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Some remarks on the automorphic spectrum of
the inner forms of $\text{SL}(N)$

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Abstract
In this survey article, we start by reviewing Arthur’s conjectures
for the multiplicities of $L^2$-automorphic representations in the discrete
spectrum. We also give a sketch of the main ideas thereof, as exem-
plified in Arthur’s endoscopic classification for classical groups, and
then discuss its relation with the Hiraga-Saito theory for the group
$\text{SL}(N)$ and its inner forms. This is based on a talk given in the RIMS
workshop “Automorphic Representations and Related Topics”, Kyôto
2013.

1 Multiplicities in the discrete spectrum
Let $F$ be a number field and $\mathbb{A} := \mathbb{A}_F$ its ring of adèles. Fix an algebraic
closure $\bar{F}$ of $F$. We define $\Gamma_F := \text{Gal}(\bar{F}/F)$ and denote its Weil group by
$W_F$. The Weil-Deligne group of $F$ is denoted by $W_F'$.

For a connected reductive $F$-group $G$, one of the main concerns of the
theory of $L^2$-automorphic forms is to study the right regular representation
of $G(\mathbb{A})$ on

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})) \oplus \text{(continuous spectrum)}$$

where $G(\mathbb{A})^1$ is the kernel of the Harish-Chandra homomorphism $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$.

It is known that the discrete part $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$ decomposes into
$\bigoplus \pi m(\pi)\pi$ with multiplicities $m(\pi) < \infty$ for all $\pi = \bigotimes\pi_v$. Our main goal
is the study of $m(\pi)$. In this article, we adopt the usual convention that
the archimedean components of $\pi$ are viewed as Harish-Chandra modules.
Assume hereafter:

1) the existence of the automorphic Langlands group $L_F \to W_F$ (we shall
write $L'_F := W_F \times \text{SU}(2)$);
2) $G$ is quasisplit.

The first assumption is of course too extravagant; we use it only to streamline the exposition. In particular, we can then talk about the A-parameters $\psi: L'_F \rightarrow \mathbf{1}G := \hat{G} \rtimes W_F$. The $\hat{G}$-conjugacy classes of A-parameters are expected to parametrize packets of automorphic representations of $G(\mathbb{A})$.

The internal structure of A-packets is expected to be controlled by the groups

\begin{align*}
S_{\psi} & := \left\{ \hat{g} \in \hat{G} : \hat{g}\psi\hat{g}^{-1} = a \cdot \psi, \ a \in \ker^1(W_F, Z_{\hat{G}}) \right\}, \\
S_{\psi,\text{ad}} & := S_{\psi}/Z_{\hat{G}}, \\
S_{\psi} & := \pi_0(S_{\psi,\text{ad}}, 1).
\end{align*}

The idea is that elements in $S_{\psi}$ gives rise to endoscopic data of $G$ by which $\psi$ factors through.

The group $S_{\psi} \times L'_F$ acts on $\hat{\mathfrak{g}} := \text{Lie}(\hat{G})$, which gives a representation

\[ \tau_{\psi} = \bigoplus_{\alpha} (\lambda_{\alpha} \otimes \mu_{\alpha} \otimes \nu_{\alpha}) \] (decomposition into irreducibles)

where the exterior tensor products are taken with respect to the product $S_{\psi} \times L_F \times SU(2)$. The relevance of these objects are explained as follows.

i) We define a sign character $\epsilon_{\psi}: S_{\psi} \rightarrow \{\pm 1\}$ by setting

\[ \epsilon_{\psi}(x) := \prod_{\alpha} \det(\lambda_{\alpha}(s)) \]

where $s \in S_{\psi}$ projects to $x \in S_{\psi}$, and the index $\alpha$ ranges over those with $\mu_{\alpha}$ symplectic and $\epsilon(\frac{1}{2}, \mu_{\alpha}) = 1$.

