction, namely that infinitely many groups Δ contain a non-central normal subgroup of Γ (which then has to be of infinite index). For instance, for $G = \mathrm{SL}(2,\mathbb{R})$ we can find a descending sequence of finite index normal subgroups Γ_n of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ such that for all n the multiplicity in $L^2(\Gamma_n\backslash G)$ of either one of the two lowest discrete series representations of G (or equivalently, the genus of the corresponding Riemann surface) is equal to one [25]. Similarly, one can find a descending sequence of normal subgroups Γ_n of $\mathrm{SL}(2,\mathbb{Z})$ such that the limiting measure of the sequence (μ_{Γ_n}) has a strictly positive density on the entire complementary spectrum $\Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$ [27].

Note that in these examples we have $c_{\Gamma_n}(\gamma)/[\Gamma : \Gamma_n] = 1$ for $\gamma \in \bigcap_n \Gamma_n$, and that therefore the (normalized) geometric side of the trace formula (which has to be regularized since the lattices are not cocompact) does *not* converge to $f(1)$. By Margulis's normal subgroup theorem non-central normal subgroups of infinite index do not exist for irreducible lattices Γ in semisimple Lie groups G of real rank at least two and without compact factors ([21, p. 4, Theorem 4'], cf. also [ibid., IX.6.14]).

In any case, for the non-cocompact lattices $\mathrm{SL}(N,\mathbb{Z}) \subset \mathrm{SL}(N,\mathbb{R})$ we can show the following:

Theorem 2. *The collection of measures μ_Γ, where Γ runs over all congruence subgroups of* $\mathrm{SL}(N,\mathbb{Z})$ *not containing the central element* -1, *has the limit multiplicity property.*

Note that for $N \geq 3$ every finite index subgroup of $\mathrm{SL}(N,\mathbb{Z})$ is a congruence subgroup. The result generalizes to the lattices $\mathrm{SL}(N,\mathfrak{o}_F)$, where F is a number field.

We note also that it might be interesting to investigate the behavior of $c_\Delta(\gamma)$ further for special collections of finite index subgroups Δ with non-trivial intersection. In this situation, there will be further secondary terms in the limit (besides the dominating Plancherel term $f(1)$ considered here).

4. A quick review of Arthur's trace formula

In order to prove Theorem 2, we switch to an adelic setup and use Arthur's trace formula [3, 4, 5, 6, 7, 8, 9].

From now on let G be an arbitrary reductive group defined over $\mathbb{Q}$. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$ be the locally compact adele ring of $\mathbb{Q}$. For every place v of $\mathbb{Q}$ (i.e. $v = \infty$ or a prime) let $|\cdot|_v$ be the normalized absolute value on $\mathbb{Q}_v$ and let $|\cdot|_{\mathbb{A}} = \prod_v |\cdot|_v$ be the adelic norm. We have the product formula $|x| = 1$ for $x \in \mathbb{Q}^*$.

We fix a maximal $\mathbb{Q}$-split torus T_0 of G and let M_0 be its centralizer, which is a minimal Levi subgroup defined over F. We also fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v = \mathbf{K}_\infty \mathbf{K}_{\mathrm{fin}} \subset G(\mathbb{A})$ that is admissible with respect to M_0 [5, §1]. Denote by A_0 the identity component of $T_0(\mathbb{R})$, which is viewed as a subgroup of $T_0(\mathbb{A})$ via the diagonal embedding of $\mathbb{R}$ into F_∞.

