<table>
<thead>
<tr>
<th>Title</th>
<th>PERIODS OF AUTOMORPHIC FORMS: THE CASE OF (GL$_{n+1} \times$ GL$_n$,GL$_n$) (Automorphic Representations and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
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PERIODS OF AUTOMORPHIC FORMS:
THE CASE OF \((\text{GL}_{n+1} \times \text{GL}_n, \text{GL}_n)\)

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This note is a report on a joint work with Shunsuke Yamana [2]. Details will appear elsewhere.

Let \(G\) be a connected reductive algebraic group over a number field \(F\) and \(G'\) a closed subgroup of \(G\) over \(F\). Let \(\mathcal{A}(G)\) and \(\mathcal{A}(G')\) denote the spaces of automorphic forms on \(G(\mathbb{A})\) and \(G'(\mathbb{A})\) respectively. We will consider the period integral

\[
P^{G'}(\varphi \otimes \varphi') := \int_{G'(F) \backslash G'(\mathbb{A})} \varphi(g)\varphi'(g) \, dg
\]

for \(\varphi \in \mathcal{A}(G)\) and \(\varphi' \in \mathcal{A}(G')\). Let \(\pi \subset \mathcal{A}(G)\) and \(\pi' \subset \mathcal{A}(G')\) be irreducible subrepresentations. If \(P^{G'}(\varphi \otimes \varphi')\) converges for all \(\varphi \in \pi\) and \(\varphi' \in \pi'\), then

\[
P^{G'}|_{\pi \otimes \pi'} \in \text{Hom}_{\triangle G'(A)}(\pi \otimes \pi', \mathbb{C})
\]

We say that \(\pi \otimes \pi'\) is \(\Delta G'\)-distinguished (with respect to \(P^{G'}\)) if \(P^{G'}|_{\pi \otimes \pi'} \neq 0\).

In this note, we consider the case \(G = \text{GL}_{n+1}\) and \(G' = \text{GL}_n\), which was studied by Jacquet, Piatetski-Shapiro and Shalika.

**Theorem 1** (Jacquet-Piatetski-Shapiro-Shalika). If \(\varphi \in \mathcal{A}^{\text{cusp}}(G)\) and \(\varphi' \in \mathcal{A}^{\text{cusp}}(G')\), then

\[
P^{G'}(\varphi \otimes \varphi'_s) = I(s, \varphi, \varphi') := \int_{N'(A) \backslash G'(A)} W^\psi(g, \varphi) W^{\overline{\psi}}(g, \varphi') |\det g|^s \, dg.
\]

Here, \(\mathcal{A}^{\text{cusp}}(G)\) and \(\mathcal{A}^{\text{cusp}}(G')\) denote the spaces of cusp forms on \(G(\mathbb{A})\) and \(G'(\mathbb{A})\) respectively, \(\varphi'_s = \varphi' \cdot |\det|^s\) for \(s \in \mathbb{C}\), \(N \subset G\) and \(N' \subset G'\) are upper triangular unipotent subgroups, \(W^\psi(g, \varphi)\) is a Whittaker function (with respect to a nontrivial character \(\psi\) of \(F\backslash \mathbb{A}\)) defined by

\[
W^\psi(g, \varphi) = \int_{N(F) \backslash N(A)} \varphi(ug) \overline{\psi(u_{1,2} + u_{2,3} + \cdots + u_{n,n+1})} \, du
\]

and \(W^{\overline{\psi}}(g, \varphi')\) is defined similarly. The left-hand side converges for all \(s\) and the right-hand side converges for \(\Re s \gg 0\). Moreover, if \(\varphi =
$\otimes_{v} \varphi_{v} \in \pi \subset \mathcal{A}^{cusp}(G)$ and $\varphi' = \otimes_{v} \varphi'_{v} \in \pi' \subset \mathcal{A}^{cusp}(G')$, then

$$I(s, \varphi, \varphi') = L\left(s + \frac{1}{2}, \pi \times \pi'\right) \prod_{v} \frac{I(s, W_{\psi_{v}}^{\varphi_{v}}, W_{\overline{\psi}_{v}}^{\overline{\varphi}_{v}})}{L\left(s + \frac{1}{2}, \pi_{v} \times \pi'_{v}\right)}.$$ 

In particular, $\pi \otimes \pi'$ is $\Delta G'$-distinguished if and only if $L\left(\frac{1}{2}, \pi \times \pi'\right) \neq 0$.

