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ON CONFLUENT HYPERGEOMETRIC FUNCTIONS AND REAL ANALYTIC
SIEGEL MODULAR FORMS OF DEGREE 2

TAKUYA MIYAZAKI

We consider a vector-valued version of the confluent hypergeometric functions on the real symplectic groups, [11]. We characterize their vanishing in certain cases in Section 1, and give them another expressions of Fourier-Jacobi type in Section 2. They are applied to study Fourier-Jacobi expansions of certain real analytic Eisenstein series and also to construct a real analytic Siegel modular form.

1. VANISHING OF INTEGRALS

Let \( G \) be the real symplectic group of degree \( n \) with a maximal compact subgroup \( K \simeq U(n) \). We put \( \mu_g(i) = C i + D \) for \( g = \begin{pmatrix} C & D \\ \ast & \ast \end{pmatrix} \in G \) with \( i = \sqrt{-1}1_n \). Let \( \varphi(x) \) be a polynomial on complex symmetric matrices \( x \in S(\mathbb{C}) \) of size \( n \), and let \( \ell \) be an even integer. Then we define a function \( \varphi_{\ell}(g,s) := \det(\mu_g(i))^{-\ell^2} \det(\overline{\mu_g(i)})^{-\ell^2} \varphi(\mu_g(i)^{-1} \mu_g(\overline{i})) \) of \( g \in G \) and \( s \in \mathbb{C} \). A natural action of \( K \) on \( S(\mathbb{C}) \), and hence on \( \varphi(x) \), shows that \( \varphi_{\ell}(g,s) \) defines a \( K \)-finite vector in \( I_P(s) \), a degenerate principal series representation induced from the Siegel maximal parabolic subgroup \( P \) of \( G \).

For a real symmetric nonsingular \( n \) by \( n \) matrix \( B \in S(\mathbb{R}) \) we define an integral

\[
W_B(g,s)(\varphi_{\ell}) := \int_{S(\mathbb{R})} e^{-\operatorname{tr}(Bx)} \varphi_{\ell}(w_2n(x)g,s) dx
\]

with \( e(t) = e^{2\pi it} \), \( w_2 := \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \) and \( n(x) := \begin{pmatrix} 1_n \\ 0_n \\ 1_n \end{pmatrix} \). This is the confluent hypergeometric function associated with \( \varphi_{\ell}(g,s) \in I_P(s) \).

If the parameter \( s \) is specialized to an integer, then \( I_P(s) \) will become reducible. In that case we can obtain a vanishing criterion of (1.1) depending on each \( \varphi_{\ell}(g,s) \in I_P(s) \) and the signature \((p,q)\) of \( B \in S(\mathbb{R}) \). A typical example of this can be stated as following. Assume that \( n = 2 \) and that \( \varphi(x) \) belongs to an irreducible \( U(2) \)-module of highest weight \((r,0)\) with an even integer \( r \geq 0 \). We understand \( \det(x) \) is of weight \((2,2)\). In particular \( \varphi_{\ell}(g,s) \) is of weight \((r - \ell, -\ell)\).

**Proposition 1.1** ([6], [14]). Let \( n = 2 \) and \( s = d + 1 \) with a positive even integer \( d \). Assume that \( \varphi \) is of weight \((r,0)\). Then \( W_B(g,d+1)(\varphi_{\ell}) \) is vanishing in the following cases.

(i) \( r - \ell < d \) and \( -\ell \leq -d \), and \((p,q) = (2,0)\).

(ii) \( r - \ell \geq d \) and \( -\ell \leq -d \), and \((p,q) = (0,2)\) or \((2,0)\).

(iii) \( r - \ell \geq d \) and \( -\ell > -d \), and \((p,q) = (0,2)\).

As the complements we can prove that \( W_B(g,d+1)(\varphi_{\ell}) \neq 0 \) if \( r - \ell \geq d \) and \( -\ell \leq -d \), and \((p,q) = (1,1)\), for example. For results in higher degrees, see [7].
Proof. The proof proceeds as follows. It suffices to discuss the vanishing of
\[(1.2) \int_{S(\mathbb{R})} e(-\text{tr}(Bx)) \det(\epsilon(x))^{-\frac{d-r+\ell+1}{2}} \det(\epsilon(x))^{-\frac{d-\ell+1}{2}} \varphi(\epsilon(x)^{-1} \overline{\epsilon(x)}) dx, \quad \epsilon(x) = 1_{2} - ix.\]

Here we remark \(\epsilon(x)^{-1} \overline{\epsilon(x)} = 2\epsilon(x)^{-1} - 1_{2}\). Then the following lemma is crucial.

