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GLOBAL COEFFICIENTS IN THE FINE GEOMETRIC EXPANSION OF ARTHUR'S TRACE FORMULA FOR $\mathrm{GL}(n)$

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1. INTRODUCTION

In this note we will give an upper bound for the absolute value of the global coefficients $a^M(\gamma, S)$ appearing in the fine geometric expansion of Arthur's trace formula for $\mathrm{GL}(n)$, and sketch a proof of this bound, see [10] for details. Moreover, we will indicate how this upper bound might be useful in the proof of a Weyl's law for Hecke operators on $\mathrm{GL}(n)$.

Let F be a number field with ring of adeles $\mathbb{A}_F$ and let G be a reductive group defined over F. Arthur's trace formula for G is an identity of distributions

$$J_{\mathrm{geom}}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f) = J_{\mathrm{spec}}(f)$$

between the so-called geometric and spectral side on some space of test functions f (for example smooth and compactly supported functions on $G(\mathbb{A}_F)^1$). Here $\mathcal{O}$ denotes the set of certain equivalence classes which are parametrised by conjugacy classes of semisimple elements in $G(F)$, and $\mathfrak{X}$ is the set of spectral data for $G(F)$, cf. [4]. In [2, Theorem 8.1] Arthur obtains the following fine expansion for $J_{\mathfrak{o}}(f)$: There exist coefficients $a^M(\gamma, S) \in \mathbb{C}$ such that for all $f \in C_c^\infty(G(\mathbb{A}_F)^1)$ we have

$$J_{\mathfrak{o}}(f) = \sum_M \frac{|W^M|}{|W^G|} \sum_\gamma a^M(\gamma, S) J_M^G(\gamma, f) \tag{1}$$

provided that S is a sufficiently large finite set of places of F (with respect to $\mathfrak{o}$ and the support of f). Here M runs over the finite set of Levi subgroups of G containing a fixed minimal Levi subgroup, and $\gamma \in M(F) \cap \mathfrak{o}$ runs over a system of representatives for the so-called (M, S)-equivalence classes [4, §19]. In the case of $G = \mathrm{GL}_n$ this equivalence relation is given by $M(F)$-conjugation and does not depend on S. Further, W^G denotes the Weyl group of G, and the distributions $J_M^G(\gamma, f)$ can be defined as weighted orbital integrals [3].

The coefficients $a^M(\gamma, S)$ depend on the normalisation of measures on the adelic and v-adic points of G and its subgroups. Exact formulas for them are only known in special cases, namely for semisimple γ in arbitrary $M \subseteq G$ by [2, Theorem 8.2], and for arbitrary M and γ in the case of GL_n, SL_n with $n \leq 3$, and $\mathrm{Sp}_2 \subseteq \mathrm{GL}_4$, cf. [7, 5, 6].

Here we will give an upper bound for the absolute value of these coefficients (with respect to some fixed choice of measures) for Levi subgroups of GL_n and arbitrary γ. Such upper bounds are needed - among other things - for establishing asymptotics for traces of Hecke operators on GL_n with uniform error term along the lines of [8]. The main idea how this upper bound might be used in a proof of such a result will be sketched in §5 below.

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2. Main result

We first need to find an upper bound for coefficients for unipotent γ, since by definition [2, (8.1)] the general case is reduced to the unipotent one.

Coefficients for unipotent elements. Fix an integer $n \geq 1$ and let $G = \mathrm{GL}_n$. Let T_0 denote the torus of diagonal elements in G, and U_0 the unipotent subgroup consisting of upper triangular matrices with 1's on the diagonal. Then $P_0 = T_0U_0 \subseteq G$ is a minimal parabolic subgroup and we call an arbitrary parabolic subgroup $P \subseteq G$ standard if $P_0 \subseteq P$. Let $\mathcal{L}$ be the set of all Levi subgroups $M \subseteq G$ containing T_0. If v is a non-archimedean place of F, let $\mathcal{O}_{F_v}$ denote the ring of integers in the local field F_v, and $\mathbf{K}_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ the usual maximal compact subgroup in $G(F_v)$. If v is a real place, we take $\mathbf{K}_v = \mathrm{O}(n)$, and if v is complex, we take $\mathbf{K}_v = \mathrm{U}(n)$ as maximal compact subgroups. We let $\mathbf{K} = \prod_v \mathbf{K}_v \subseteq G(\mathbb{A}_F)$ be the usual maximal compact subgroup in $G(\mathbb{A}_F)$.

We denote by $\mathcal{U}_M$ the variety of unipotent elements in M, and by $\mathfrak{U}^M$ the finite set of M-conjugacy classes on $\mathcal{U}_M$. Let S_∞ denote the set of archimedean places of F. Fix a finite set of places S of F with $S_\infty \subseteq S$, and let $F_S = \prod_{v\in S} F_v$ be the product over the completions F_v of F at v. We write $S_f = S\backslash S_\infty$ for the set of non-archimedean places contained in S. The set of unipotent elements $\mathcal{U}_G \subseteq G$ constitutes exactly one equivalence class $\mathfrak{o}_{\mathrm{unip}} = \mathcal{U}_G(F) \in \mathcal{O}$. The distribution associated with $\mathfrak{o}_{\mathrm{unip}}$ is the unipotent distribution

$$J_{\mathrm{unip}} = J_{\mathfrak{o}_{\mathrm{unip}}} : C_c^\infty(G(\mathbb{A}_F)^1) \longrightarrow \mathbb{C}$$

studied in [1]. Specialising [1, Theorem 8.1] to GL_n, there are uniquely determined numbers $a^M(\mathcal{V}, S) \in \mathbb{C}$ for $\mathcal{V} \in \mathfrak{U}^M$ such that

$$J_{\mathrm{unip}}(f) = \sum_{M\in\mathcal{L}} \frac{|W^M|}{|W^G|} \sum_{\mathcal{V}\in\mathfrak{U}^M} a^M(\mathcal{V}, S) J_M^G(\mathcal{I}_M^G\mathcal{V}, f) \tag{2}$$

