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WHITTAKER-FOURIER COEFFICIENTS OF CUSPIDAL REPRESENTATIONS ON CLASSICAL GROUPS

EREZ LAPID AND ZHENGYU MAO

Let $G$ be a quasi-split reductive group defined over a number field $F$. Fix a Borel subgroup $B$ of $G$ defined over $F$ and let $N$ be its unipotent radical. Fix a non-degenerate character of $N(A)$, trivial on $N(F)$. The $\psi_N$-Whittaker coefficient of an automorphic form $\varphi$ on $G(F)\backslash G(A)$ is defined by

$$W^{\psi_N}(\varphi) = \int_{N(F)\backslash N(A)} \varphi(u)\psi_N^{-1}(u) \, du.$$  

When $\varphi$ belongs to a $\psi_N$-generic automorphic cuspidal representation $\pi$ we would like to compare $W^{\psi_N}(\varphi)$ with the Petersson inner product. More precisely, we have a relation

$$|W(\varphi)|^2 = c_\pi \lim_{s=1} \frac{\Delta_S^S(s)}{L^S(s, \pi, \text{Ad})} \int_{N(F_S)}^{\text{st}} (\pi(u)\varphi, \varphi)_{G(F)\backslash G(A)} \psi_N(u)^{-1} \, du$$

where

- $S$ is a sufficiently large finite set of places of $F$ containing all archimedean ones and the places where either $G$, $\pi$ or $\psi_N$ are ramified.
- $\Delta_S^S(s)$ is a certain partial $L$-function which depends only on $G$.
- $L^S(s, \pi, \text{Ad})$ is the partial $L$-function of $\pi$ with respect to the adjoint $L$-function of $L^G$.
- The local integral makes sense by a suitable regularization. (In the $p$-adic case it is simply the integral over a sufficiently large compact open subgroup of $N(F_n)$.)

The constant $c_\pi$ depends on the automorphic realization of $\pi$ as well as the Haar measures chosen. It follows from the Casselman–Shalika formula [CS80] that $c_\pi$ does not depend on $S$. It is convenient to choose the Haar measures by $\text{vol}(G(F)\backslash G(A)) = \text{vol}(N(F)\backslash N(A)) = 1$. The measure on $N(F_S)$ is chosen so that under the decomposition $N(A) = N(F_S) \times N(F^S)$, $\text{vol}(N(F^S) \cap K^S) = 1$ where $K^S$ is a suitable maximal compact subgroup of $G(A^S)$. Implicit here is the assumption that the limit $\lim_{s=1} \frac{\Delta_S^S(s)}{L^S(s, \pi, \text{Ad})}$ exists and is non-zero.

In the case where $G = \text{GL}_n$, the theory of Rankin-Selberg integrals for $\text{GL}_n \times \text{GL}_n$ developed by Jacquet–Piatetski-Shapiro–Shalika (cf. [Jac01, §2]) together with local unfolding shows that $c_\pi = 1$ for any cuspidal representation $\pi$. In the case where $G = \text{SL}_n$ it easily follows that $c_\pi = |X(\bar{\pi})|^{-1}$ if $\pi$ is the $\psi_N$-generic irreducible constituent of the restriction of functions from a cuspidal representation $\bar{\pi}$ of $\text{GL}_n(A)$ to $\text{SL}_n(F)\backslash \text{SL}_n(A)$. Here $X(\bar{\pi})$ is the finite group of Hecke character $\chi$ such that $\bar{\pi} \otimes \chi = \bar{\pi}$.

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Our work concerns the value of $c_{\pi}$ for the identity component of classical groups.
By the works of Ginzburg–Rallis–Soudry [GRS11] and Cogdell–Kim–Piatetski-Shapiro–Shahidi [CKPSS04] the generic cuspidal representations of a classical group $G$ are parameterized by isobaric representations $\Pi = \pi_1 \boxtimes \cdots \boxtimes \pi_k$ of $GL_N$ (with $N$ determined by $G$) where $\pi_1, \ldots, \pi_k$ are distinct cuspidal representations of $GL_{n_1}, \ldots, GL_{n_k}$ of a certain type. Moreover, Ginzburg–Rallis–Soudry give an automorphic realization of such $\pi$’s in terms of $\Pi$ (namely, $\pi$ is the descent of $\Pi$). We conjecture that for this $\pi$ we have $c_{\pi} = 2^{1-k}$ unless $G = SO(2n)$ and all the local components of $\pi_v$ are invariant under $O(2n)$, in which case $c_{\pi} = 2^{2-k}$. This relation is analogous to a conjecture of Ichino–Ikeda [II10]. There is also an analogous conjecture for the metaplectic two-fold cover for $Sp_n(\mathbb{A})$. In this case we expect that

$$|\mathcal{W}(\varphi)|^2 = 2^{-k} \prod_{i=1}^{n} \zeta_F(2i) \frac{L^S(1/2, \Pi)}{L^S(1, \Pi, \text{sym}^2)} \int_{N(F_S)}^\text{st} \langle \pi(u)\varphi, \varphi \rangle_{G(F)\backslash G(A)^1} \psi_N(u)^{-1} \, du$$

where $\Pi$ is as before with all $\pi_i$’s satisfying $L(1/2, \pi_i)L^S(1, \pi_i, \wedge^2) = \infty$ and $\pi$ is the descent of $\Pi$. The case $n = 1$ is essentially a reformulation of a well-known result of Waldspurger [Wal81].

In the cases of odd orthogonal, unitary and metaplectic groups we reduce the conjecture to a local statement. We also have partial results toward the local statement.

References


