PERIODS OF AUTOMORPHIC FORMS: THE CASE OF $(GL_{n+1} \times GL_n, GL_n)$ (Automorphic Representations and Related Topics)

Author(s): ICHINO, ATSUSHI

Citation: 数理解析研究所講究録 (2013), 1871: 1-5

Issue Date: 2013-12

URL: http://hdl.handle.net/2433/195476

Type: Departmental Bulletin Paper

Publisher: Kyoto University
PERIODS OF AUTOMORPHIC FORMS:
THE CASE OF $(GL_{n+1} \times GL_n, GL_n)$

市野 篤史 (ICHINO, ATSUSHI)

This note is a report on a joint work with Shunsuke Yamana [2].
Details will appear elsewhere.

Let $G$ be a connected reductive algebraic group over a number field $F$ and $G'$ a closed subgroup of $G$ over $F$. Let $\mathcal{A}(G)$ and $\mathcal{A}(G')$ denote the spaces of automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. We will consider the period integral

$$P^{G'}(\varphi \otimes \varphi') := \int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g) \varphi'(g) dg$$

for $\varphi \in \mathcal{A}(G)$ and $\varphi' \in \mathcal{A}(G')$. Let $\pi \subset \mathcal{A}(G)$ and $\pi' \subset \mathcal{A}(G')$ be irreducible subrepresentations. If $P^{G'}(\varphi \otimes \varphi')$ converges for all $\varphi \in \pi$ and $\varphi' \in \pi'$, then

$$P^{G'}|_{\pi \otimes \pi'} \in \text{Hom}_{\triangle G'(\mathbb{A})}(\pi \otimes \pi', \mathbb{C})$$

We say that $\pi \otimes \pi'$ is $\Delta G'$-distinguished (with respect to $P^{G'}$) if $P^{G'}|_{\pi \otimes \pi'} \neq 0$.

In this note, we consider the case $G = GL_{n+1}$ and $G' = GL_n$, which was studied by Jacquet, Piatetski-Shapiro and Shalika.

**Theorem 1** (Jacquet-Piatetski-Shapiro-Shalika). If $\varphi \in \mathcal{A}^{\text{cusp}}(G)$ and $\varphi' \in \mathcal{A}^{\text{cusp}}(G')$, then

$$P^{G'}(\varphi \otimes \varphi'_s) = I(s, \varphi, \varphi') := \int_{N'(\mathbb{A}) \backslash G'(\mathbb{A})} W^{\psi}(g, \varphi) W^{\bar{\psi}}(g, \varphi') |\det g|^s \, dg$$

Here, $\mathcal{A}^{\text{cusp}}(G)$ and $\mathcal{A}^{\text{cusp}}(G')$ denote the spaces of cusp forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively, $\varphi'_s = \varphi' \cdot |\det|^s$ for $s \in \mathbb{C}$, $N \subset G$ and $N' \subset G'$ are upper triangular unipotent subgroups, $W^\psi(g, \varphi)$ is a Whittaker function (with respect to a nontrivial character $\psi$ of $F\backslash \mathbb{A}$) defined by

$$W^\psi(g, \varphi) = \int_{N(F) \backslash N(\mathbb{A})} \varphi(ug) \bar{\psi}(u_{1,2} + u_{2,3} + \cdots + u_{n,n+1}) \, du$$

and $W^{\bar{\psi}}(g, \varphi')$ is defined similarly. The left-hand side converges for all $s$ and the right-hand side converges for $\Re s \gg 0$. Moreover, if $\varphi = \ldots$
\( \otimes_v \varphi_v \in \pi \subset \mathcal{A}^{cusp}(G) \) and \( \varphi' = \otimes_v \varphi_v' \in \pi' \subset \mathcal{A}^{cusp}(G') \), then

\[
I(s, \varphi, \varphi') = L \left( s + \frac{1}{2}, \pi \times \pi' \right) \prod_v \frac{I(s, W_{\varphi_v}^{\psi_v}, W_{\varphi_v'}^{\overline{\psi}_v})}{L \left( s + \frac{1}{2}, \pi_v \times \pi'_v \right)}.
\]

In particular, \( \pi \otimes \pi' \) is \( \Delta G' \)-distinguished if and only if

\[
L \left( \frac{1}{2}, \pi \times \pi' \right) \neq 0.
\]

The last assertion is a special case of the Gan-Gross-Prasad conjecture [1]. We also remark that \( I(s, \varphi, \varphi') \) makes sense for any automorphic forms \( \varphi \) and \( \varphi' \). Our main result is an extension of the above theorem.