holds for all functions $f \in C_c^\infty(G(\mathbb{A}_F)^1)$ of the form $f_S \otimes \mathbf{1}_{\mathbf{K}^S}$ with $f_S \in C_c^\infty(G(F_S)^1)$ and $\mathbf{1}_{\mathbf{K}^S}$ the characteristic function of $\mathbf{K}^S = \prod_{v\notin S} \mathbf{K}_v \subseteq \prod_{v\notin S} G(F_v)$. Here $\mathcal{I}_M^G\mathcal{V}$ denotes the unipotent conjugacy class in $G(F)$ induced from $\mathcal{V}$ and $J_M^G(\mathcal{I}_M^G\mathcal{V}, f) := J_M^G(u, f)$ for some (and hence any) $u \in \mathcal{V}$. The expansion of the general distribution in (1) is a generalisation of this equality.

The absolute value of the constants depends on a choice of measures on the adelic and v-adic points of G and its subgroups. This can already be seen for GL_1: There is only one coefficient and it equals $a^{\mathrm{GL}_1}(1, S) = \mathrm{vol}(F^\times\backslash\mathbb{A}_F^1)$. We choose the usual (resp., twice the usual) Lebesgue measures on F_v if v is real (resp., complex), and normalise the measure such that $\mathrm{vol}(\mathcal{O}_{F_v}) = \mathbb{N}(\mathcal{D}_v)^{-\frac{1}{2}}$ if v is non-archimedean ($\mathbb{N}(\mathcal{D}_v)$ = the norm of the local different of $F_v/\mathbb{Q}_p$). Similarly, we take the usual multiplicative measure on F_v if v is archimedean, and $\zeta_{F,v}(1)|x|_v^{-1}dx_v$ if v is non-archimedean ($\zeta_{F,v}$ = local factor of Dedekind zeta function). The (up to normalisation unique) Haar measure on the maximal compact subgroup is normalised to give it volume 1. By taking the usual coordinates in the torus and the unipotent subgroup of upper triangular matrices, we can define a measure on the minimal standard parabolic subgroup and, using Iwasawa decomposition, also on arbitrary standard parabolic subgroups and $G(F_v)$. Globally, we take the (unnormalised) product measures, and take the measure defined by $1 \to G(\mathbb{A}_F)^1 \to G(\mathbb{A}_F) \xrightarrow{|\det|} \mathbb{R}_{>0} \to 1$ on $G(\mathbb{A}_F)^1$.

With respect to these measures the following bound on the coefficients associated with the unipotent elements can then be proved.

Theorem 1. *Let $n, d \in \mathbb{Z}_{\geq 1}$. There exist non-negative constants $\kappa = \kappa(n,d)$ and $C = C(n,d)$ such that for every number field F of degree $[F:\mathbb{Q}] = d$ and absolute discriminant D_F the following holds: For every finite set of places S of F with $S \supseteq S_\infty$, all $M \in \mathcal{L}$, and all unipotent orbits $\mathcal{V} \in \mathfrak{U}^M$, we have*

$$\left|a^M(\mathcal{V}, S)\right| \leq C D_F^{\kappa} \sum_{\substack{s_v \in \mathbb{Z}_{\geq 0},\ v \in S_f:\\ \sum s_v = \eta}} \prod_{v \in S_f} \left| \frac{\zeta_{F,v}^{(s_v)}(1)}{\zeta_{F,v}(1)} \right| \tag{3}$$

with respect to the measures described above. The sum here runs over tuples of integers $s_v \geq 0$ for $v \in S_f$ such that the sum $\sum_{v \in S_f} s_v$ equals $\eta = \dim \mathfrak{a}_0^M$, the semisimple rank of M.

Remarks.

- The term $\left|\frac{\zeta_{F,v}^{(s_v)}(1)}{\zeta_{F,v}(1)}\right|$ in (3) is of the same order as $\frac{(\log q_v)^{s_v}}{q_v - 1}$ for q_v the cardinality of the residue field of the local field F_v. In particular, the sum over the logarithmic derivatives of the zeta functions in (3) could be replaced by

$$\sum_{\substack{s_v \in \mathbb{Z}_{\geq 0},\ v \in S_f:\\ \sum s_v = \eta}} \prod_{\substack{v \in S_f:\\ s_v > 0}} \frac{(\log q_v)^{s_v}}{q_v - 1}.$$

  However, the examples discussed below suggest that it is more canonical to use the logarithmic derivatives of the zeta function.
- At least for the examples $G = \mathrm{GL}_2$ and $G = \mathrm{GL}_3$ the logarithmic factor is sharp, cf. §4.
- If one keeps track of all constants occurring in the proof of the theorem, one can at least extract a polynomial upper bound for κ in n and d.

It is natural to ask for the minimal possible κ in (3) and the examples in §4 suggest that any $\kappa > 0$ will do. More precisely, we conjecture the following about the actual size of the coefficients.

Conjecture 2. *For every $\kappa > 0$ and all $n, d \in \mathbb{Z}_{\geq 1}$ there exists a constant $C = C(n,d,\kappa) \geq 0$ such that*

(i)

$$\left|a^M(\mathcal{V}, S)\right| \leq C D_F^{\kappa} \sum_{\substack{s_v \in \mathbb{Z}_{\geq 0},\ v \in S_f:\\ \sum s_v = \eta}} \prod_{v \in S_f} \left| \frac{\zeta_{F,v}^{(s_v)}(1)}{\zeta_{F,v}(1)} \right|, \tag{4}$$

