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On quotients of Hom-functors

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1. Introduction

A hom-functor on a category $C$ is the functor $\text{Hom}(-, X)$ for an object $X$ of $C$. We consider the quotient functor $\text{Hom}(-, X)/G$ by a subgroup $G$ of $\text{Aut}X$. We are interested in replacing hom-functors in the definitions of limit and adjoint by quotients of hom-functors.

2. Limit

We recall the definition of limit in terms of hom-functor. $\text{Set}$ denotes the category of sets. For a small category $C$, $[C^{\text{op}}, \text{Set}]$ denotes the category of contravariant functors $C \to \text{Set}$. $[C^{\text{op}}, \text{Set}]$ has limits. For instance, the product $F \times G$ of $F$ and $G$ in $[C^{\text{op}}, \text{Set}]$ is given by

$$(F \times G)(A) = F(A) \times G(A) \quad \text{for } A \in C.$$ 

And the final object $1$ of $[C^{\text{op}}, \text{Set}]$ is given by

$$1(A) = \{1\} \quad \text{for } A \in C.$$ 

For $X \in C$, the hom-functor $h_X$ is defined by

$$h_X(A) = \text{Hom}(A, X).$$ 

A functor $F : C^{\text{op}} \to \text{Set}$ is said to be representable if $F \cong h_X$ for some $X$. For $X_1, X_2, Z \in C$ we have

$$Z \text{ is a product of } X_1 \text{ and } X_2 \iff h_Z \cong h_{X_1} \times h_{X_2}.$$ 

Therefore

product of two objects exists in $C$

$\iff$ product of two representable functors is representable.

And similarly

a final object exists in $C \iff 1$ is representable.

The existence of a limit in $C$ is thus expressed as the representability of a limit of hom-functors. We first aim to replace representability by familial representability.
3. Sum of hom-functors

A functor $F: C^{\text{op}} \to \text{Set}$ is said to be familially representable if

$$ F \cong \coprod h_{X_i} $$

for some family $X_i$ of objects in $C$ ([Carboni and Johnstone]).

**Theorem 1.** Let $C$ be a finite category. The following conditions are equivalent to each other.

(i) $h_X \times h_Y$ and 1 are familially representable ($\forall X, Y \in C$).

(ii) Finite limits of hom-functors are familially representable.

(iii) Pushouts and coequalizers exist in $C$.

(iv) Finite connected limits exist in $C$.

Moreover these conditions imply that all morphisms of $C$ are epimorphisms.

Remark. "(iii) $\Rightarrow$ (iv)" is generally true.

For the proof of the theorem we may follow the proof of the general representability theorem in [Freyd and Scedrov]. It simplifies owing to our finiteness assumption. We may also use the characterization of familially representable functors ([Leinster]).

An interest with such categories comes from an attempt to define general Burnside rings. Suppose that $C$ satisfies (i) of Theorem 1. For any $X, Y \in C$ we take isomorphisms

$$ h_X \times h_Y \cong \coprod h_{Z_i} $$

and

$$ 1 \cong \coprod h_{W_j}. $$

Then the free abelian group based on the isomorphism classes of objects of $C$ becomes a ring by setting

$$ [X][Y] = \sum [Z_i], $$

$$ 1 = \sum [W_j]. $$

Here $[X]$ stands for the isomorphism class of an object $X$. This ring may be called the Burnside ring of $C$.

4. The Burnside ring of a finite category

Let $C$ be a finite category. Assume that $C$ satisfies the following conditions.

(B1) For every $X, Y \in C$ there exists a unique family of integers $c_{Z}^{XY}$ such that

$$ |\text{Hom}(A, X)||\text{Hom}(A, Y)| = \sum_{Z} c_{Z}^{XY} |\text{Hom}(A, Z)| \quad (\forall A \in C). $$

(Here $|S|$ stands for the cardinality of a set $S$.)
(B2) There exists a unique family of integers $d_Z$ such that

$$1 = \sum_Z d_Z |\text{Hom}(A, Z)| \quad (\forall A \in C).$$

Then the free abelian group based on the isomorphism classes of objects of $C$ becomes a ring:

$$[X][Y] = \sum_Z c_Z^{XY}[Z],$$

$$1 = \sum_Z d_Z[Z].$$

**Theorem.** ([Yoshida]) Assume that a finite category $C$ satisfies the following conditions.

(Y1) $C$ has the unique epi-mono factorization property.

(Y2) $C$ has the coequalizer

$$\text{Coeq}(X \xrightarrow{\alpha} X)$$

for any $\alpha \in \text{Aut} X$.

Then $C$ satisfies (B1) and (B2).

The following diagram shows the relationship between Theorem 1 and Yoshida’s theorem:

$$[X][Y] = \sum_Z c_Z^{XY}[Z],$$

pushout, coequalizer exist $\implies 1 = \sum_Z d_Z[Z],$

$\downarrow$

$$1 = \sum_Z d_Z[Z].$$

A problem will be to characterize categories satisfying (B1) and (B2).

Here are examples of generalized Burnside rings. Let $G$ be a finite group.

(1) Let $C$ be the category whose objects are $G$-sets $G/H$ for all subgroups $H$, and whose morphisms are $G$-maps. Then $C$ satisfies the condition of Theorem 1. The resulting ring is the ordinary Burnside ring of $G$.