Lemma 1.2 (A generalized binomial expansion formula). Assume \(\varphi\) is of weight \((r_1, r_2)\). Then
\[\varphi(1_{2} + x) = \sum_{(r'_1, r'_2)} \varphi_{r'_1, r'_2}(x),\]
where \(\varphi_{r'_1, r'_2}(x)\) is a polynomial belonging to the \(U(2)\)-module of weight \((r'_1, r'_2)\) with \(0 \leq r'_1 \leq r_1\) and \(0 \leq r'_2 \leq r_2\).

This can be proved by constructing a basis of \(U(2)\)-modules by using Jack polynomials of two variables. Then the above binomial expansion is reduced to the corresponding property of Jack polynomials which was established by Lassalle [5], Kaneko [3]. See Yokokawa [14] for details, and [7] for the proof in higher degree case.

According to the lemma, (1.2) can be written as a sum of
\[(1.3) \int_{S(\mathbb{R})} e(-\text{tr}(Bx)) \det(\epsilon(x))^{-\frac{d+r+\ell+1}{2}} \det(\epsilon(x))^{-\frac{d-r+\ell+1}{2}} \varphi_{r'_1, 0}(\epsilon(x)^{-1}) \psi(\epsilon(x)^{-1} \overline{\epsilon(x)}) dx\]
with \(r'_1 \leq r_1\). Each of these integrals can be studied by following the arguments by Shimura [11] and [12], Proposition 3.1. It implies indeed that (1.3) are vanishing for all \(r'_1 \leq r_1\) and \(-\ell \leq -d\) and \((p, q) = (2, 0)\), for example. Thus the vanishing of (1.2) is concluded in this case. On the other hand, (1.2) can be rewritten in another form as
\[(1.2) = \int_{S(\mathbb{R})} e(-\text{tr}(Bx)) \det(\epsilon(x))^{-\frac{d+r+\ell+1}{2}} \det(\epsilon(x))^{-\frac{d-r+\ell+1}{2}} \varphi_{r'_1, 0}(\epsilon(x)^{-1}) dx\]
with an appropriate \(\psi\) of weight \((r, 0)\). By repeating the previous arguments, this expression yields that (1.2) is vanishing if \(r - \ell \geq d\) and \(-\ell \leq -d\) and \((p, q) = (0, 2)\). This combined with the above gives the assertion in (ii) of the proposition. \(\square\)

2. Expressions of Fourier-Jacobi Type

Let us take \(\varphi = 1\) of weight \((0, 0)\) for brevity, and put \(s = d + 1\) and \(\ell = d\) in (1.1). Then we have
\[(1.1) = \det(a)^{2-d} \int_{S(\mathbb{R})} e(-\text{tr}(B[a]x)) \det(\epsilon(x))^{-\frac{1}{2}} \det(\epsilon(x))^{-d-\frac{1}{2}} dx\]
when \(g = m(a) := \begin{pmatrix} a & 0_2 \\ 0_2 & a^{-1} \end{pmatrix}\), \(a = \left( \begin{array}{cc} \sqrt{v} \\ \sqrt{v'} \end{array} \right) \in \text{GL}_2(\mathbb{R})\), \(v, v' > 0\) and \(q \in \mathbb{R}\). Also let us put coordinates on \(x \in S(\mathbb{R})\) as \(x = \begin{pmatrix} u' & p \\ p & u \end{pmatrix}\).

Assume that \(B\) is of the form \(B = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda n \\ \frac{1}{n} & 1 \end{pmatrix} \) with \(\lambda = 0\) or \(\frac{1}{2}\) and \(\ell, n \in \mathbb{Z}\) (index 1) and is nondegenerate.
Proposition 2.1. With the above setting (2.1) is equal to
\[
(2\pi v')^{d+1} e^{-2\pi v'}(2\pi v)^{d+1} \int_0^\infty e^{-4\pi \sqrt{}}t \Omega(4\pi|\det(B)|v, 4\pi(\frac{q}{v} + \ell + \lambda)^2v; \frac{1+t}{1+t})(1+t)^{-\frac{d}{2}}-dt,
\]
when \(\det(B) = n - \lambda^2 < 0\). On the other hand, it is vanishing, when \(\det(B) > 0\). Here we are defining
\[
\Omega(x,y;w) := (1-w)^{\frac{1}{2}}\exp(-\frac{x+y}{2})\sum_{\kappa=0}^{\infty}\frac{\Gamma(\kappa+1)}{\Gamma(d+\frac{1}{2}+\kappa)}L_{\kappa}^{d-z^{1}}(x)L_{\kappa}^{-z}(y)w^{\kappa},
\]
with \(|w| < 1\) using the Laguerre polynomials \(L_{\kappa}^{v}(z)\).