ii) It is expected that to $\psi$ is associated an A-packet $\Pi_{\psi}$ of representations of $G(\mathbb{A})$, together with a map

\[ S_{\psi} \times \Pi_{\psi} \rightarrow \mathbb{C}^\times, \quad (x, \pi) \mapsto \langle x, \pi \rangle. \]

iii) Set

\[ m_{\psi}(\pi) := \frac{1}{|S_{\psi}|} \sum_{x \in S_{\psi}} \epsilon_{\psi}(x)\langle x, \pi \rangle. \]

Now we can state Arthur’s conjecture on the multiplicities [1].
Conjecture 1.1. For every admissible irreducible representation $\pi$ of $G(\mathbb{A})$, we have

$$m(\pi) = \sum_{\psi} m_{\psi}(\pi)$$

where $\psi$ ranges over the $\hat{G}$-conjugacy classes of $A$-parameters.

Remark 1.2. We note that in many cases (e.g. the classical groups), this formula is expected to come from a decomposition into direct sums:

$$L_{\text{disc}}^{2}(G(F) \backslash G(A)^{1}) = \bigoplus_{\psi} L_{\psi}^{2}.$$ 

Consequently, every $\pi$ in the discrete $L^{2}$ spectrum should belong to at most one $A$-packet, say that corresponding to $\psi$, and we expect $m(\pi) = m(\psi)$.

2 Known cases

Arthur's conjectures are largely inspired by his study of the trace formula: see [3] for an excellent introduction. Here are a few known cases.

A. For the quasisplit groups $SO(2n+1)$, $Sp(2n)$, this is proved in [5], by using the selfdual irreducible cuspidal automorphic representations of $GL(n)$ as a substitute for the $A$-parameters. In particular, there is no need to assume the existence of $L_{F}$. This is done by realizing these classical groups as elliptic endoscopic groups for the twisted space $GL(n)$.

B. For the quasisplit groups $SO(2n)$, a coarse version "up to outer automorphisms" is proved in [5], in which one can only identify the $O(2n)$-orbits of $\psi$.

C. The case of $U(3)$ is proved earlier by Rogawski [12].

D. Arthur's machine is adopted to the quasisplit unitary groups $U(n)$ by Chung Pang Mok [11]. There is no ambiguity of outer automorphisms.

E. For the group $SL(N)$, Hiraga and Saito [8] have obtained the multiplicity formula for the generic spectrum by using the representations of $GL(N)$ as substitutes of the $A$-parameters as before. They also obtained coarser results for the inner forms of $SL(N)$.

As regards the classical groups $SO$, $Sp$ and $U$, it would be interesting to consider the non-quasisplit cases as well, as alluded in [5, Chapter 9]. Some modifications of the definitions of $S$-groups are needed. The same remark certainly applies to $SL(N)$ and its inner forms.

We will return to these issues later.
3 Arthur’s approach

Grosso modo, Arthur’s approach in [5, Chapter 4] can be summarized by the triad

multiplicity formula

global intertwining relation ——— stable multiplicity formula

in which any two terms imply the third one. The so-called stable multiplicity formula is a stable variant of our objective, the multiplicity formula. It pertains only to quasisplit groups. Note that in the Endoscopic Classification for classical $G$, these three properties are proved altogether in a long interlocking argument.

4 Stable multiplicity formula

Let $S$ be a union of connected components of a reductive $\mathbb{C}$-group; these components generate a group $\langle S \rangle$, whose neutral component is denoted by $S^\circ$. Fix a maximal torus $T$ in $S^\circ$ and set

$$W^\circ := W(S^\circ, T),$$
$$W := W(S, T) = N_S(T)/T.$$ 

As usual, put $\alpha_T := \text{Hom}(X^*(T), \mathbb{R})$ and set

$$W_{\text{reg}} := \{ w \in W : \det(w - 1|\alpha_T) \neq 0 \}.$$ 

Fix a Borel subgroup of $S^\circ$ containing $T$. For each $w \in W$, set

$$\varepsilon(w) := (-1)^{\# \{ \alpha \in \Sigma(S^\circ, T) : \alpha > 0, \alpha w < 0 \}}$$

where $\Sigma(S^\circ, T)$ is the set of roots of $(S^\circ, T)$. We also write $\varepsilon^G(w)$ to emphasize the ambient group $G$. The first goal is to “stabilize” the expression

$$i(S) := \frac{1}{|W^\circ|} \sum_{w \in W_{\text{reg}}} \varepsilon(w)|\det(w - 1)|^{-1}.$$ 

