ON CONFLUENT HYPERGEOMETRIC FUNCTIONS AND REAL ANALYTIC
SIEGEL MODULAR FORMS OF DEGREE 2

TAKUYA MIYAZAKI

We consider a vector-valued version of the confluent hypergeometric functions on the real symplectic groups, [11]. We characterize their vanishing in certain cases in Section 1, and give them another expressions of Fourier-Jacobi type in Section 2. They are applied to study Fourier-Jacobi expansions of certain real analytic Eisenstein series and also to construct a real analytic Siegel modular form.

1. VANISHING OF INTEGRALS

Let $G$ be the real symplectic group of degree $n$ with a maximal compact subgroup $K \simeq U(n)$. We put $\mu_g(i) = Cl + D$ for $g = (C,D) \in G$ with $i = \sqrt{-1}I_n$. Let $\varphi(x)$ be a polynomial on complex symmetric matrices $x \in S(\mathbb{C})$ of size $n$, and let $\ell$ be an even integer. Then we define a function $\varphi_{\ell}(g,s) := \det(\mu_g(i))^{-t_2\ell} \det(\mu_g(i))^{-t_2\ell} \varphi(\mu_g(i))^{1-1\mu_g(i))}$ of $g \in G$ and $s \in \mathbb{C}$. A natural action of $K$ on $S(\mathbb{C})$, and hence on $\varphi(x)$, shows that $\varphi_{\ell}(g,s)$ defines a $K$-finite vector in $I_P(s)$, a degenerate principal series representation induced from the Siegel maximal parabolic subgroup $P$ of $G$.

For a real symmetric nonsingular $n$ by $n$ matrix $B \in S(\mathbb{R})$ we define an integral

$$W_B(g,s)(\varphi_{\ell}) := \int_{S(\mathbb{R})} e(-\text{tr}(Bx)) \varphi_{\ell}(w_2n(x)g,s) dx$$

with $e(t) = e^{2\pi it}$, $w_2 := \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$ and $n(x) := \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix}$. This is the confluent hypergeometric function associated with $\varphi_{\ell}(g,s) \in I_P(s)$.

If the parameter $s$ is specialized to an integer, then $I_P(s)$ will become reducible. In that case we can obtain a vanishing criterion of (1.1) depending on each $\varphi_{\ell}(g,s) \in I_P(s)$ and the signature $(p,q)$ of $B \in S(\mathbb{R})$. A typical example of this can be stated as following. Assume that $n = 2$ and that $\varphi(x)$ belongs to an irreducible $U(2)$-module of highest weight $(r,0)$ with an even integer $r \geq 0$. We understand $\det(x)$ is of weight $(2,2)$. In particular $\varphi_{\ell}(g,s)$ is of weight $(r - \ell, -\ell)$.

**Proposition 1.1** ([6], [14]). Let $n = 2$ and $s = d + 1$ with a positive even integer $d$. Assume that $\varphi$ is of weight $(r,0)$. Then $W_B(g,d+1)(\varphi_{\ell})$ is vanishing in the following cases.

(i) $r - \ell < d$ and $-\ell \leq -d$, and $(p,q) = (2,0)$.

(ii) $r - \ell \geq d$ and $-\ell \leq -d$, and $(p,q) = (0,2)$ or $(2,0)$.

(iii) $r - \ell \geq d$ and $-\ell > -d$, and $(p,q) = (0,2)$.

As the complements we can prove that $W_B(g,d+1)(\varphi_{\ell}) \neq 0$ if $r - \ell \geq d$ and $-\ell \leq -d$, and $(p,q) = (1,1)$, for example. For results in higher degrees, see [7].
Proof. The proof proceeds as follows. It suffices to discuss the vanishing of
\[
\int_{S(\mathbb{R})} e(-\text{tr}(Bx)) \det(\epsilon(x))^{-\frac{d+r-l+1}{2}} \det(\overline{\epsilon(x)})^{-\frac{d-r+l+1}{2}} \varphi(\epsilon(x)^{-1}) dx,
\]
where \( \epsilon(x)^{-1}\overline{\epsilon(x)} = 2\epsilon(x)^{-1} - 1_{2} \).

Then the following lemma is crucial.

Lemma 1.2 (A generalized binomial expansion formula). Assume \( \varphi \) is of weight \((r_{1}, r_{2})\). Then
\[
\varphi(1_{2} + x) = \sum_{(r'_{1}, r'_{2})} \varphi_{r'_{1}, r'_{2}}(x),
\]
where \( \varphi_{r'_{1}, r'_{2}}(x) \) is a polynomial belonging to the \( U(2) \)-module of weight \((\sqrt{r'}, r_{2}')\) with \( 0 \leq r'_{1} \leq r_{1} \) and \( 0 \leq r'_{2} \leq r_{2} \).