**Theorem 2 (I-Yamana).** Let \( \varphi \in \mathcal{A}(G) \) and \( \varphi' \in \mathcal{A}(G') \). Then

\[
P_{\text{reg}}^{G'}(\varphi \otimes \varphi_s') = I(s, \varphi, \varphi')
\]

as meromorphic functions of \( s \). Here, \( P_{\text{reg}}^{G'} \) is the regularized period integral defined below.

As immediate consequences, we obtain the following corollaries.

**Corollary 3.**

1. \( P_{\text{reg}}^{G'} \) is \( \Delta G'(\mathbb{A}) \)-invariant.
2. \( P_{\text{reg}}^{G'}(\varphi \otimes \varphi_s') = 0 \) unless \( \varphi \) and \( \varphi' \) are generic.

**Corollary 4.** Assume that \( \pi \) and \( \pi' \) are induced from irreducible cuspidal automorphic representations of Levi subgroups of \( G \) and \( G' \) respectively. Then \( \pi \otimes \pi' \) is \( \Delta G' \)-distinguished (with respect to \( P_{\text{reg}}^{G'} \)) if and only if

\[
L \left( \frac{1}{2}, \pi \times \pi' \right) \neq 0.
\]

**Corollary 5.** Let \( \varphi \in \pi \subset \mathcal{A}^{\text{disc}}(G) \) and \( \varphi' \in \pi' \subset \mathcal{A}^{\text{disc}}(G') \). Here, \( \mathcal{A}^{\text{disc}}(G) \) and \( \mathcal{A}^{\text{disc}}(G') \) denote the spaces of square integrable automorphic forms on \( G(\mathbb{A}) \) and \( G'(\mathbb{A}) \) respectively. Assume that \( \pi \) is not \( 1 \)-dimensional. Then \( P_{\text{reg}}^{G'}(\varphi \otimes \varphi') \) converges and

\[
P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = 0
\]

unless \( \pi \) and \( \pi' \) are cuspidal.

The original motivation was to study the Gan-Gross-Prasad conjecture in the non-tempered case. We also expect an application to the spectral expansion of the relative trace formula of Jacquet-Rallis [4]. In what follows, we will explain the definition of \( P_{\text{reg}}^{G'} \) and the proof of Theorem 2.
Following Jacquet, Lapid and Rogawski [3], we define $P_{\text{reg}}^{G'}$. The construction is based on truncation. Recall that Arthur’s truncation is given by

$$\Lambda^{T}(\varphi) = \sum_{P}(-1)^{\dim \mathfrak{a}_{P}} \sum_{\gamma \in P \backslash G} \varphi_{P}(\gamma g) \hat{\tau}_{P}(H_{P}(\gamma g) - T),$$

which is rapidly decreasing. Here, $P = MU$ is a standard parabolic subgroup of $G$, $\varphi_{P}$ is the constant term of $\varphi$ along $P$, $\mathfrak{a}_{P} = \text{Hom}(X^{*}(M), \mathbb{R})$, $\mathfrak{a}_{P}^{*} = X^{*}(M) \otimes \mathbb{R}$, $\mathfrak{a}_{P} = \mathfrak{a}_{P}^{G} \oplus \mathfrak{a}_{G}$ is the canonical decomposition, $H_{P} : G(\mathbb{A}) \to \mathfrak{a}_{P}$ is a function such that $e^{\langle \chi, H_{P}(m) \rangle} = |\chi(m)|_{\mathbb{A}}$ for $\chi \in X^{*}(M)$, $m \in M(\mathbb{A})$ and extended by the Iwasawa decomposition, $T \in \mathfrak{a}_{0}^{G} = \mathfrak{a}_{B}^{G}$ is sufficiently positive with the standard Borel subgroup $B$, and $\hat{\tau}_{P}$ is the characteristic function of the obtuse cone in $\mathfrak{a}_{P}$ spanned by coroots. The integral $P^{G'}(\Lambda^{T}(\varphi \otimes \varphi))$ converges but is hard to compute. Thus we adopt more suitable “mixed truncation” given by

$$\Lambda_{m}^{T}(\varphi) = \sum_{P}(-1)^{\dim \mathfrak{a}_{P}} \sum_{\gamma \in P \backslash PWG'} \varphi_{P}(\gamma g) \hat{\tau}_{P}(H_{P}(\gamma g) - T),$$

where $W$ is the Weyl group of $G$.