We write $\mathcal{L}$ for the (finite) set of Levi subgroups containing M_0. Let $M \in \mathcal{L}$. We write T_M for the split part of the identity component of the center of M. Set $A_M = A_0 \cap T_M(\mathbb{R})$ and $W(M) = N_{G(F)}(M)/M$. Denote by $\mathfrak{a}_M^*$ the $\mathbb{R}$-vector space spanned by the lattice $X^*(M)$ of $\mathbb{Q}$-rational characters of M and let $\mathfrak{a}_{M,\mathbb{C}}^* = \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$. We write $\mathfrak{a}_M$ for the dual space of $\mathfrak{a}_M^*$, which is spanned by the co-characters of T_M. Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m)\rangle} = |\chi(m)|_{\mathbb{A}} = \prod_v |\chi(m_v)|_v$$

for any $\chi \in X^*(M)$ and denote by $M(\mathbb{A})^1 \subset M(\mathbb{A})$ the kernel of H_M. Let $\mathcal{L}(M)$ be the set of Levi subgroups containing M and $\mathcal{P}(M)$ the set of parabolic subgroups of G with Levi part M. Denote by Σ_M the set of reduced roots of T_M on the Lie algebra of G. For any $\alpha \in \Sigma_M$ we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co-root. Let $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ be the discrete part of $L^2(A_M M(F)\backslash M(\mathbb{A}))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $M(\mathbb{A})$. We denote by $\Pi_{\mathrm{disc}}(M(\mathbb{A}))$ the countable set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ into irreducibles.

4.1. **Intertwining operators.** Now let $P \in \mathcal{P}(M)$. We write $\mathfrak{a}_P = \mathfrak{a}_M$. Let U_P be the unipotent radical of P. Denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of T_M on the Lie algebra $\mathfrak{u}_P$ of U_P. Denote by δ_P the modulus function of $P(\mathbb{A})$.

Let $\bar{\mathcal{A}}^2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(M(F)U_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))\ \forall x \in G(\mathbb{A})\}$$

with respect to the natural inner product on this space.

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$.

For any $P \in \mathcal{P}(M)$ let $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $U_P(\mathbb{A})$- and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$- and $\mathfrak{z}$-finite vectors, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$. Let $\rho(P, \lambda)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$.

For $P, Q \in \mathcal{P}(M)$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

be the standard *intertwining operator* [7, §1]. These operators satisfy the following properties.

(1) $M_{P|P}(\lambda) \equiv \mathrm{Id}$ for all $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$.
(2) For any $P, Q, R \in \mathcal{P}(M)$ we have $M_{R|P}(\lambda) = M_{R|Q}(\lambda) \circ M_{Q|P}(\lambda)$ for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. In particular, $M_{Q|P}(\lambda)^{-1} = M_{P|Q}(\lambda)$.
(3) $M_{Q|P}(\lambda)^* = M_{P|Q}(-\bar{\lambda})$ for any $P, Q \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. In particular, $M_{Q|P}(\lambda)$ is unitary for $\lambda \in i\mathfrak{a}^*_M$.
(4) If $P|^\alpha Q$ then $M_{Q|P}(\lambda)$ depends only on $\langle \lambda, \alpha^\vee \rangle$.

For any $P \in \mathcal{P}(M)$ we have a canonical isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_\mathbb{C}, K_\infty)$-modules

$$j_P : \mathrm{Hom}(\pi, L^2(A_M M(F)\backslash M(\mathbb{A}))) \otimes \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) \to \mathcal{A}^2_\pi(P).$$

Suppose that $P|^\alpha Q$. The operator $M_{Q|P}(\pi, s) := M_{Q|P}(\pi, s\varpi)$ admits a normalization by a global factor $n_\alpha(\pi, s)$ which is a meromorphic function in s. We may write

$$M_{Q|P}(\pi, s) \circ j_P = n_\alpha(\pi, s) \cdot j_Q \circ (\mathrm{Id} \otimes R_{Q|P}(\pi, s)) \tag{1}$$

where $R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_v, s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ ([7, §6], cf. [23, (2.17)]).