(ii)

$$\left|\frac{a^M(\mathcal{V}, S)}{a^{\mathbf{M}_{M,\mathcal{V}}}(\mathbf{1}^{\mathbf{M}_{M,\mathcal{V}}}, S)}\right| \leq C D_F^{\kappa} \sum_{\substack{s_v \in \mathbb{Z}_{\geq 0},\ v \in S_f:\\ \sum s_v = \eta}} \prod_{v \in S_f} \left| \frac{\zeta_{F,v}^{(s_v)}(1)}{\zeta_{F,v}(1)} \right| \tag{5}$$

for all $M \in \mathcal{L}$, $\mathcal{V} \in \mathfrak{U}^M$, all number fields F of degree $[F:\mathbb{Q}] = d$, and all finite set of places S of F with $S \supseteq S_\infty$. Here for a unipotent conjugacy class $\mathcal{V} \in \mathfrak{U}^M$ the Levi subgroup $\mathbf{M}_{M,\mathcal{V}} \subseteq M$ is chosen such that $\mathcal{V} \subseteq M$ is induced from the trivial conjugacy class $\mathbf{1}^{\mathbf{M}_{M,\mathcal{V}}}$ in $\mathbf{M}_{M,\mathcal{V}}$.

Remarks.

- Every unipotent conjugacy class in M is induced in such a way, as in $\mathrm{GL}(n)$ all unipotent conjugacy classes are Richardson classes.
- The denominator $a^{\mathbf{M}_{M,\mathcal{V}}}(\mathbf{1}^{\mathbf{M}_{M,\mathcal{V}}}, S)$ on the left hand side of (5) equals the volume of the quotient $\mathbf{M}_{M,\mathcal{V}}(F)\backslash\mathbf{M}_{M,\mathcal{V}}(\mathbb{A}_F)^1$ (cf. [1, Corollary 8.5]) and is in particular independent of the set S. It is conceivable that the quotient on the left hand side of (5) is independent of the choice of global measure on the various groups involved, but only depends on the local measures.
- If we consider the trivial conjugacy class $\mathbf{1}^M$ in some Levi subgroup $M \in \mathcal{L}$, then the associated Levi subgroup $\mathbf{M}_{M,\mathbf{1}^M}$ is contained in the G-conjugacy class of M. Therefore, Conjecture 2(ii) is trivially true for the trivial conjugacy class in any Levi subgroup because of $a^{\mathbf{M}_{M,\mathcal{V}}}(\mathbf{1}^{\mathbf{M}_{M,\mathcal{V}}}, S) = a^M(\mathbf{1}^M, S)$. The coefficient for the trivial conjugacy class is given by a volume [1, Corollary 8.5], which in our normalisation of measures is given by a product of residues and values at integers of the Dedeking zeta function attached to F. Hence also in this case Conjecture 2(i) holds because of the upper bound provided by the Brauer-Siegel Theorem.
- Conjecture 2(i) also holds for all Levi subgroups and unipotent conjugacy classes in GL_2 and GL_3 (see below). However, all examples so far suggest that the second inequality (5) has the more canonical form.
- Both parts of the conjecture are equivalent if the lower bound of the Brauer-Siegel Theorem holds for F (for example, if F is a normal extension of $\mathbb{Q}$, or if we assume GRH).
- There is a more structural reason, why $a^{\mathbf{M}_{M,\mathcal{V}}}(\mathbf{1}^{\mathbf{M}_{M,\mathcal{V}}}, S)$ should appear as the "main" part of $a^M(\mathcal{V}, S)$: In the cases where an exact formula for the coefficients is known, these coefficients are given in terms of derivatives of certain (local) zeta functions associated with the unipotent orbits. This should be possible more generally, suggesting that there are indeed terms of the form $\sum_{s_v}\prod_v \left|\frac{\zeta_{F,v}^{(s_v)}(1)}{\zeta_{F,v}(1)}\right|$, but also that $a^{\mathbf{M}_{M,\mathcal{V}}}(\mathbf{1}^{\mathbf{M}_{M,\mathcal{V}}}, S)$ should occur naturally in an exact formula for $a^M(\mathcal{V}, S)$.

Coefficients for arbitrary elements. If $\gamma \in M(F)$ is arbitrary, the coefficient $a^M(\gamma, S)$ is defined in terms of coefficients $a^H(u, S)$ for $H \subseteq M$ certain reductive subgroups and $u \in \mathcal{U}_H(F)$ unipotent (see [2, (8.1)]). From our main result we can deduce the following bound for general coefficients. By definition, $a^M(\gamma, S) = 0$ if the semisimple part $\gamma_s \in M(F)$ of γ in its Jordan decomposition is not elliptic in $M(F)$.

Corollary 3. *For any $n, d \in \mathbb{Z}_{\geq 1}$ there exist $\kappa = \kappa(n,d) \geq 0$ and $C = C(n,d) \geq 0$ such that the following holds. Let F and S be as in Theorem 1. Let $M \in \mathcal{L}$ and $\gamma \in M(F)$, and suppose that all eigenvalues of γ are algebraic integers (in some algebraic closure of $\mathbb{Q}$). Further, let $M_1(F) \subseteq M(F)$ be the unique Levi subgroup such that $\gamma_s \in M_1(F)$ is regular elliptic in $M_1(F)$. Then, if γ_s is elliptic in $M(F)$,*

$$|a^M(\gamma, S)| \leq C|\operatorname{discr}^{M_1}(\gamma_s)|_\infty^\kappa \sum_{\substack{s_v \in \mathbb{Z}_{\geq 0},\ v \in S_f:\\ \sum s_v = \eta}} \prod_{v \in S_f} \left|\frac{\zeta_{F,v}^{(s_v)}(1)}{\zeta_{F,v}(1)}\right| \tag{6}$$

with respect to the measures defined above. Here $|\operatorname{discr}^{M_1}(\gamma_s)|_\infty$ is the norm of the discriminant of γ_s in $M_1(F)$ as an element of F over $\mathbb{Q}$, and $\eta = \dim \mathfrak{a}_{M_{1,\gamma_s}}^{M_{\gamma_s}}$.

One can of course also formulate an analogue of Conjecture 2 for arbitrary coefficients.

Remarks.