**Theorem 4.1.** There exist unique constants $\sigma(S_1)$ for each connected reductive $\mathbb{C}$-group $S_1$, such that

i) $\sigma(S_1) = \sigma(S_1/Z_1)/|Z_1|$ for every central subgroup $Z_1$, this means in particular that $\sigma(S_1) = 0$ if $S_1$ is not semisimple;
ii) for $S$ as above, we have

$$i(S) = \sum_{s \in S/\mathsf{conj} \atop \#Z(S_s) < \infty} |\pi_0(S_s, 1)|^{-1} \sigma(S_s^o)$$

where $S_s := Z_S(s)$.

Assume hereafter in this section that $G$ is quasisplit. By assuming the local Langlands correspondence and the endoscopic character relations, to each $A$-parameter $\psi$ for $G$ we may attach a stable distribution $f \mapsto f(\psi)$ on $G(\mathbb{A})$. It satisfies $f(\psi) = \prod_v f_v(\psi_v)$ if $f = \prod_v f_v \in C_c^\infty(G(\mathbb{A})^1)$ and $\psi_v$ is the local $A$-parameter deduced from $\psi$.

On the other hand, recall the stable trace formula for $G$, written as

$$I_{\text{disc}}^G(f) = \sum_{G'} \iota(G, G') S_{\text{disc}}^{G'}(f')$$

(cf. [2, §7]), where

- $I_{\text{disc}}^G$ is the discrete part of Arthur’s invariant trace formula for $G$;
- $G'$ ranges over the elliptic endoscopic data of $G$, identified somehow abusively with the associated endoscopic group;
- $\iota(G, G')$ are explicit positive constants;
- $f' \in C_c^\infty(G'(\mathbb{A}))$ is a Langlands-Shelstad transfer of $f$;
- $S_{\text{disc}}^{G'}$ is the discrete part of the stabilized trace formula for $G'$, which is a stable distribution on $G'(\mathbb{A})$.

Remark 4.2. In Arthur’s works, he has to introduce a parameter $t > 0$ and consider the distributions $I_{\text{disc},t}^G$, etc., to ensure absolute convergence. We deliberately omit this technical complication.

Now one can state the conjectural stable multiplicity formula.

Conjecture 4.3. For each $f \in C_c^\infty(G(\mathbb{A})^1)$, we have

$$S_{\text{disc}}^{G}(f) = \sum_{\psi} |S_{\psi}|^{-1} \sigma((S_{\psi,\text{ad}})^o) \varepsilon^G(\psi) f(\psi).$$

As mentioned above, this is essentially proved for the groups $SO$, $Sp$ and $U$. 
5 Intertwining relations

We consider only the local intertwining relation, as the global version [5, Corollary 4.2.1] is simply the product of its local avatars. Our main reference is [4].