4.2. **The spectral side of the trace formula.** Arthur's trace formula provides two alternative expressions for a certain distribution J on $G(\mathbb{A})^1$ which depends on the choice of M_0 and $\mathbf{K}$ and is non-invariant (except in the case where G is anisotropic modulo the center). The distribution $J(f)$ is defined for test functions $f \in C_c^\infty(G(\mathbb{A})^1)$. It has a geometric expansion as a sum of contributions of classes of elements of $G(\mathbb{Q})$ with respect to a certain equivalence relation that is weaker than conjugacy and depends in general on the support of f. The contribution of a central element γ is simply $\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) f(\gamma)$, as expected. For our purposes it is better to use the preliminary variants of this expansion contained in Arthur's work. We will not go into details in this sketch, but we will describe the recent reformulation of the spectral side in [16, 18] (which implies its absolute convergence, an issue that had been left unresolved in Arthur's work).

Let $L \supset M$ be Levi subgroups in $\mathcal{L}$, $P \in \mathcal{P}(M)$, and let $m = \dim \mathfrak{a}_L^G$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. We set

$$\mathcal{L}_1(M) = \{M_1 \in \mathcal{L}(M) : \dim \mathfrak{a}_M^{M_1} = 1\}, \quad \mathcal{F}_1(M) = \bigcup_{M_1 \in \mathcal{L}_1(M)} \mathcal{P}(M_1).$$

For any $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee) \in \mathfrak{B}_{P,L}$ let $\mathrm{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by $\underline{\beta}$ and let

$$\begin{aligned}\Xi_L(\underline{\beta}) &= \{(Q_1, \dots, Q_m) \in \mathcal{F}_1(M)^m : \quad \beta_i^\vee \in \mathfrak{a}_M^{Q_i},\ i = 1, \dots, m\} \\ &= \{(\langle P_1, P_1'\rangle, \dots, \langle P_m, P_m'\rangle) : \quad P_i|^{\beta_i} P_i',\ i = 1, \dots, m\}.\end{aligned}$$

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu \in \mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu \in \mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda) = M_{P_2|P_1}(\lambda) D_\varpi M_{P_1|P_2}(\lambda) : \mathcal{A}^2(P_2) \to \mathcal{A}^2(P_2),$$

where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$.

For any m-tuple $\mathcal{X} = (\langle P_1, P_1'\rangle, \dots, \langle P_m, P_m'\rangle) \in \Xi_L(\underline{\beta})$ with $P_i|^{\beta_i} P_i'$, denote by $\Delta_{\mathcal{X}}(P, \lambda)$ the expression

$$\begin{aligned}\frac{\mathrm{vol}(\underline{\beta})}{m!} M_{P_1'|P}(\lambda)^{-1} \delta_{P_1|P_1'}(\lambda) M_{P_1'|P_2'}(\lambda) \cdots \\ \delta_{P_{m-1}|P_{m-1}'}(\lambda) M_{P_{m-1}'|P_m'}(\lambda) \delta_{P_m|P_m'}(\lambda) M_{P_m'|P}(\lambda).\end{aligned}$$

In [18, pp. 179-180] we defined a (purely combinatorial) map $\mathcal{X}_L : \mathfrak{B}_{P,L} \to \mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta}) \in \Xi_L(\underline{\beta})$ for all $\underline{\beta} \in \mathfrak{B}_{P,L}$. (The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ which lies outside a prescribed finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.)

For any $s \in W(M)$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing s. We recall that $\mathfrak{a}_{L_s} = \{H \in \mathfrak{a}_M \mid sH = H\}$. Set $\iota_s = |\det(s-1)_{\mathfrak{a}_M^{L_s}}|^{-1}$. For $P \in \mathcal{P}(M)$ and $s \in W(M)$ let $M(P, s) : \mathcal{A}^2(P) \to \mathcal{A}^2(P)$ be as in [7, p. 1309], a unitary operator which commutes with the operators $\rho(P, \lambda, h)$ for $\lambda \in i\mathfrak{a}_{L_s}^*$. We can now state the refined spectral expansion.

Theorem 3 ([18]). *For any $f \in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$J(f) = \sum_{[M]} J_{\mathrm{spec},M}(f),$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where $J_{\mathrm{spec},M}(f)$ is defined as

$$\frac{1}{|W(M)|}\sum_{s\in W(M)}\iota_s\sum_{\underline{\beta}\in\mathfrak{B}_{P,L_s}}\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*}\mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\lambda)M(P,s)\rho(P,\lambda,f))\,d\lambda$$

with $P\in\mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent.