- Although equality (1) only holds if the set S is sufficiently large with respect to $\mathfrak{o}$ in the sense of [2, p. 203], the coefficients $a^M(\gamma, S)$ are well-defined for any finite set S containing the archimedean places.
- In view of our anticipated application to the Weyl law for Hecke operators, the most important property of the bound (6) is the explicit dependence on the set S and the discriminant of γ, cf. also §5.
- The coefficients are invariant under scaling by scalars, i.e., $a^M(\alpha\gamma, S) = a^M(\gamma, S)$ for every $\alpha \in F^\times$, so that Corollary 3 in fact gives a bound for every $\gamma \in M(F)$.

3. Sketch of proof

The proof of Theorem 1 is by induction on the semisimple rank of M. The initial case is $M = T_0$. So $\mathfrak{U}^{T_0} = \{\mathbf{1}^{T_0}\}$ and $a^{T_0}(\mathbf{1}^{T_0}, S) = \mathrm{vol}(T_0(F)\backslash T_0(\mathbb{A}_F)^1) = (\lambda_{-1}^F)^n$ by [1, Corollary 8.5]) giving our desired bound by the Brauer-Siegel Theorem. Here $\zeta_F(s)$ is the Dedekind zeta function attached to F and $\zeta_F(s) = \lambda_{-1}^F(s-1)^{-1} + \lambda_0^F + \lambda_1^F s + \dots$ its Laurent-expansion around $s = 1$.

For the induction step we use (2) with respect to M instead of G. We rewrite it as

$$(7) \qquad \sum_{\mathcal{V} \in \mathfrak{U}^M} a^M(\mathcal{V}, S) J_M^M(\mathcal{V}, f) = J_{\mathrm{unip}}^M(f) - \sum_{L \subsetneq M} \frac{|W^L|}{|W^M|} \sum_{\mathcal{V} \in \mathfrak{U}^L} a^L(\mathcal{V}, S) J_L^M(\mathcal{I}_L^M \mathcal{V}, f).$$

By the induction hypothesis an upper bound of the desired form for the absolute value of the coefficients $a^L(\mathcal{V}, S)$ appearing on the right hand side is known. To make (7) usable, we first need to choose good test functions.

Choice of test functions. The coefficients on the left hand side can be separated by the use of appropriate test functions: The distributions $J_M^M(\mathcal{V}, f)$ are unweighted orbital integrals, more precisely,

$$J_M^M(\mathcal{V}, f) = \int_{U_\mathcal{V}^M(F_S)} \int_{\mathbf{K}_S} f(k^{-1}xk)dk\, dx,$$

where $P_\mathcal{V}^M = L_\mathcal{V}^M U_\mathcal{V}^M \subseteq M$ is the standard parabolic subgroup in M such that the conjugacy class in $M(F)$ of the orbit of $L_\mathcal{V}^M(F)$ on $U_\mathcal{V}^M(F)$ is $\mathcal{V}$. Such a parabolic subgroup exists, since in $\mathrm{GL}(n)$ (and hence also in every $M \in \mathcal{L}$) every unipotent conjugacy class is a Richardson class. If $f = \prod_{v \in S} f_v \in C_c^\infty(G(F_S))$, the integral giving $J_M^M(\mathcal{V}, f)$ factorises as a product of local orbital integrals $J_{M,v}^M(\mathcal{V}, f_v)$. In particular, if $J_{M,v}^M(\mathcal{V}, f_v) = 0$ for one v, then $J_M^M(\mathcal{V}, f) = 0$. Let $J_{M,\infty}^M(\mathcal{V}, \cdot) = \prod_{v \in S_\infty} J_{M,v}^M(\mathcal{V}, \cdot)$. Note that $F_{S_\infty} \simeq \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ for r_1 the number of real embeddings and r_2 the number of pairs of complex embeddings of F. Now the set of distributions $\{J_{M,\infty}^M(\mathcal{V}, \cdot)\}_{\mathcal{V} \in \mathfrak{U}^M}$ is independent over $C_c^\infty(G(F_{S_\infty}))$. Hence we can fix functions $f_{\mathcal{V},\infty} \in C_c^\infty(G(F_{S_\infty}))$ only depending on the signature (r_1, r_2) of F such that $J_{M,\infty}^M(\mathcal{V}_1, f_{\mathcal{V}_2,\infty}) = 0$ unless $\mathcal{V}_1 = \mathcal{V}_2$ in which case it equals 1. We then define the test functions by $f_\mathcal{V} = f_\mathcal{V} \cdot \mathbf{1}_{\mathbf{K}_{S_f}} \in C_c^\infty(G(F_S))$, $\mathbf{K}_{S_f} := \prod_{v \in S_f} \mathbf{K}_v$, so that $J_M^M(\mathcal{V}_1, f_{\mathcal{V}_2}) = 0$ unless $\mathcal{V}_1 = \mathcal{V}_2$. Thus this set of test functions separates the coefficients on the left hand side of (2).

Estimating the unipotent weighted orbital integrals. To get an upper bound on the absolute value of the coefficients, we still need to find an appropriate upper bound for the right hand side of (7) for the set of test functions just fixed. To that end, one first bounds the weighted orbital integrals $J_L^M(\mathcal{I}_L^M \mathcal{V}_1, f_{\mathcal{V}_2})$ for $\mathcal{V}_1 \in \mathfrak{U}^L$, $\mathcal{V}_2 \in \mathfrak{U}^G$. In contrast to $L = M$, the weighted orbital integrals do not factor as a product of local integrals, but using Arthur's descent formula for (M,L)-families, it suffices to consider the local distributions, which for GL_n are of the form

$$J_{L,v}^{M}(\mathcal{I}_L^M\mathcal{V},f_v)=\int_{U^M_{\mathcal{I}_L^M\mathcal{V}}(F_v)}\int_{\mathbf{K}_v} f(k^{-1}xk)w_{L,\mathcal{V},v}^{M}(x)dk\,dx,$$

where $w_{L,\mathcal{V},v}^{M}:\mathcal{I}_L^M\mathcal{V}(F_v)\longrightarrow\mathbb{C}$ is a certain (logarithmic) weight function described in [3]. Now the fact that $G=\mathrm{GL}(n)$, and hence also M, is $\mathbb{Q}$-split, implies that the weight functions vary "functorially" in non-archimedean v so that we can bound these integrals in a way depending explicitly on F and S for our special test functions $f_{\mathcal{V}_2,v}=\mathbf{1}_{\mathbf{K}_v}$ if v is non-archimedean. The fact that the set of archimedean parts of the test functions depends only on the signature of F (of which there are of course only finitely many for fixed degree $d=[F:\mathbb{Q}]$), allows us to find estimates uniform in F of fixed degree.