The last assertion is a special case of the Gan-Gross-Prasad conjecture [1]. We also remark that $I(s, \varphi, \varphi')$ makes sense for any automorphic forms $\varphi$ and $\varphi'$. Our main result is an extension of the above theorem.

**Theorem 2** (I-Yamana). Let $\varphi \in \mathcal{A}(G)$ and $\varphi' \in \mathcal{A}(G')$. Then

$$P_{\text{reg}}^{G'}(\varphi \otimes \varphi_{s}') = I(s, \varphi, \varphi')$$

as meromorphic functions of $s$. Here, $P_{\text{reg}}^{G'}$ is the regularized period integral defined below.

As immediate consequences, we obtain the following corollaries.

**Corollary 3.**

1. $P_{\text{reg}}^{G'}$ is $\Delta G' (\mathbb{A})$-invariant.
2. $P_{\text{reg}}^{G'}(\varphi \otimes \varphi_{s}') = 0$ unless $\varphi$ and $\varphi'$ are generic.

**Corollary 4.** Assume that $\pi$ and $\pi'$ are induced from irreducible cuspidal automorphic representations of Levi subgroups of $G$ and $G'$ respectively. Then $\pi \otimes \pi'$ is $\Delta G'$-distinguished (with respect to $P_{\text{reg}}^{G'}$) if and only if

$$L\left(\frac{1}{2}, \pi \times \pi'\right) \neq 0.$$ 

**Corollary 5.** Let $\varphi \in \pi \subset \mathcal{A}^{\text{disc}}(G)$ and $\varphi' \in \pi' \subset \mathcal{A}^{\text{disc}}(G')$. Here, $\mathcal{A}^{\text{disc}}(G)$ and $\mathcal{A}^{\text{disc}}(G')$ denote the spaces of square integrable automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. Assume that $\pi$ is not 1-dimensional. Then $P_{\text{reg}}^{G'}(\varphi \otimes \varphi')$ converges and $P_{\text{reg}}^{G'}(\varphi \otimes \varphi') = 0$ unless $\pi$ and $\pi'$ are cuspidal.

The original motivation was to study the Gan-Gross-Prasad conjecture in the non-tempered case. We also expect an application to the spectral expansion of the relative trace formula of Jacquet-Rallis [4]. In what follows, we will explain the definition of $P_{\text{reg}}^{G'}$ and the proof of Theorem 2.
Following Jacquet, Lapid and Rogawski [3], we define $P_{reg}^{G'}$. The construction is based on truncation. Recall that Arthur's truncation is given by

$$\Lambda^{T}\varphi(g) = \sum_{P}(-1)^{\dim \mathfrak{a}_{P}^{G}} \sum_{\gamma \in P \backslash G} \varphi_{P}(\gamma g) \hat{\tau}_{P}(H_{P}(\gamma g) - T),$$

which is rapidly decreasing. Here, $P = MU$ is a standard parabolic subgroup of $G$, $\varphi_{P}$ is the constant term of $\varphi$ along $P$, $\mathfrak{a}_{P} = \text{Hom}(X^{*}(M), \mathbb{R})$, $\mathfrak{a}_{P}^{*} = X^{*}(M) \otimes \mathbb{R}$, $\mathfrak{a}_{P} = \mathfrak{a}_{P}^{G} \oplus \mathfrak{a}_{G}$ is the canonical decomposition, $H_{P} : G(\mathbb{A}) \to \mathfrak{a}_{P}$ is a function such that $e^{\langle \chi, H_{P}(m) \rangle} = |\chi(m)|_{\mathbb{A}}$ for $\chi \in X^{*}(M)$, $m \in M(\mathbb{A})$ and extended by the Iwasawa decomposition, $T \in \mathfrak{a}_{0}^{G} = \mathfrak{a}_{B}^{G}$ is sufficiently positive with the standard Borel subgroup $B$, and $\hat{\tau}_{P}$ is the characteristic function of the obtuse cone in $\mathfrak{a}_{P}$ spanned by coroots. The integral $P^{G'}(\Lambda^{T}\varphi \otimes \varphi')$ converges but is hard to compute. Thus we adopt more suitable “mixed truncation” given by

$$\Lambda_{m}^{T}\varphi(g) = \sum_{P}(-1)^{\dim \mathfrak{a}_{P}^{G}} \sum_{\gamma \in P \backslash PWG'} \varphi_{P}(\gamma g) \hat{\tau}_{P}(H_{P}(\gamma g) - T),$$

where $W$ is the Weyl group of $G$.