Note that here the term corresponding to $M=G$ is simply $J_{\mathrm{spec},G}(f)=\mathrm{tr}\,R_{\mathrm{disc}}(f)$.

5. Conditions on intertwining operators

We now formulate two conditions on the behavior of the intertwining operators $M_{Q|P}$ that are necessary for our approach to work. We call these properties (TWN) (for *tempered winding numbers*) and (BD) (for *bounded degree*). The first property is global in nature. The second property is a local one (although strictly speaking it includes a uniformity property in the finite place p). While the first property is directly connected to well-known and only partially solved problems in the theory of automorphic L-functions, the second property seems more accessible. Unfortunately, also here we have only partial results.

Fix a faithful $\mathbb{Q}$-rational representation $\rho: G\to\mathrm{GL}(V)$ and a $\mathbb{Z}$-lattice Λ in the representation space V such that the stabilizer of $\hat{\Lambda}=\hat{\mathbb{Z}}\otimes\Lambda\subset\mathbb{A}_{\mathrm{fin}}\otimes V$ in $G(\mathbb{A}_{\mathrm{fin}})$ is the group $\mathbf{K}_{\mathrm{fin}}$. For any $N\geq 1$ let

$$\mathbf{K}(N)=\{g\in G(\mathbb{A}_{\mathrm{fin}})\;:\;\rho(g)v\equiv v\pmod{N\hat{\Lambda}},\quad v\in\hat{\Lambda}\}$$

be the principal congruence subgroup of level N, an open normal subgroup of $\mathbf{K}_{\mathrm{fin}}$. The groups $\mathbf{K}(N)$ form a neighborhood base of the identity element in $G(\mathbb{A}_{\mathrm{fin}})$. For an open subgroup K of $\mathbf{K}_{\mathrm{fin}}$ let the *level* of K be the minimum integer N with $\mathbf{K}(N)\subset K$. Analogously, define $\mathrm{level}(K_p)$ for open subgroups $K_p\subset\mathbf{K}_p$.

More generally, for any closed algebraic subgroup H of G defined over $\mathbb{Q}$ we define $\mathrm{level}^H(K)$ to be the level of $K\cap H(\mathbb{A}_{\mathrm{fin}})\subset H(\mathbb{A}_{\mathrm{fin}})$ (with respect to $\rho|_H$).

We will use the notation $A\ll B$ to mean that there exists a constant c (independent of the parameters under consideration) such that $|A|\leq cB$.

As in [23], for any $\pi\in\Pi(M(\mathbb{R}))$ we define $\Lambda_\pi=\sqrt{\lambda_\pi^2+\lambda_\tau^2}$, where τ is a lowest $\mathbf{K}_\infty$-type of $\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi)$ and λ_π and λ_τ denote the corresponding Casimir eigenvalues. Roughly speaking, Λ_π measures the size of π. For $M\in\mathcal{L}$, $\alpha\in\Sigma_M$ and $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}))$ let $n_\alpha(\pi,s)$ be the global normalizing factor defined by (1).

Let U_α be the unipotent subgroup of G corresponding to α and let L_α be the group generated by $U_{\pm\alpha}$. It is a closed connected $\mathbb{Q}$-simple normal subgroup of the group $M_\alpha\in\mathcal{L}(M)$ generated by M and $U_{\pm\alpha}$ [10, Proposition 4.11] and $M\cap L_\alpha$ is a maximal Levi subgroup of L_α defined over $\mathbb{Q}$.