Estimating the unipotent contribution. It remains to find an upper bound for $|J^M_{\text{unip}}(f)|$ for f varying in the set of test functions as before. For that one basically uses the ideas of [1], but one has to make the dependence on the field F explicit. This constitutes the main and most technical part of the proof. It involves an explicit version of reduction theory for $\mathrm{GL}(n)$ over number fields and solving a certain lattice point problem over the adele ring of F.

Remark. A more careful choice and analysis of the special test functions $f_{\mathcal{V}}$ might lead to a better upper bound on the contribution of $J_L^M(\mathcal{I}_L^M\mathcal{V},f)$ too the exponent κ in (3). However, it is doubtful that with our methods one can get much closer to Conjecture 2. The main reason is that in order to estimate the global distributions $J^M_{\text{unip}}(f)$ in the last step, we bound integrals over $M(F)\backslash M(\mathbb{A}_F)^1$ by integrals over large compact sets which inevitably leads to the addition of non-trivial powers of D_F on the right hand side of (7).

4. Examples: Coefficients for GL(2) and GL(3)

Coefficients for GL_2. There are two unipotent conjugacy classes in $\mathrm{GL}_2(\mathbb{Q})$: The trivial class $\mathcal{V}_{\text{triv}}=\mathbf{1}^{\mathrm{GL}_2}$, and the regular class $\mathcal{V}_{\text{reg}}$ generated by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Moreover, $\mathcal{L}$ consist of only T_0 and G, and an explicit formula for the coefficients can be found for example in [7, §16]. In particular,

$$a^{T_0}(\mathcal{V}_{\text{triv}},S)=\mathrm{vol}(T_0(F)\backslash T_0(\mathbb{A}_F)^1)=(\lambda_{-1}^F)^2,$$
$$a^{G}(\mathcal{V}_{\text{triv}},S)=\mathrm{vol}(G(F)\backslash G(\mathbb{A}_F)^1)=\lambda_{-1}^F\zeta_F(2).$$

The only non-trivial case belongs to the pair $(G,\mathcal{V}_{\text{reg}})$ in which case, $\mathbf{M}_{\mathrm{GL}_2,\mathcal{V}_{\text{reg}}}=T_0$ and

$$a^{\mathrm{GL}_2}(\mathcal{V}_{\text{reg}},S)=\mathrm{vol}(T_0(F)\backslash T_0(\mathbb{A}_F)^1)\frac{\lambda_0^S}{\lambda_{-1}^S}=a^{T_0}(\mathbf{1}^{T_0},S)\frac{\lambda_0^S}{\lambda_{-1}^S}.$$

Now

$$\frac{\lambda_0^S}{\lambda_{-1}^S}=\frac{\lambda_0^F}{\lambda_{-1}^F}-\sum_{v\in S_f}\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)}=\frac{\lambda_0^F}{\lambda_{-1}^F}+\sum_{v\in S_f}\left|\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)}\right| \tag{8}$$

so that

$$\left|a^G(\mathcal{V}_{\text{reg}},S)\right|=\lambda_{-1}^F\lambda_0^F+(\lambda_{-1}^F)^2\sum_{v\in S_f}\left|\frac{\zeta_{F,v}'(1)}{\zeta_{F,v}(1)}\right|.$$

The coefficients λ_{-1}^F and λ_0^F can be bounded by $\ll_d D_F^\varepsilon$ for every $\varepsilon > 0$ (for λ_{-1}^F this is part of the Siegel-Brauer Theorem, and in general follows from Cauchy's formula and estimates on $\zeta_F(s)$ in the critical strip) so that in this case the first part (4) of Conjecture 2 holds.

Coefficients for GL_3. Up to conjugation, there are three Levi subgroups in $\mathcal{L}$: T_0, M_1, and GL_3, where $M_1 = \mathrm{GL}_2 \times \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_3$ (diagonally embedded). There are three different orbits in $\mathfrak{U}^{\mathrm{GL}_3}$: the trivial conjugacy class $\mathbf{1}^{\mathrm{GL}_3}$, the subregular $\mathcal{V}_{\text{s-r}}$, and the regular conjugacy class $\mathcal{V}_{\text{reg}}$. The relations between the Levi subgroups and the conjugacy classes are as follows:

L	$\mathcal{V} \in \mathfrak{U}^L$	$\mathcal{I}_L^{\mathrm{GL}_3}\mathcal{V} \in \mathfrak{U}^{\mathrm{GL}_3}$	$\mathbf{M}_{L,\mathcal{V}}$
T_0	$\mathbf{1}^{T_0}$	$\mathcal{V}_{\text{reg}}$	T_0
M_1	$\mathbf{1}^{M_1}$	$\mathcal{V}_{\text{s-r}}$	M_1
M_1	$\mathcal{V}_{\text{s-r}}^{M_1}$	$\mathcal{V}_{\text{reg}}$	T_0
GL_3	$\mathbf{1}^{\mathrm{GL}_3}$	$\mathbf{1}^{\mathrm{GL}_3}$	GL_3
GL_3	$\mathcal{V}_{\text{s-r}}$	$\mathcal{V}_{\text{s-r}}$	M_1
GL_3	$\mathcal{V}_{\text{reg}}$	$\mathcal{V}_{\text{reg}}$	T_0