We note that \(\nu^{d}eL_{\kappa}^{d-z^{1}}(4\pi|\det(B)|v)\) is the Whittaker functions of the antiholomorphic discrete series representation \(\overline{\pi}_{d+^{1}2}\) of \(\overline{SL_{2}}(\mathbb{R})\) of \(SO(2)-\)type (= weight) \(-d-\frac{1}{2}-2\kappa\), and its product with \(\mathcal{V}t_{e^{-2\pi(_{v}^{q}+\ell+\lambda)^{2_{\mathcal{V}}}}L_{\kappa}^{-S}}(4\pi(_{v}^{q}+\ell+\lambda)^{2}v)\), which is of weight \(\frac{1}{2}+2\kappa\), gives the Whittaker function of weight \(-d\) belonging to a discrete series representation of the real Jacobi group. This means that \((2\pi v')^{\frac{d+1}{2}}\Omega(4\pi|\det(B)|v, 4\pi(\frac{q}{v} + \ell + \lambda)^2v; \frac{1+t}{1+t})\) is a generating series of Whittaker functions of weight \(-d\) on the real Jacobi group. Moreover, we should remark the generalized Hille-Hardy formula [Erdélyi 1, Rangarajan 9, and Srivastava 13]:
\[
\Omega(x,y;w) = \Gamma(d+\frac{1}{2})^{-1}\exp(-\frac{x+y}{2}\cdot\frac{1+w}{1-w})\Phi_{3}(d,d+\frac{1}{2};\frac{xw}{1-w}, \frac{xyw}{(1-w)^{2}}),
\]
where \(\Phi_{3}(\beta, \gamma,X, Y)\) is an Humbert’s confluent hypergeometric function, [2], Vol. I, p.225, (22). Then we can estimate the right hand side, cf. Shimomura [10], which is essential to verify the convergence of the integral expression in the proposition.

3. A SCALAR VALUED EISENSTEIN SERIES

We can apply the local formula in Proposition 2.1 to study the Fourier-Jacobi expansion of a scalar-valued Eisenstein series. Define at every finite prime \(p\)
\[
\Lambda_p(n(x_p)m(a_p)k_p) := |\det(a_p)|_{p}^{d+1}
\]
with \(n(x_p)m(a_p) \in P(\mathbb{Q}_p)\) and \(k_p \in G(\mathbb{Z}_p)\), and
\[
\Lambda_{\infty}(g_{\infty}) = \det(\mu_{g_{\infty}}(i))^{-z}\det(\overline{\mu_{g_{\infty}}(i)})^{-d-z}1l
\]
with an even integer \(d \geq 4\). We set \(\Lambda(g) := \Lambda(\infty)\Pi_p\Lambda_p(g_p), g \in G(A)\), and define
\[
E(g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Lambda(\gamma g).
\]
It is a scalar valued Eisenstein series. We set \(g = n(x_{\infty})m(a_{\infty})\Pi_p k_p\) with \(x_{\infty} = \left(\begin{array}{ll}u' & p \\ p & u\end{array}\right)\) and \(a_{\infty} = \left(\begin{array}{ll}q/\sqrt{v} & 0 \\ \sqrt{v} & \sqrt{v}\end{array}\right)\), and consider the Fourier-Jacobi expansion
\[
E(g) = \sum_{m \in \mathbb{Z}} \text{FJ}_m(\tau, z; v' + \frac{q^2}{v})e(mu'), \quad \tau = u + iv, \quad z = p + iq.
\]
Proposition 3.1. Let $m = 1$. Then there exists a family $\{ \phi^\kappa_1(\tau, z) : \kappa = 0, 1, 2, \ldots \}$ of real analytic Jacobi form of index 1 and weight $-d$ satisfying the following properties.

(i) $\phi^0_1(\tau, z)$ is a skew holomorphic Jacobi Eisenstein series of index 1 and weight $-d$.
(ii) $\phi^\kappa_1(\tau, z)$ is obtained by differentiating $\phi^0_1(\tau, z)$ by $k$ times.
(iii) The generating series

$$
\phi^\Sigma_1(\tau, z; w) := (1 - w)^{\frac{1}{2}} \sum_{\kappa=0}^\infty \frac{\Gamma\left(\kappa + 1\right)}{\Gamma\left(d + \frac{1}{2} + \kappa\right)} \phi^\kappa_1(\tau, z) w^\kappa, \quad |w| < 1
$$

converges absolutely.
(iv) The coefficient $\text{FJ}_1(\tau, z; v + \frac{2}{\tau})$ of index 1 is equal to

$$
(2\pi v')^{d+1} e^{-2\pi v'} \int_0^\infty e^{-4\pi t'} \phi_{\Sigma 1}\left(\frac{\tau}{1+t'}, \frac{z}{1+t}\right) \left(1+t\right)^{-\frac{1}{2}d^2 - \frac{1}{2}d} dt.
$$

This result refines Kohnen's limit formula, [4]. Also by applying suitable operator, (3.1) yields a description of every coefficient of a positive index. As concerns the coefficients of negative indices we will meet another ingredient that did not appear in the case of positive index.