Definition 1. We say that the group G satisfies property (TWN) (tempered winding numbers) if for any $M \in \mathcal{L}$, $M \neq G$, and any finite subset $\mathcal{F} \subset \Pi(\mathbf{K}_{M,\infty})$ there exists an integer $k > 1$ such that for any $\alpha \in \Sigma_M$ and any $\epsilon > 0$ we have

$$\int_{i\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \, ds \ll_{\mathcal{F},\epsilon} (1 + \Lambda_{\pi_\infty})^k \left(\mathrm{level}^{M \cap L_\alpha}(K_M)\right)^\epsilon$$

for all open compact subgroups K_M of $\mathbf{K}_{M,\mathrm{fin}}$ and all $\pi = \pi_\infty \otimes \pi_{\mathrm{fin}} \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ such that π_∞ contains a $\mathbf{K}_{M,\infty}$-type in the set $\mathcal{F}$ and $\pi_{\mathrm{fin}}^{K_M} \neq 0$.

Since the normalizing factors $n_\alpha(\pi, s)$ arise from co-rank one situations, property (TWN) is hereditary for Levi subgroups. The known properties of Rankin-Selberg L-functions [24] imply the following result.

Theorem 4. *The groups* $\mathrm{GL}(n)$ *and* $\mathrm{SL}(n)$ *satisfy (TWN).*

Recall that the matrix coefficients of the local normalized intertwining operators $R_{Q|P}(\pi_p, s)^{K_p}$ are rational functions of p^s. Moreover, their denominators can be controlled in terms of π_p, and the degrees of these denominators are bounded in terms of G only. For any Levi subgroup $M \in \mathcal{L}$ let G_M be the closed subgroup of G generated by the unipotent radicals U_P, $P \in \mathcal{P}(M)$. It is a connected semisimple normal subgroup of G [10, Proposition 4.11].

Definition 2. We say that G satisfies property (BD) (bounded degree) if there exists a constant c (depending only on G and ρ), such that for any $M \in \mathcal{L}$, $M \neq G$, and adjacent parabolic subgroups P, $Q \in \mathcal{P}(M)$, any prime p, any open subgroup $K_p \subset \mathbf{K}_p$ and any smooth irreducible representation π_p of $M(\mathbb{Q}_p)$, the degrees of the numerators of the linear operators $R_{Q|P}(\pi_p, s)^{K_p}$ are bounded by $c \log_p \mathrm{level}^{G_M}(K_p)$ if $\mathbf{K}_p$ is hyperspecial, and by $c(1+\log_p \mathrm{level}^{G_M}(K_p))$, otherwise.

Property (BD) is discussed in detail in [19]. It is hereditary for Levi subgroups. The main result of [19] (Theorem 1, taken together with Proposition 6) is the following.

Theorem 5. *The groups* $\mathrm{GL}(n)$ *and* $\mathrm{SL}(n)$ *satisfy (BD).*

The relevance of (BD) to our approach lies in the following consequence.

Proposition 1. *Suppose that G satisfies (BD). Let $M \in \mathcal{L}$ and P, $Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Then for all open subgroups $K \subset \mathbf{K}_{\mathrm{fin}}$ and all $\tau \in \Pi(\mathbf{K}_\infty)$ we have*

$$\int_{i\mathbb{R}} \| R_{Q|P}(\pi, s)^{-1} R'_{Q|P}(\pi, s) \big|_{I_P^G(\pi)^{\tau,K}} \| (1 + |s|^2)^{-1} \, ds$$
$$\ll 1 + \log(1 + \|\tau\|) + \log \mathrm{level}^{G_M}(K).$$

This proposition follows from the generalizations of Bernstein's inequality contained in [11], taking the control of the denominators of the $R_{Q|P}(\pi_p, s)$ into account.

6. Application to the limit multiplicity problem

In this section we will explain our proof strategy for the limit multiplicity problem. It is based on induction over the Levi subgroups of a given reductive group G. However, the suitable inductive property is not the spectral limit property but a property that we call polynomial boundedness (PB). It appears implicitly already in the work of Delorme [14]. We will first explain this property and then outline the proof. Note that the space $C^\infty(G(\mathbb{R})^1)_{c,\mathrm{fin}}$ is the union of spaces $C^\infty(G(\mathbb{R})^1)_{r,\mathcal{F}}$, where $r > 0$ restricts the support of the function and $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$ the possible $\mathbf{K}_\infty$-types.