The first, second and fourth case are trivial so that we are left with the remaining cases $\mathcal{V}_{\text{reg}} \subseteq \mathrm{GL}_3$, $\mathcal{V}_{\text{reg}}^{M_1} \subseteq M_1$, and $\mathcal{V}_{\text{s-r}} \subseteq \mathrm{GL}_3$. For them we get from [5, Lemma 4] and [9, Lemma 9] that

$$a^{\mathrm{GL}_3}(\mathcal{V}_{\text{reg}}, S) = \mathrm{vol}(T_0(F)\backslash T_0(\mathbb{A}_F)^1)\left(\left(\frac{\lambda_0^S}{\lambda_{-1}^S}\right)^2 + \frac{\lambda_1^S}{\lambda_{-1}^S}\right) = a^{T_0}(\mathbf{1}^{T_0}, S)\left(\left(\frac{\lambda_0^S}{\lambda_{-1}^S}\right)^2 + \frac{\lambda_1^S}{\lambda_{-1}^S}\right),$$

$$a^{M_1}(\mathcal{V}_{\text{reg}}^{M_1}, S) = \mathrm{vol}(T_0(F)\backslash T_0(\mathbb{A}_F)^1)\frac{\lambda_0^S}{\lambda_{-1}^S} = a^{T_0}(\mathbf{1}^{T_0}, S)\frac{\lambda_0^S}{\lambda_{-1}^S},$$

and

$$a^{\mathrm{GL}_3}(\mathcal{V}_{\text{s-r}}, S) = \mathrm{vol}(M_1(F)\backslash M_1(\mathbb{A}_F)^1)\frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)} = a^{M_1}(\mathbf{1}^{M_1}, S)\frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)}.$$

The second coefficient is already covered by the considerations for GL_2. For the coefficient associated with the subregular conjugacy class in GL_3, we get

$$\left|\frac{a^{\mathrm{GL}_3}(\mathcal{V}_{\text{s-r}}, S)}{\mathrm{vol}(M_1(F)\backslash M_1(\mathbb{A}_F)^1)}\right| = \left|\frac{\zeta_F^{S\prime}(2)}{\zeta_F^S(2)}\right| \ll_d 1$$

so that for this coefficient both parts of Conjecture 2 hold without any condition on the field. For the coefficient associated with the regular conjugacy class in GL_3, the first part of Conjecture 2 follows from a similar computation as (8) by using the upper bounds for the coefficients λ_{-1}^F, λ_0^F, and λ_1^F.

5. Weyl law for Hecke operators

A main motivation for finding an upper bound for the coefficients $a^M(\gamma, S)$ is its prospected applicability in the proof of a Weyl law for Hecke operators on $G = \mathrm{GL}(n)$. We want to sketch the main idea how our bound could be used. We work over $\mathbb{Q}$ from now on and write $\mathbb{A} = \mathbb{A}_\mathbb{Q}$.

Let $\Pi_{\text{cusp}}(G(\mathbb{A})^1)$ denote the set of irreducible unitary representations occurring discretely in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. If $\pi \in \Pi_{\text{cusp}}(G(\mathbb{A})^1)$, then π occurs with multiplicity 1 in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ and can be written as $\pi = \pi_\infty \otimes \pi_f$ with π_∞ (resp., π_f) an irreducible unitary representation of $G(\mathbb{R})$ (resp., $G(\mathbb{A}_f)$). Let $\mathcal{H}_{\pi_\infty}$ (resp., $\mathcal{H}_{\pi_f}$) denote the Hilbert space attached to π_∞ (resp.,

π_f). Let $K_f \subseteq G(\mathbb{A}_f)$ be an open compact subgroup, and let $\sigma \in \widehat{\mathbf{K}_\infty}$ be a $\mathbf{K}_\infty$-type ($\widehat{\mathbf{K}_\infty}$= unitary dual of $\mathbf{K}_\infty$). Let V_σ denote the (finite-dimensional) Hilbert space on which σ acts, and let $\Pi_{\text{cusp}}(G(\mathbb{A})^1)_\sigma$ be the set of all $\pi \in \Pi(G(\mathbb{A})^1)$ for which $(\mathcal{H}_{\pi_\infty} \otimes V_\sigma)^{\mathbf{K}_\infty} \neq 0$, i.e., for which σ is a $\mathbf{K}_\infty$-type of π. Here $(\mathcal{H}_{\pi_\infty} \otimes V_\sigma)^{\mathbf{K}_\infty}$ denotes the space of $\mathbf{K}_\infty$-fixed vectors in $\mathcal{H}_{\pi_\infty} \otimes V_\sigma$. Let $T \in C_c^\infty(K_f\backslash G(\mathbb{A}_f)/K_f)$ be an element of the Hecke algebra. Then one wants to find an asymptotic for the sum

$$(9) \qquad \sum_{\substack{\pi \in \Pi_{\text{cusp}}(G(\mathbb{A})^1)_\sigma \\ \|\lambda_{\pi_\infty}\| \leq X}} \dim \left(\mathcal{H}_{\pi_\infty} \otimes V_\sigma\right)^{\mathbf{K}_\infty} \operatorname{tr} T(\pi_f)$$

as $X \to \infty$, where $\lambda_{\pi_\infty} \in \left(\mathfrak{a}_0^G\right)^*$ denotes the Casimir eigenvalue of π_∞. Moreover, for intended applications of this (namely, to the theory of low-lying zeros of L-functions), a sufficiently good error term in X and $\deg T = \int_{G(\mathbb{A}_f)} |T(x)| dx$ is needed.