4. Vector-valued Siegel modular forms

One can generalize the results in Section 3 to a vector-valued Eisenstein series. We take a polynomial belonging to the $U(2)$-module $V(d)$ of weight $(2d, 0)$ and put

$$
\Lambda_{\infty}(g_\infty)(\varphi) := \varphi_d(g_\infty, d + 1)
$$

using the notation in Section 1. Then we set $\Lambda(g)(\varphi) := \Lambda_{\infty}(g_\infty)(\varphi) \prod_p \Lambda_p(g_p)$ and define

$$
E(g)(\varphi) := \sum_{g \in \Gamma(\mathbb{Q}) \backslash G(\mathbb{Q})} \Lambda(g)(\varphi).
$$

This belongs to the $U(2)$-module of weight $(d, -d)$ according to the right $K$-translation.

Proposition 1.1 implies that the Siegel-Fourier expansion of (4.1) is supported on those $B$ of signature $(1, 1)$, and besides, $(1, 0)$, $(0, 1)$, and $B = 0_2$. Now we are concerned with the Fourier-Jacobi expansion. Then it turns out that this vector valued Eisenstein series has suitable symmetry for its coefficients of positive and negative indices and that we can treat them in a parallel way. Indeed, the coefficient of indices $\pm 1$ can be described by suitably modifying the expressions (3.1). Besides these, we can also describe the coefficient of index 0, thus the Fourier-Jacobi expansion of $E(g)(\varphi)$ is understood well explicitly. See [8] for the details.

Our method can be extended to study other Siegel-type Fourier series of degree 2. Keep that $\varphi$ varies in $V(d)$ and consider $W_B(g)(\varphi) := W_B(g, d + 1)(\varphi_d)$ defined in (1.1). Besides it, let $h(\tau)$ be a cusp form of weight $d + \frac{1}{2}$ for $\Gamma_0(4)$ that corresponds to a normalized cuspidal eigenform of weight $2d$ for $\text{SL}_2(\mathbb{Z})$ by Shimura correspondence. Consider its Fourier expansion

$$
h(\tau) = \sum_{\ell=1}^\infty c(\ell)e(\ell\tau).
$$
Let us define

\[ F(g_{\infty}k; \varphi) := \sum_{B} F_B(g_{\infty}k)(\varphi) \text{ for } g_{\infty}k \in G(\mathbb{R}) \prod_p G(\mathbb{Z}_p), \]

where the coefficients \( F_B(g_{\infty}k; \varphi) \) are determined by

(i) If \( D_B := -\det(2B) > 0 \), then

\[ F_B(g_{\infty}k; \varphi) := \left( \sum_{t | e_B} t^d c \left( \frac{e_B}{t^2} \right) \right) D_B^{1-d} W_B(g_{\infty})(\varphi), \]

where \( e_B := \gcd(m, r, n) \) for \( B = \begin{pmatrix} m & r/2 & n \\ 0 & 0 & 0 \end{pmatrix} \) with \( m, n, r \in \mathbb{Z}. \)

(ii) If \( D_B < 0 \), or if \( \text{rank}(B) = 1 \), then \( F_B(g_{\infty}k; \varphi) := 0. \)

(iii) If \( B = 0_2 \), then

\[ F_{0_2}(g_{\infty}k; \varphi) := \sum_{0 \neq \ell \in \mathbb{Z}} \left( \sum_{t | \ell} t^{d-1} c \left( \frac{\ell}{t^2} \right) \right) |\ell|^{1-2d} W_{\ell}(g_{\infty})(\varphi), \]

where we put

\[ W_{\ell}(g_{\infty})(\varphi) := \int_0^\infty e(-\ell s) \int_0^\infty \Lambda_{\infty} \left( w_1 n \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) m \begin{pmatrix} 0 & 1 \\ 1 & s \end{pmatrix} g_{\infty} \] \[ dtds \]

with \( w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \)

The compact group \( K \cong U(2) \) acts on \( \{ F(g_{\infty}k; \varphi) \mid \varphi \in V(d) \} \) by the right translation, which has the weight \((d, -d)\). Using our local formulas we can rewrite (4.2) into a series of Fourier-Jacobi type and study its transformation property for the action of Jacobi group. Then we get the following result by repeating the argument in the holomorphic case, [15].

**Theorem 4.1** ([8], Theorem 9.4). For every \( \varphi \in V(d) \) (4.2) satisfies

\[ F(\gamma g_{\infty}k; \varphi) = F(g_{\infty}k; \varphi) \]

for all \( \gamma \in \text{Sp}(2, \mathbb{Z}), \) thus it defines a real analytic Siegel modular form of degree 2.

**References**


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