Definition 3. Let $\mathfrak{M}$ be a set of Borel measures on $\Pi(G(\mathbb{R})^1)$. We call $\mathfrak{M}$ *polynomially bounded* (PB), if there exists $r > 0$ such that for each finite set $\mathcal{F} \subset \Pi(K_\infty)$ the supremum $\sup_{\mu\in\mathfrak{M}}|\mu(\hat{f})|$ is a continuous seminorm on $C^\infty(G(\mathbb{R})^1)_{r,\mathcal{F}}$. (It is equivalent to demand the same condition for all $r > 0$.)

We note that this property is equivalent to a more intuitive boundedness condition which involves the natural partition of $\Pi(G(\mathbb{R})^1)$ into subsets $\Pi(G(\mathbb{R})^1)_{\underline{\delta}}$ parametrized by discrete data $\underline{\delta}$ and the Casimir eigenvalue λ_π of a representation $\pi \in \Pi(G(\mathbb{R})^1)$. Namely, (PB) for a set $\mathfrak{M}$ is equivalent to the condition that for all $\underline{\delta} \in \mathcal{D}$ there exists $N_{\underline{\delta}} > 0$ such that $\mu(\{\pi \in \Pi(G(\mathbb{R})^1)_{\underline{\delta}} : |\lambda_\pi| \le R\}) \ll_{\underline{\delta}} (1+R)^{N_{\underline{\delta}}}$ for all $\mu \in \mathfrak{M}$ and $R > 0$. This equivalence is needed in the proof of the spectral statement below. See [20, §6] for more details.

We first reformulate the main result more generally and put it into the adelic framework. For any open compact subgroup K of $G(\mathbb{A}_{\mathrm{fin}})$ let μ_K^G be the measure on $\Pi(G(\mathbb{R})^1)$ given by

$$\mu_K = \frac{1}{\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K)} \sum_{\pi\in\Pi(G(\mathbb{R})^1)} \dim \mathrm{Hom}_{G(\mathbb{R})^1}(\pi, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K))\, \delta_\pi$$
$$= \frac{\mathrm{vol}(K)}{\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)} \sum_{\pi\in\Pi_{\mathrm{disc}}(G(\mathbb{A})^1)} m_\pi \dim(\pi_{\mathrm{fin}})^K\, \delta_{\pi_\infty}.$$

Theorem 6. *Let G be reductive over $\mathbb{Q}$ and G' be simply connected. Assume that G satisfies properties (TWN) and (BD). Let $\mathfrak{K}$ be a collection of open subgroups $K \subset \mathbf{K}$ such that any non-trivial element of $Z(G)(\mathbb{Q})$ is contained in only finitely many $K \in \mathfrak{K}$ and* $\mathrm{level}^H(K) \to \infty$ *for all connected semisimple $H \lhd G$, $H \neq 1$. Then the measures μ_K, $K \in \mathfrak{K}$, have property (LM).*

We remark that if G is itself semisimple and simply connected and has no $\mathbb{Q}$-simple factor H with $H(\mathbb{R})$ compact, then we have the strong approximation theorem for $G(\mathbb{A}_{\mathrm{fin}})$ (cf. [28, Theorem 7.12]) and can easily pass back to the setup of lattices in $G(\mathbb{R})$. In particular, taking $G = \mathrm{SL}(n)$, we can deduce Theorem 2 from Theorem 6.

We prove the theorem by establishing that $\mu_K(\hat{f}) \to f(1)$ for all test functions $f \in C^\infty_{c,\mathrm{fin}}(G(\mathbb{R})^1)$. For this, we apply Arthur's trace formula to the test functions $f \otimes \mathbf{1}_K$.