The "trivial" Hecke operator. If $T = \mathbf{1}_{K_f}$ is the characteristic function of K_f, an asymptotic for the above equation was found in [11] for arbitrary $\mathbf{K}_\infty$-type σ, but without error term. In [8] an asymptotic with error estimate was proven for $T = \mathbf{1}_{K_f}$ and $\sigma = \text{id}$ under a mild restriction on K_f. More precisely, the main result in [8] says that

$$(10) \qquad \sum_{\substack{\pi \in \Pi_{\text{cusp}}(G(\mathbb{A})^1)_{\text{id}} \\ \lambda_{\pi_\infty} \in t\Omega}} \dim \mathcal{H}_\pi^K = \frac{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_f)}{|W^G|} \int_{t\Omega} \beta(\lambda) d\lambda + O(t^{d-1}(\log t)^{\max\{n,3\}})$$

as $t \to \infty$, where $\Omega \subseteq i\left(\mathfrak{a}_0^G\right)^*$ is a W^G-invariant compact domain with piecewise C^2-boundary, $\mathcal{H}_\pi = \mathcal{H}_{\pi_\infty} \otimes \mathcal{H}_{\pi_f}$, $K = \mathbf{K}_\infty K_f$, $\beta(\lambda)$ is the Plancherel measure belonging to $\mathrm{GL}_n(\mathbb{R})$, and $d = \frac{n(n-1)}{2}$ is the dimension of the symmetric space $G(\mathbb{R})^1/\mathbf{K}_\infty$.

Let us recall the main ideas of the proof of this result from [8]: The main tool is Arthur's trace formula. Thus one has to find a good test function f, or rather a family of test functions f_λ depending on the continuous parameter $\lambda \in \left(\mathfrak{a}_0^G\right)^*$. A natural choice is to take $\mathbf{1}_{K_f}$ as the non-archimedean part and a compactly supported spherical function F_λ as the archimedean part. The support of F_λ can be chosen such that on the geometric side of the trace formula only the unipotent contribution survives, i.e., $J_{\text{geom}}(f_\lambda) = J_{\text{unip}}(f_\lambda)$ for $f_\lambda = F_\lambda \cdot \mathbf{1}_{K_f}$. Hence the trace formula has the form

$$J_{\text{unip}}(f_\lambda) = J_{\text{disc}}(f_\lambda) + J_{\text{non-disc}}(f_\lambda),$$

where $J_{\text{disc}}(f_\lambda)$ (resp., $J_{\text{non-disc}}(f_\lambda)$) denotes the contribution of the discrete (resp. non-discrete) part of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ to the spectral side $J_{\text{spec}}(f_\lambda)$. Then one shows that $\int_{t\Omega} J_{\text{non-disc}}(f_\lambda) d\lambda$ is of order $O(t^{d-1} \log t)$, and the difference of the left hand side of (10) and $\int_{t\Omega} J_{\text{disc}}(f_\lambda) d\lambda$ is of order $O(t^{d-1})$ as $t \to \infty$. Hence the remaining task is to show that as $t \to \infty$

$$(11) \qquad \int_{t\Omega} J_{\text{unip}}(f_\lambda) d\lambda = \frac{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_f)}{|W^G|} \int_{t\Omega} \beta(\lambda) d\lambda + O\left(t^{d-1}(\log t)^{\max\{n,3\}}\right).$$

To this end, one uses the fine expansion of the unipotent part (2), and shows that the integrals $\int_{t\Omega} J_M^G(u, f_\lambda) d\lambda$ are of order $O\left(t^{d-1}(\log t)^{\max\{n,3\}}\right)$ if $(M, u) \neq (G, 1)$. The size of the constants $a^M(u, S)$ is not relevant in this case, since S is fixed and there are only finitely many constants. Noting that the first term on the right hand side of (11) essentially equals the integral $\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) \int_{t\Omega} f_\lambda(1) d\lambda$ then finishes the proof.

General Hecke operators. Again, we want to use Arthur's trace formula to find an asymptotic (with remainder) for (9). The main difference of the above case to our situation of an arbitrary Hecke operator T is that it is no longer possible to choose the archimedean part of the test function such that all but the unipotent contribution on the geometric side vanish. Instead there are finitely many $\mathfrak{o} \in \mathcal{O}$ for which the distributions $J_{\mathfrak{o}}(F_\lambda \cdot T)$ are non-zero, and the set of such $\mathfrak{o}$ depends on T. Let $\mathcal{O}_T \subseteq \mathcal{O}$ denote the set of all such $\mathfrak{o}$. Some of key properties of $\mathfrak{o} \in \mathcal{O}_T$ (e.g., the discriminant or determinant of the elements of $\mathfrak{o}$) can be bounded in terms of the degree $\deg T = \int_{G(\mathbb{A}_f)} |T(x)| dx$ of the Hecke operator. It is not difficult to see that we may assume $K_f = \mathbf{K}_f$, and further by Cartan decomposition that T is equal to the characteristic function of a double coset $\mathbf{K}_f a_T \mathbf{K}_f$ for some diagonal element $a_T \in T_0(\mathbb{Q})$ with integral entries.

Suppose for simplicity that our $\mathbf{K}_\infty$-type is trivial, $\sigma = \mathrm{id}$. We fix a family of spherical test function F_λ, $\lambda \in \left(\mathfrak{a}_0^G\right)^*$, at the archimedean place as before (but without any constraints at its support now), and set $f_\lambda = F_\lambda \cdot T$. It follows similarly as for $T = \mathbf{1}_{K_f}$ that there exist $a, b \geq 0$ such that

$$\int_{t\Omega} J_{\mathrm{spec}}(f_\lambda) d\lambda = \sum_{\substack{\pi \in \Pi_{\mathrm{cusp}}(G(\mathbb{A})^1)_{\mathrm{id}} \\ \lambda_{\pi_\infty} \in t\Omega}} \operatorname{tr} T(\pi_f) + O\big((\deg T)^a t^{d-1} (\log t)^b\big)$$