This can be proved by constructing a basis of \( U(2) \)-modules by using Jack polynomials of two variables. Then the above binomial expansion is reduced to the corresponding property of Jack polynomials which was established by Lassalle [5], Kaneko [3]. See Yokokawa [14] for details, and [7] for the proof in higher degree case.

According to the lemma, (1.2) can be written as a sum of
\[
\int_{S(\mathbb{R})} e(-\text{tr}(Bx)) \det(\epsilon(x))^{-\frac{d+r-l+1}{2}} \det(\overline{\epsilon(x)})^{-\frac{d-r+l+1}{2}} \varphi_{r',0}(\epsilon(x)^{-1}) dx
\]
with \( r' \leq r \). Each of these integrals can be studied by following the arguments by Shimura [11] and [12], Proposition 3.1. It implies indeed that (1.3) are vanishing for all \( r' \leq r \), if \( r - \ell \geq d \) and \( -\ell \leq -d \) and \((p, q) = (2, 0)\), for example. Thus the vanishing of (1.2) is concluded in this case. On the other hand, (1.2) can be rewritten in another form as
\[
(1.2) = \int_{S(\mathbb{R})} e(-\text{tr}(Bx)) \det(\epsilon(x))^{-\frac{d+r-l+1}{2}} \det(\overline{\epsilon(x)})^{-\frac{d-r+l+1}{2}} \psi(\epsilon(x)^{-1}) dx
\]
with an appropriate \( \psi \) of weight \((r, 0)\). By repeating the previous arguments, this expression yields that (1.2) is vanishing if \( r - \ell \geq d \) and \( -\ell \leq -d \) and \((p, q) = (0, 2)\). This combined with the above gives the assertion in (ii) of the proposition. \( \square \)

2. Expressions of Fourier-Jacobi Type

Let us take \( \varphi = 1 \) of weight \((0, 0)\) for brevity, and put \( s = d + 1 \) and \( \ell = d \) in (1.1). Then we have
\[
(1.1) = \det(a)^{2-d} \int_{S(\mathbb{R})} e(-\text{tr}(B[a]x)) \det(\epsilon(x))^{-\frac{1}{2}} \det(\overline{\epsilon(x)})^{-d-\frac{1}{2}} dx
\]
when \( g = m(a) := \left( \begin{array}{cc} a & 0_{2} \\ 0_{2} & a^{-1} \end{array} \right) \), \( a = \left( \begin{array}{cc} \sqrt{v'} & q/\sqrt{v} \\ 0 & \sqrt{v} \end{array} \right) \in \text{GL}_{2}(\mathbb{R}) \), \( v, v' > 0 \) and \( q \in \mathbb{R} \). Also let us put coordinates on \( x \in S(\mathbb{R}) \) as \( x = \left( \begin{array}{cc} u' & p \\ p & u \end{array} \right) \).

Assume that \( B \) is of the form\( B = \left( \begin{array}{cc} 1 & \ell \\ \ell & 1 \end{array} \right) \left( \begin{array}{cc} 1 & \lambda \\ \lambda & n \end{array} \right) \left( \begin{array}{cc} 1 & \ell \\ \ell & 1 \end{array} \right) \) with \( \lambda = 0 \) or \( \frac{1}{2} \) and \( \ell, n \in \mathbb{Z} \) (index 1) and is nondegenerate.
Proposition 2.1. With the above setting (2.1) is equal to
\[(2\pi v')^{d+1} e^{-2\pi v' (2\pi v') \frac{d+1}{2}} \int_0^\infty e^{-4\pi v' t} \Omega (4\pi |\det(B)| v, 4\pi (\frac{d}{2} + \ell + \lambda)^2 v; \frac{t}{1+t}) (1+t)^{-\frac{1}{2}} t^{d-\frac{1}{2}} dt,\]
when \(\det(B) = n - \lambda^2 < 0\). On the other hand, it is vanishing, when \(\det(B) > 0\). Here we are defining
\[\Omega(x, y; w) := (1-w)^{\frac{1}{2}} \exp(-\frac{x+y}{2}) \sum_{\kappa=0}^{\infty} \frac{\Gamma(\kappa+1)}{\Gamma(d+\frac{1}{2}+\kappa)} L_{\kappa}^{-d}(x) L_{\kappa}^{-d}(y) w^\kappa l\]
with \(|w| < 1\) using the Laguerre polynomials \(L_{\kappa}^{d}(z)\).