**Lemma 6.**

1. $\Lambda_{m}^{T}(\varphi)$ is rapidly decreasing on $G'(F) \backslash G'(\mathbb{A})$.
2. $P^{G'}(\Lambda_{m}^{T}(\varphi \otimes \varphi)) = \sum_{\lambda} p_{\lambda}(T)e^{\langle \lambda, T \rangle}$, where the right-hand side is a finite sum with $\lambda \in (\mathfrak{a}_{0,\mathbb{C}}^{G})^{*}$ and $p_{\lambda} \in \mathbb{C}[\mathfrak{a}_{0}]$.

We define

$$P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = p_{0}(T)$$

if the exponents of $\varphi$ and $\varphi'$ avoid some finitely many hyperplanes. It turns out that $p_{0}(T)$ is constant, i.e., independent of $T$. If $\varphi \in \mathcal{A}_{\text{cusp}}(G)$, then $\Lambda_{m}^{T}(\varphi) = \varphi$, so that $P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = P^{G'}(\varphi \otimes \varphi')$. This identity holds more generally if the exponents of $\varphi$ and $\varphi'$ satisfy some finitely many negativity conditions. We can define $P_{\text{reg}}^{G'}(\varphi \otimes \varphi_{s}')$ for generic $s$ and obtain a meromorphic function of $s$.

Following Lapid and Rogawski [5], we prove Theorem 2. We may assume that $\varphi$ is a cuspidal Eisenstein series. We want to unfold $P^{G'}(\varphi \otimes \varphi')$ by using the Fourier expansion

$$\varphi(g) = \sum_{i=0}^{n} \sum_{\gamma \in P_{i}' \backslash G'} W_{Q_{i}}^{\psi}(\gamma g, \varphi_{Q_{i}}).$$
Here,

\[ Q_i = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}^{i} \right\}_{n-1-i} \subset G, \]

\[ P'_i = \left\{ \begin{pmatrix} * & * \\ 0 & \nabla \end{pmatrix}^{i} \right\}_{n-i} \subset G', \]

and \( W^\psi_{Q_i} \) is the Whittaker function for the \( GL_{n+1-i} \) part. If \( \varphi \in \mathscr{A}^{cusp}(G) \), then only the term \( i = 0 \) survives. Since \( P'_0 = N' \) and \( W^\psi_{Q_0} = W^\psi \), we can unfold \( P^G' (\varphi \otimes \phi') \) to get \( I(s, \varphi, \varphi') \). In general, we cannot unfold. Instead, we compute the convergent integral \( P^G' (\theta_\phi \otimes \varphi') \) in two ways. Here, \( \phi(\lambda) = f(\lambda) \cdot \varphi \) for \( \lambda \in (a^n_{P,\mathbb{C}})^* \) with \( f \in \mathcal{P}\mathcal{W}((a^n_{P,\mathbb{C}})^*) \) and \( \varphi \in \mathscr{A}^{cusp}_P(G) \), \( \theta_\phi \) is a pseudo Eisenstein series given by

\[
\theta_\phi(g) = \int_{\Re \lambda = \kappa} f(\lambda) E(g, \varphi, \lambda) \, d\lambda
\]

with sufficiently positive \( \kappa \in (a^n_{P,\mathbb{C}})^* \) and an Eisenstein series

\[
E(g, \varphi, \lambda) = \sum_{\gamma \in P \backslash G} \varphi(\gamma g) e^{\langle \lambda, H_P(\gamma g) \rangle}.
\]

We can show that

\[
P^G' (\theta_\phi \otimes \varphi'_s) = \int_{\Re \lambda = \kappa} f(\lambda) P^G'_{reg} (E(\varphi, \lambda) \otimes \varphi'_s) \, d\lambda
\]

under some mild condition of \( f \). We can unfold \( P^G' (\theta_\phi \otimes \varphi'_s) \) to get

\[
\int_{\Re \lambda = \kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi') \, d\lambda + \sum_{i=1}^{n} \cdots
\]

where the last sum vanishes under another mild condition of \( f \). The upshot is

\[
\int_{\Re \lambda = \kappa} f(\lambda) P^G'_{reg} (E(\varphi, \lambda) \otimes \varphi'_s) \, d\lambda = \int_{\Re \lambda = \kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi') \, d\lambda
\]

for sufficiently many \( f \) which allows us to extract the desired identity.

REFERENCES


DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA OIWAKE-CHO, SAKYO-KU, KYOTO 606-8502, JAPAN

E-mail address: ichino@math.kyoto-u.ac.jp