Let $F$ be a local field of characteristic zero, $G$ be a connected reductive $F$-group and $M$ a Levi subgroup of $G$. To each A-parameter $\psi : W'_F \times \text{SU}(2) \rightarrow LG$, we may define the groups $S_\psi, S_\psi$. Moreover, if $\psi$ factors through $LM \hookrightarrow LG$, say via $\psi_M : W'_F \times \text{SU}(2) \rightarrow LM$, then by assuming the local Langlands correspondence, we may form the A-packet $\Pi_{\psi_M}$. Let $\sigma \in \Pi_{\psi_M}$ and $w \in N_G(M)(F)$ such that $w \pi := \pi \circ \text{Ad}(w^{-1})$ is isomorphic to $\pi$. Fix such an isomorphism $\pi(w) : w \pi \sim \pi$. The variety $Mw$ becomes a $M$-bitorsor under multiplication by $M$, as $w$ normalizes $M$. That is, $Mw$ is a twisted space in the sense of Labesse [10]. The assignment $\eta Mw \mapsto \pi(\eta)\pi(w)\pi(m')$ gives rise to an irreducible representation of the twisted space $Mw$ (see loc. cit.) Denote it by $\pi_w$.

Assume that $\psi_M$ is invariant under the Weyl element associated to $w$. Then $\psi_M$ can be plugged into the formalism of twisted endoscopy [9] for $Mw$. Define $\Pi_{\psi_M}^w \subset \Pi_{\psi_M}$ to be the $w$-fixed elements in $\Pi_{\psi_M}$.

Let $(M', s, \ldots)$ be an elliptic endoscopic datum of the twisted space $Mw$ by which $\psi_M$ factors through via $\psi' : W'_F \times \text{SU}(2) \rightarrow LM'$. Consider a “lifting” of the elliptic endoscopic datum to $G$, upon replacing $s$ by $s' \in sZ_{\hat{Mw}}^\Gamma_F/Z_{\hat{G}}^\Gamma_F$:

\[
\begin{align*}
G' \longrightarrow (G, \text{inner twist by } \text{Ad}(w)) & \longrightarrow G \text{ untwisted} \\
\text{Levi} & \\
\hat{M}' & \longrightarrow \hat{Mw} \\
\text{Levi}
\end{align*}
\]

where the dashed line means connection via elliptic endoscopic datum. We also assume that an $L$-embedding $LM' \hookrightarrow LG$ is chosen.

Conjecture 5.1. Given a lifting as above, there exists a canonical map

\[ \Delta : \text{transfer factor for } (G', G) \]

\[ \Delta_w : \text{twisted transfer factor for } (M', Mw), \]

and there exist explicit constants $c(\psi_{M,w})$ depending on the choice of an additive character $\theta_F : F \rightarrow \mathbb{C}^\times$, which should satisfy a global product formula, such that

\[ f'(\psi') \rightarrow c(\psi_{M,w}) \sum_{\pi \in \Pi_{\psi_M}^w} \Delta_w(\psi'_w, \pi_w) \text{tr} (R_P(\pi_w, \psi_M)I_P(\pi, f)) \]
for all \( f \in C_c^\infty(G(F)) \) where

- \( \Delta_w(\psi'_w, \pi_w) \) is the spectral transfer factor corresponding to the geometric one \( \Delta_w; \)
- \( I_P(\pi) \) is the normalized parabolic induction with respect to a parabolic subgroup \( P = MU; \)
- \( R_P(\pi_w, \psi_M) \) is the normalized intertwining operator attached to \( \pi_w \in \Pi^w_{\psi_M} \) and \( \theta_F; \)
- \( \psi' \in C_c^\infty(G'(F)) \) is a transfer of \( f. \)

Note that \( R_P(\pi_w, \psi_M) \) and \( \Delta_w(\psi'_w, \pi_w) \) depends on the choice of \( \pi(w) : w\pi \sim \pi. \) But the ambiguities cancel with each other in the final expression. If \( G \) is quasisplit, we can normalize things by Whittaker models.

**Remark 5.2.** (a) For classical groups including the unitary groups, this conjecture can be simplified somehow and is proved in [5, 11]; note that the case of \( \text{SO}(2n) \) is more delicate. (b) By taking \( M = G \) and \( w = 1 \), we revert to the endoscopic character formula for \( A \)-packets:

\[
f'(\psi') = \sum_{\pi \in \Pi^w_{\psi_M}} \Delta(\psi', \pi)f(\pi)
\]

where \( f(\pi) := tr\pi(f). \) (c) This local intertwining relation is used to construct general \( A \)-packets, as well as the relevant character identities, from the “elliptic” ones.