We note that \(\nu^{d} e L_{\kappa}^{d}(4\pi |\det(B)| v)\) is the Whittaker functions of the antiholomorphic discrete series representation \(\overline{\pi}_{d+\frac{1}{2}}\) of \(\overline{SL_2}(\mathbb{R})\) of \(SO(2)\)-type (= weight) \(-d-\frac{1}{2}\). Moreover, we should remark the generalized Hille-Hardy formula (Erdélyi [1], Rangarajan [9], and Srivastava [13]):
\[\Omega(x, y; w) = \Gamma(d+\frac{1}{2})^{-1} \exp(-\frac{x+y}{2} \cdot \frac{1+w}{1-w}) \Phi_3(d, d+\frac{1}{2}; \frac{xw}{1-w}, \frac{xyw}{(1-w)^2})\]
where \(\Phi_3(\beta, \gamma, X, Y)\) is an Humbert’s confluent hypergeometric function, [2], Vol. I, p.225, (22). Then we can estimate the right hand side, cf. Shimomura [10], which is essential to verify the convergence of the integral expression in the proposition.

3. A SCALAR VALUED EISENSTEIN SERIES

We can apply the local formula in Proposition 2.1 to study the Fourier-Jacobi expansion of a scalar-valued Eisenstein series. Define at every finite prime \(p\)
\[\Lambda_p(n(x_p)m(a_p)k_p) := |\det(a_p)|_{p}^{d+1}\]
with \(n(x_p)m(a_p) \in P(\mathbb{Q}_p)\) and \(k_p \in G(\mathbb{Z}_p)\), and
\[\Lambda_{\infty}(g_{\infty}) := \det(\mu_{g_{\infty}}(i))^{-\frac{1}{2}} \det(\overline{\mu_{g_{\infty}}(i)})^{-d-\frac{1}{2}}\]
with an even integer \(d \geq 4\). We set \(\Lambda(g) := \Lambda(\gamma_{\infty}) \prod_{p} \Lambda_p(g_p), g \in G(A)\), and define
\[E(g) := \sum_{g \in P(\mathbb{Q}) \setminus G(\mathbb{Q})} \Lambda(g)\]
It is a scalar valued Eisenstein series. We set \(g = n(x_{\infty})m(a_{\infty}) \prod_p k_p\) with \(x_{\infty} = \left(\begin{array}{ll}u' & p \\ p & u\end{array}\right)\) and \(a_{\infty} = \left(\begin{array}{cc} q/\sqrt{v} & 0 \\ \sqrt{v} & \sqrt{v}\end{array}\right)\), and consider the Fourier-Jacobi expansion
\[E(g) = \sum_{m \in \mathbb{Z}} FJ_m(\tau, z; v' + \frac{q^2}{v}) e(mu'), \quad \tau = u+iv, \quad z = p+iq.\]
**Proposition 3.1.** Let \( m = 1 \). Then there exists a family \( \{ \phi_{\kappa}^{\kappa}(\tau, z) \mid \kappa = 0, 1, 2, \ldots \} \) of real analytic Jacobi form of index 1 and weight \(-d\) satisfying the following properties.

(i) \( \phi_{0}^{0}(\tau, z) \) is a skew holomorphic Jacobi Eisenstein series of index 1 and weight \(-d\).

(ii) \( \phi_{\kappa}^{\kappa}(\tau, z) \) is obtained by differentiating \( \phi_{0}^{0}(\tau, z) \) by \( k \) times.

(iii) The generating series

\[
\phi_{1}^{\kappa}(\tau, z; w) := (1 - w)^{1} \sum_{\kappa=0}^{\infty} \frac{\Gamma(\kappa + 1)}{\Gamma(d + \frac{1}{2} + \kappa)} \phi_{1}^{\kappa}(\tau, z) w^{\kappa}, \quad |w| < 1
\]

converges absolutely.

(iv) The coefficient \( FJ_{1}(\tau, z; v' + \frac{z^{2}}{v}) \) of index 1 is equal to

\[
(2\pi v')^{\frac{d+1}{2}} e^{-2\pi v'} \int_{0}^{\infty} e^{-4\pi \nu' t} \phi_{1}^{\Sigma} \left( \tau, z; \frac{t}{1+t} \right) (1+t)^{-\frac{1}{2}d-\frac{1}{2}} dt.
\]

This result refines Kohnen’s limit formula, [4]. Also by applying suitable operator, (3.1) yields a description of every coefficient of a positive index. As concerns the coefficients of negative indices we will meet another ingredient that did not appear in the case of positive index.