The key auxiliary assertion is: the measures μ_K, where $K \subset \mathbf{K}$ ranges over all open subgroups, satisfy property (PB). The proof of this assertion is based on the following geometric and spectral estimates. For each k let $\mathfrak{B}_k$ be a fixed basis of the degree $\leq k$ part of the enveloping algebra of $\operatorname{Lie} G(\mathbb{R})^1 \otimes \mathbb{C}$.

- We have the following *geometric statement:* there exists an integer $k \geq 0$ such that for all $\Omega \subset G(\mathbb{R})^1$ we have

$$|J(f \otimes \mathbf{1}_K)| \ll \sum_{X \in \mathfrak{B}_k} \sup|f * X| \qquad \forall K \subset \mathbf{K}, \ f \in C^\infty(\Omega),$$

  and moreover

$$J_{\mathrm{nc}}(f \otimes \mathbf{1}_K) = J(f \otimes \mathbf{1}_K) - \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) \sum_{\gamma \in Z(G)(\mathbb{Q})} (f \otimes \mathbf{1}_K)(\gamma) \to 0$$

  if $\operatorname{level}^H(K) \to \infty$ for all connected semisimple normal subgroups $H \lhd G$, $H \neq 1$.
- We have the following *spectral statement:* Assume (PB) for $M \neq G$, (TWN) and (BD) for G. Then for all $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$ there exist k and $\varepsilon > 0$ such that:

$$J_{\mathrm{spec},M}(f \otimes \mathbf{1}_K) \ll \operatorname{level}^{G_M}(K)^{-\varepsilon} \sum_{X \in \mathfrak{B}_k} |f * X|_{L^1(G(\mathbb{R})^1)}$$

  for all $f \in C^\infty_{c,\mathcal{F}}(G(\mathbb{R})^1)$. In particular, under these assumptions we have $J_{\mathrm{spec},M}(f \otimes \mathbf{1}_K) \to 0$ if $\operatorname{level}^{G_M}(K) \to \infty$.

We can now prove (PB) for all Levi subgroups $M \in \mathcal{L}$ and the collection $\{\mu_K^M : K \subset \mathbf{K}_M\}$ by induction, assuming (TWN) and (BD) for G.

Let finally $\mathfrak{K}$ be a collection of open subgroups of $\mathbf{K}$ as above, namely such that any non-trivial element of $Z(G)(\mathbb{Q})$ is contained in only finitely many $K \in \mathfrak{K}$ and $\operatorname{level}^H(K) \to \infty$ for all connected semisimple $H \lhd G$, $H \neq 1$. Let $f \in C^\infty_{c,\mathrm{fin}}(G(\mathbb{R})^1)$ be arbitrary. Then we have

$$\frac{1}{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)} J(f \otimes \mathbf{1}_K) \to f(1), \quad K \in \mathfrak{K},$$

because of the geometric statement, while

$$J(f \otimes \mathbf{1}_K) - \operatorname{tr} R_{\mathrm{disc}}(f \otimes \mathbf{1}_K) \to 0$$

because of the spectral statement (applied to all $M \in \mathcal{L}$, $M \neq G$). Taken together we obtain property (LM) for the μ_K, $K \in \mathfrak{K}$.

The proof of the spectral statement is based on the refined spectral expansion of Theorem 3 and the estimates of Definition 1 and Proposition 1. The main point is that the spectral expansion of Theorem 3 involves only first derivatives of the intertwining operators. The proof in [20] deals only with principal congruence subgroups. The necessary group theoretic arguments to deal with the general case will appear in [17].

References

[1] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. On the growth of L^2-invariants for sequences of lattices in lie groups. arXiv:1210.2961.

[2] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. On the growth of Betti numbers of locally symmetric spaces. *C. R. Math. Acad. Sci. Paris*, 349(15-16):831–835, 2011.

[3] James Arthur. A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbf{Q})$. *Duke Math. J.*, 45(4):911–952, 1978.

[4] James Arthur. A trace formula for reductive groups. II. Applications of a truncation operator. *Compositio Math.*, 40(1):87–121, 1980.