as $t \to \infty$. The main problem is to analyse the contribution from the geometric side. For this purpose, one wants to use the fine geometric expansion (1) for $J_{\mathfrak{o}}(f_\lambda)$ and $\mathfrak{o} \in \mathcal{O}_T$. However, the expansion (1) only holds if S is sufficiently large with respect to $\mathfrak{o}$. For $\mathfrak{o} \in \mathcal{O}_T$ this can be made more precise as follows: Let S_T consist of all non-archimedean places v at which $T_v \neq \mathbf{1}_{\mathbf{K}_v}$. There exists a finite set of places S_0 independent of T such that $S^T := S_0 \cup S_T$ is sufficiently large in the sense that (1) holds for every $\mathfrak{o} \in \mathcal{O}_T$. Hence,

$$J_{\mathrm{geom}}(f_\lambda) = \sum_{\mathfrak{o} \in \mathcal{O}_T} \sum_{M \in \mathcal{L}} \frac{|W^M|}{|W^G|} \sum_{\gamma \in M(\mathbb{Q}) \cap \mathfrak{o}/\sim} a^M(\gamma, S^T) J_M^G(\gamma, f_\lambda),$$

where $\sim$ denotes $M(\mathbb{Q})$-conjugacy. As we want an error term depending on T, we can not assume that the coefficients $a^M(\gamma, S)$ are bounded by an unknown constant anymore, but need to know how the coefficients vary with T. Our assumption on T ensures that only γ with eigenvalues being algebraic integers may contribute non-trivially to the above sum, and there exists a constant $a_1 > 0$ depending only on n such that the absolute value of the discriminant of every such element γ is bounded by $O\big((\deg T)^{a_1}\big)$. In particular, there exist $a_2, c > 0$ depending only on n such that for every such γ we have

$$|a^M(\gamma, S^T)| \leq c (\deg T)^{a_2}.$$

Hence, one is basically left to estimate the weighted orbital integrals $J_M^G(f_\lambda)$, or rather their integral over $\lambda \in t\Omega$ (in a uniform way with respect to $\deg T$). This can be hopefully done by using methods from [12, §6] for the non-archimedean, and from [8] for the archimedean case.

6. Generalisation to other groups

In the proof of Theorem 1 we used several things which are specific to $\mathrm{GL}(n)$. However, some of the problems arising for more general groups might be solved by more careful considerations. We list the main differences and explain where the problem for general groups lie.

$\mathbb{Q}$-split: The only crucial point where we used that GL_n is $\mathbb{Q}$-split, was when we estimated the v-adic weight functions in the weighted orbital integrals. If G is $\mathbb{Q}$-split, the (finite) set of weight functions at the archimedean places v varies "functorially" with v so that basically there are only finitely many weight functions overall. This allows for uniform estimates. However, a slight modification of the argument should still work at least for quasi-split groups.

(G,S)-equivalence classes: A more serious problem is that of the (G,S)-equivalence relation on $\mathcal{U}_G(F)$. Recall that $\gamma_1,\gamma_2\in\mathcal{U}_G(F)$ are (G,S)-equivalent if γ_1 is $G(F_S)$-conjugate to γ_2. For $G=\mathrm{GL}(n)$, it was already pointed out that this is the same as $G(F)$-conjugacy and is in particular independent of S. In general, however, this relation depends on S, and the set of $G(F_S)$-conjugacy classes on $\mathcal{U}_G(F)$ might grow when S becomes larger. Already for $G=\mathrm{SL}(2)$, $F=\mathbb{Q}$, the (G,S)-equivalence and the conjugacy classes do not agree anymore. For the induction step one would now need to use the general equation (1),

$$\sum_{\mathcal{V}\in\mathcal{U}_G(F)/(G,S)} a^G(\mathcal{V},S)J_G^G(\mathcal{V},f)$$

$$=J_{\mathrm{unip}}(f)-\sum_{M\in\mathcal{L},M\neq G}\frac{|W^M|}{|W^G|}\sum_{\mathcal{V}\in\mathcal{U}_M(F)/(M,S)} a^M(\mathcal{V},S)J_M^G(\mathcal{I}_M^G\mathcal{V},f),$$

where $\mathcal{U}_M(F)/(M,S)$ denotes the set of (M,S)-equivalence classes in $\mathcal{U}_M(F)$.

Again, we want to find a finite set of test functions $f_{\mathcal{V}}\in C_c^\infty(G(F_S))$ indexed by equivalence classes $\mathcal{V}\in\mathcal{U}_G(F)/(G,S)$ such that if we plug in $f=f_{\mathcal{V}}$, only the term belonging to $\mathcal{V}$ survives on the left hand side. For $G=\mathrm{GL}(n)$ it was sufficient to choose functions separating the weighted orbital integrals at the archimedean places only. This is in general not sufficient, and one also has to fix test functions at the non-archimedean places in S varying with $\mathcal{V}\in\mathcal{U}_M(F)/(M,S)$. One then also needs to estimate the non-archimedean weighted orbital integrals for these fixed test functions which is considerably more difficuilt compared to finding a (trivial) bound for the non-archimedean weighted orbital integrals over $\mathbf{1}_{\mathbf{K}_{S_f}}$, which was sufficient for $\mathrm{GL}(n)$. Also, the estimation of the unipotent contribution $|J_{\mathrm{unip}}^M(f)|$ will become more difficuilt, since then the lattice point problem will also depend on S.

Richardson classes: To study the weighted orbital integrals $J_L^M(\mathcal{I}_L^M\mathcal{V},f)$, we expressed them as integrals over the unipotent radical of a certain standard parabolic subgroup by using the fact that all unipotent conjugacy classes in $\mathrm{GL}(n)$ are Richardson classes. For a general group, there might exist unipotent conjugacy classes which are not Richardson. However, it should be possible to find a sufficiently explicit form of $J_L^M(\mathcal{I}_L^M\mathcal{V},f)$ as an integral over the unipotent radical of the Jacobson-Morozov parabolic subgroup associated with $\mathcal{I}_L^M\mathcal{V}$ (against a certain measure which has to be determined).

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