### 4. Vector-valued Siegel Modular Forms

One can generalize the results in Section 3 to a vector-valued Eisenstein series. We take a polynomial belonging to the U(2)-module \( V(d) \) of weight (2d, 0) and put

\[
\Lambda_{\infty}(g_{\infty})(\varphi) := \varphi_{d}(g_{\infty}, d+1)
\]

using the notation in Section 1. Then we set \( \Lambda(g)(\varphi) := \Lambda_{\infty}(g_{\infty})(\varphi) \prod_{p} \Lambda_{p}(g_{p}) \) and define

\[
E(g)(\varphi) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Lambda(\gamma g)(\varphi).
\]

This belongs to the U(2)-module of weight \((d, -d)\) according to the right \( K \)-translation.

Proposition 1.1 implies that the Siegel-Fourier expansion of (4.1) is supported on those \( B \) of signature \((1, 1)\), and besides, \((1, 0)\), \((0, 1)\), and \( B = 0_2 \). Now we are concerned with the Fourier-Jacobi expansion. Then it turns out that this vector valued Eisenstein series has suitable symmetry for its coefficients of positive and negative indices and that we can treat them in a parallel way. Indeed, the coefficient of indices \( \pm 1 \) can be described by suitably modifying the expressions (3.1). Besides these, we can also describe the coefficient of index 0, thus the Fourier-Jacobi expansion of \( E(g)(\varphi) \) is understood well explicitly. See [8] for the details.

Our method can be extended to study other Siegel-type Fourier series of degree 2. Keep that \( \varphi \) varies in \( V(d) \) and consider \( W_{B}(g)(\varphi) := W_{B}(g, d+1)(\varphi_{d}) \) defined in (1.1). Besides it, let \( h(\tau) \) be a cusp form of weight \( d + \frac{1}{2} \) for \( \Gamma_{0}(4) \) that corresponds to a normalized cuspidal eigenform of weight \( 2d \) for \( \text{SL}_{2}(\mathbb{Z}) \) by Shimura correspondence. Consider its Fourier expansion

\[
h(\tau) = \sum_{\ell=1}^{\infty} c(\ell) e(\ell \tau).
\]
Let us define

\[(4.2) \quad F(g_{\infty}k; \varphi) := \sum_B F_B(g_{\infty}k)(\varphi) \text{ for } g_{\infty}k \in G(\mathbb{R}) \prod_p G(\mathbb{Z}_p),\]

where the coefficients $F_B(g_{\infty}k; \varphi)$ are determined by

(i) If $D_B := -\det(2B) > 0$, then

$$F_B(g_{\infty}k; \varphi) := \left(\sum_{t|e_B} t^d c\left(\frac{D_B}{t^2}\right)\right) D_B^{-d} W_B(g_{\infty})(\varphi),$$

where $e_B := \gcd(m, r, n)$ for $B = \begin{pmatrix} m & r/2 & n \\ r/2 & n & 0 \end{pmatrix}$ with $m, n, r \in \mathbb{Z}$.

(ii) If $D_B < 0$, or if $\text{rank}(B) = 1$, then $F_B(g_{\infty}k; \varphi) := 0$.

(iii) If $B = 0_2$, then

$$F_{0_2}(g_{\infty}k; \varphi) := \sum_{0 \neq \ell \in \mathbb{Z}} \left(\sum_{t|\ell} t^{d-1} c\left(\frac{\ell^2}{t^2}\right)\right) |\ell|^{-2d} W_{\ell}^P(g_{\infty})(\varphi),$$

where we put

$$W_{\ell}^P(g_{\infty})(\varphi) := \int_0^\infty e(-\ell s) \int_0^\infty \Lambda_\infty(w_1 n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}) m \begin{pmatrix} 0 & 1 \\ 1 & s \end{pmatrix} g_{\infty}) dt ds$$

with $w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$.

The compact group $K \simeq U(2)$ acts on $\{F(g_{\infty}k; \varphi) \mid \varphi \in V(d)\}$ by the right translation, which has the weight $(d, -d)$. Using our local formulas we can rewrite (4.2) into a series of Fourier-Jacobi type and study its transformation property for the action of Jacobi group. Then we get the following result by repeating the argument in the holomorphic case, [15].

**Theorem 4.1** ([8], Theorem 9.4). For every $\varphi \in V(d)$ (4.2) satisfies

$$F(\gamma g_{\infty}k; \varphi) = F(g_{\infty}k; \varphi)$$

for all $\gamma \in \text{Sp}(2, \mathbb{Z})$, thus it defines a real analytic Siegel modular form of degree 2.

**References**


DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY, HIYOSHI, YOKOHAMA 223-8522, JAPAN
E-mail address: miyazaki@math.keio.ac.jp