### 6 The work of Hiraga and Saito

The inner forms of \( \text{SL}(N) \) serve as a reality check for Arthur’s conjectures. Let \( F \) be a local or global field of characteristic zero. Let \( D \) be a finite-dimensional central division algebra over \( F. \) Write

\[
N = \dim_F D \cdot n
\]

and consider

\[
G^d := \text{SL}(n, D) \triangleleft \text{GL}(n, D) =: G.
\]

This construction yields all the inner forms of \( \text{SL}(N, F) \triangleleft \text{GL}(N, F). \) A familiar technique for the study of representations of \( G^d \) is to use the restriction from \( G \) to \( G^d. \) The restriction ought be dual to the \( \text{L}-\text{homomorphism} \)

\[
\text{L}G \to \text{L}G^d
\]

in view of the principle of functoriality. This is systematically done in [8], which we recall below.
When $F$ is local, for every admissible irreducible representation $\pi$ of $G(F)$, we define $\Pi_{\pi}$ to be the set of irreducible constituents of $\pi|_{G(F)}$. Note that $\pi|_{G(F)}$ is known to be semisimple of finite length. The finite sets $\Pi_{\pi}$ are our candidates for the A-packets. For those $\pi$ corresponding to a generic representation of $GL(N)$ via Jacquet-Langlands correspondence, Hiraga and Saito (a) related the internal structure of packets in terms of the $S$-groups; (b) established the endoscopic character relations conjectured by Langlands.

When $F$ is global, Hiraga and Saito studied the restriction of cusp forms. For cuspidal representations $\pi = \bigotimes \pi_{v}$ that are locally generic (up to Jacquet-Langlands correspondence), they derived a multiplicity formula à la Arthur, but with some undetermined constant in the non-quasisplit case. They made the assumption that $G^{d}$ is split at every archimedean place. Thanks to [6], this hypothesis is nowadays unnecessary.

One of the technical ingredients thereof is to reduce to the automorphic induction from $GL(\frac{N}{d}, E)$ to $GL(n, D)$, where $E/F$ is a cyclic extension of degree $d$. This reduction hinges on the seemingly folklore connection

\[
\text{Endoscopy of } G^{d} \leftrightarrow \text{Endoscopy of } G \text{ twisted by } a, \text{ for various } a
\]

where $a$ is an element in the continuous cohomology $Z^1(W_{F}, Z_{\hat{G}})$ for $F$ local (resp. $\ker^1(W_{F}, Z_{\hat{G}})$ for $F$ global). The latter box is exactly the case of automorphic induction for $E/F$, where $E/F$ is the cyclic extension corresponding to $a$ by class field theory. The required endoscopic character identities then follow from those of automorphic induction by a “restriction” procedure for endoscopy.

It seems possible to verify Arthur’s conjectures using this formalism: one may try to formulate and verify

- the local intertwining relation for $G^{d}$ or its twisted variant for automorphic induction;

- the stable multiplicity formula for $SL(N)$, which should be relatively easy.

The first obstacle is of course the extension of the local results in [8] to non-generic setting. The upshot is the character relation for automorphic induction of the Speh representations. Professor Hiraga has an unpublished proof for this using Zelevinsky involution (private communication). Granting this, it would be relatively easy to verify Arthur’s conjectures for $G^{d} = SL(N)$ such as the stable multiplicity formula.

For the non-quasisplit case, it may help us to see the necessary modifications for Arthur’s conjectures in the non-quasisplit setting, such as the use
of modified $S$-groups, etc. For example, in the study of local intertwining relations, some phenomena unseen for classical groups might appear for the inner forms of $\text{SL}(N)$, cf. [7].

All these are obviously some immature thoughts. We hope to address the relevant issues in some future papers.

References


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