**Lemma 6.**

1. $\Lambda_{m}^{T}\varphi$ is rapidly decreasing on $G'(F) \backslash G'('A)$.
2. $P^{G'}(\Lambda_{m}^{T}\varphi \otimes \varphi') = \sum_{\lambda} p_{\lambda}(T)e^{\langle \lambda, T \rangle}$, where the right-hand side is a finite sum with $\lambda \in (\mathfrak{a}_{0, \mathbb{C}}^{G})^{*}$ and $p_{\lambda} \in \mathbb{C}[\mathfrak{a}_{0}]$.

We define $P_{reg}^{G'}(\varphi \otimes \varphi') = p_{0}(T)$ if the exponents of $\varphi$ and $\varphi'$ avoid some finitely many hyperplanes. It turns out that $p_{0}(T)$ is constant, i.e., independent of $T$. If $\varphi \in \mathscr{A}^{cusp}(G)$, then $\Lambda_{m}^{T}\varphi = \varphi$, so that $P_{reg}^{G'}(\varphi \otimes \varphi') = P^{G'}(\varphi \otimes \varphi')$. This identity holds more generally if the exponents of $\varphi$ and $\varphi'$ satisfy some finitely many negativity conditions. We can define $P_{reg}^{G'}(\varphi \otimes \varphi'_{s})$ for generic $s$ and obtain a meromorphic function of $s$.

Following Lapid and Rogawski [5], we prove Theorem 2. We may assume that $\varphi$ is a cuspidal Eisenstein series. We want to unfold $P^{G'}(\varphi \otimes \varphi')$ by using the Fourier expansion

$$\varphi(g) = \sum_{i=0}^{n} \sum_{\gamma \in P_{i}^{\prime \backslash G'}} W_{Q_{i}}^{\psi}(\gamma g, \varphi_{Q_{i}}).$$
Here, 
\[ Q_i = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}^i \right\} \subset G, \]
\[ P'_i = \left\{ \begin{pmatrix} * & * \\ 0 & \nabla \end{pmatrix}^i \right\} \text{ is upper triangular unipotent} \subset G', \]
and \( W_{Q_i}^\psi \) is the Whittaker function for the \( \text{GL}_{n+1-i} \) part. If \( \varphi \in \mathcal{A}^{cusp}(G) \), then only the term \( i = 0 \) survives. Since \( P'_0 = N' \) and \( W_{Q_0}^\psi = W^\psi \), we can unfold \( P^{G'}(\varphi \otimes \phi') \) to get \( I(s, \varphi, \phi') \). In general, we cannot unfold. Instead, we compute the convergent integral \( P^{G'}(\theta_\phi \otimes \varphi') \) in two ways. Here, \( \phi(\lambda) = f(\lambda) \cdot \varphi \) for \( \lambda \in (\mathfrak{a}_{P,C}^G)^* \) with \( f \in \mathcal{P}\mathcal{W}((\mathfrak{a}_{P,C}^G)^*) \) and \( \varphi \in \mathcal{A}^{cusp}_P(G) \), \( \theta_\phi \) is a pseudo Eisenstein series given by
\[
\theta_\phi(g) = \int_{\Re \lambda = \kappa} f(\lambda) E(g, \varphi, \lambda) \, d\lambda
\]
with sufficiently positive \( \kappa \in (\mathfrak{a}_P^G)^* \) and an Eisenstein series
\[
E(g, \varphi, \lambda) = \sum_{\gamma \in P \backslash G} \varphi(\gamma g) e^{\langle \lambda, H_P(\gamma g) \rangle}.
\]
We can show that
\[
P^{G'}(\theta_\phi \otimes \varphi'_s) = \int_{\Re \lambda = \kappa} f(\lambda) P_{\text{reg}}^{G'}(E(\varphi, \lambda) \otimes \varphi'_s) \, d\lambda
\]
under some mild condition of \( f \). We can unfold \( P^{G'}(\theta_\phi \otimes \varphi'_s) \) to get
\[
\int_{\Re \lambda = \kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi') \, d\lambda + \sum_{i=1}^{n} \cdots,
\]
where the last sum vanishes under another mild condition of \( f \). The upshot is
\[
\int_{\Re \lambda = \kappa} f(\lambda) P_{\text{reg}}^{G'}(E(\varphi, \lambda) \otimes \varphi'_s) \, d\lambda = \int_{\Re \lambda = \kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi') \, d\lambda
\]
for sufficiently many \( f \) which allows us to extract the desired identity.

**REFERENCES**


[2] A. Ichino and S. Yamana, Periods of automorphic forms: The case of \( (\text{GL}_{n+1} \times \text{GL}_n, \text{GL}_n) \), preprint.


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