[5] James Arthur. The trace formula in invariant form. *Ann. of Math. (2)*, 114(1):1–74, 1981.

[6] James Arthur. On a family of distributions obtained from Eisenstein series. I. Application of the Paley-Wiener theorem. *Amer. J. Math.*, 104(6):1243–1288, 1982.

[7] James Arthur. On a family of distributions obtained from Eisenstein series. II. Explicit formulas. *Amer. J. Math.*, 104(6):1289–1336, 1982.

[8] James Arthur. A measure on the unipotent variety. *Canad. J. Math.*, 37(6):1237–1274, 1985.

[9] James Arthur. On a family of distributions obtained from orbits. *Canad. J. Math.*, 38(1):179–214, 1986.

[10] Armand Borel and Jacques Tits. Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.*, (27):55–150, 1965.

[11] Peter Borwein and Tamás Erdélyi. Sharp extensions of Bernstein's inequality to rational spaces. *Mathematika*, 43(2):413–423 (1997), 1996.

[12] David L. de George and Nolan R. Wallach. Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$. *Ann. of Math. (2)*, 107(1):133–150, 1978.

[13] David L. DeGeorge and Nolan R. Wallach. Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$. II. The tempered spectrum. *Ann. of Math. (2)*, 109(3):477–495, 1979.

[14] Patrick Delorme. Formules limites et formules asymptotiques pour les multiplicités dans $L^2(G/\Gamma)$. *Duke Math. J.*, 53(3):691–731, 1986.

[15] Jacques Dixmier. *Les C^*-algèbres et leurs représentations.* Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars Éditeur, Paris, 1969.

[16] Tobias Finis and Erez Lapid. On the spectral side of Arthur's trace formula—combinatorial setup. *Ann. of Math. (2)*, 174(1):197–223, 2011.

[17] Tobias Finis and Erez Lapid. An approximation theorem for congruence subgroups and applications. In preparation for 2013.

[18] Tobias Finis, Erez Lapid, and Werner Müller. On the spectral side of Arthur's trace formula—absolute convergence. *Ann. of Math. (2)*, 174(1):173–195, 2011.

[19] Tobias Finis, Erez Lapid, and Werner Müller. On the degrees of matrix coefficients of intertwining operators. *Pacific J. Math.*, 260(2):433–456, 2012.

[20] Tobias Finis, Erez Lapid, and Werner Müller. Limit multiplicities for principal congruence subgroups of $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. Preprint, 2012, revised 2013.

[21] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[22] Werner Müller. The trace class conjecture in the theory of automorphic forms. *Ann. of Math. (2)*, 130(3):473–529, 1989.

[23] Werner Müller. On the spectral side of the Arthur trace formula. *Geom. Funct. Anal.*, 12(4):669–722, 2002.

[24] Werner Müller. Weyl's law for the cuspidal spectrum of SL_n. *Ann. of Math. (2)*, 165(1):275–333, 2007.

[25] Morris Newman. A complete description of the normal subgroups of genus one of the modular group. *Amer. J. Math.*, 86:17–24, 1964.
[26] M. Scott Osborne and Garth Warner. *The theory of Eisenstein systems*, volume 99 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.
[27] R. Phillips and P. Sarnak. The spectrum of Fermat curves. *Geom. Funct. Anal.*, 1(1):80–146, 1991.
[28] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.
[29] François Sauvageot. Principe de densité pour les groupes réductifs. *Compositio Math.*, 108(2):151–184, 1997.
[30] Nolan R. Wallach. The spectrum of compact quotients of semisimple Lie groups. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 715–719, Helsinki, 1980. Acad. Sci. Fennica.
[31] Nolan R. Wallach. Limit multiplicities in $L^2(\Gamma\backslash G)$. In *Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989)*, volume 1447 of *Lecture Notes in Math.*, pages 31–56. Springer, Berlin, 1990.

FREIE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK, ARNIMALLEE 3, D-14195 BERLIN, GERMANY

E-mail address: `finis@math.fu-berlin.de`