(2) Let $\mathcal{F}$ be a family of subgroups of $G$ which is closed under conjugation and intersection. Let $C$ be the category whose objects are $G$-sets $G/H$ for $H \in \mathcal{F}$. Then $C$ satisfies the condition of Theorem 1.
(3) Let $\mathcal{F}$ be the set of all $p$-centric subgroups of $G$. Let $C$ be the category whose objects are $G$-sets $G/H$ for $H \in \mathcal{F}$. Then $C$ satisfies the condition that $h_X \times h_Y$ are familially representable ([Diaz and Libman], [Oda]). Further examples of $\mathcal{F}$ are found in [Oda and Sawabe].

(4) For a fusion system $\mathcal{F}$ a certain category $\mathcal{O}(\mathcal{F}^\circ)$ is defined. Then $C = \mathcal{O}(\mathcal{F}^\circ)$ satisfies the condition that $h_X \times h_Y$ are familially representable ([Puig], [Diaz and Libman]).

5. Finiteness of connected components of powers of a functor

$\text{FinSet}$ denotes the category of finite sets. Let $K$ be a finite category. We say $G \in [K, \text{FinSet}]$ is connected if $G$ is nonempty and never expressed as a sum of nonempty objects. Every $F \in [K, \text{FinSet}]$ is a sum of connected objects, each of which we call a connected component of $F$. For $F \in [K, \text{FinSet}]$ and $n \geq 0$ we have

$$F^n = F \times \cdots \times F$$

in $[K, \text{FinSet}]$.

**Theorem 2.** For $F \in [K, \text{FinSet}]$, the following are equivalent.

(i) Connected components of $F^n$ for all $n$ have only finitely many isomorphism classes.

(ii) $F(\alpha)$ is injective for every morphism $\alpha$ of $K$.

This theorem relates to Theorem 1 as follows: Let $F: K \to \text{FinSet}$ satisfy (ii) of Theorem 2. Let $C$ be a representative system of isomorphism classes of connected components of $F^n$ for all $n$. Then $C$ is finite. View $C$ as a category (a full subcategory of $[K, \text{FinSet}]$). For $X, Y \in C$, $X \times Y$ is a sum of objects of $C$ and $1$ is a sum of objects of $C$. So $C$ satisfies condition (i) of Theorem 1.

Conversely every finite category satisfying condition (i) of Theorem 1 arises this way.

6. Quotient of hom-functor

Let $C$ be a category. Let $X$ be an object of $C$ and $G$ a subgroup of $\text{Aut }X$. We define the functor $h_X/G: C^{\text{op}} \to \text{Set}$ by

$$(h_X/G)(A) = \text{Hom}(A, X)/G.$$

Here $\text{Hom}(A, X)/G$ is the quotient set relative to the natural action of $G$ on $\text{Hom}(A, X)$.

**Theorem 3.** Let $C$ be a finite category. The following conditions are equivalent to each other.

(i) $h_X \times h_Y$ and $1$ are isomorphic to sums of quotients of hom-functors $(\forall X, Y)$.

(ii) Finite limits of hom-functors are isomorphic to sums of quotients of hom-functors.

(iii) Pushouts exist in $C$.

(iv) Finite simply connected limits exist in $C$. 
These conditions imply that all morphisms of $C$ are epimorphisms.

Remark. "(iii) $\implies$ (iv)" is true for a general $C$ ([Paré]).

7. Category with pushouts

We here give an example of a category with pushouts.
Let $P$ be a partially ordered set. Suppose that a group $G$ acts on $P$:

\[ \sigma \in G, x \in P \leadsto x^\sigma \in P. \]

The category $PG$ is defined as follows.

(object) Objects of $PG$ are elements of $P$.

(morphism) For $x, y \in P$

\[ \text{Hom}_{PG}(x, y) = \{ \sigma | \sigma \in G, x \leq y^\sigma \}. \]

(composition) Composition is given by multiplication in $G$.

**Proposition.** If $P$ has pushouts, then so does $PG$.

That $P$ has pushouts means that if $z \leq x, z \leq y$, then there exists $\sup(x, y)$.

Suppose that for each $x \in P$ a subgroup $K_x$ of $G$ is given. Assume the following conditions hold.

(i) $\sigma \in K_x \implies x^\sigma = x$
(ii) $x \leq y \implies K_x \leq K_y$
(iii) $K_x^\sigma = K_{x^\sigma}$

We then define the category $D$ as follows.

(object) Objects of $D$ are elements of $P$.

(morphism) For $x, y \in P$ we set

\[ \text{Hom}_{D}(x, y) = \text{Hom}_{PG}(x, y)/K_y. \]

Here $K_y$ acts on $\text{Hom}_{PG}(x, y)$ by multiplication in $G$.

(composition) The composition of $D$ is induced by that of $PG$.

**Proposition.** If $P$ has pushouts, then so does $D$.

8. Adjoint

We recall the definition of adjoint in terms of $\text{hom}$-functor. Let $F: B \to C$ and $G: C \to B$ be functors. "$G$ is a right adjoint of $F$" means

\[ \text{Hom}_{C}(F(X), Y) \cong \text{Hom}_{B}(X, G(Y)) \]

(naturally in $X, Y$).

This isomorphism, $X$ viewed a variable, is written as

\[ \text{Hom}_{C}(F(-), Y) \cong h_{G(Y)} \]

(naturally in $Y$).
Hom\(_C(F(-), Y)\) = \(h_Y \circ F\) denoted by \(F^*(h_Y)\), this is written as
\[F^*(h_Y) \cong h_{G(Y)}.\]

Thus
\[F \text{ has a right adjoint} \quad \iff F^*(h_Y) \text{ are representable for all } Y \in C.\]

We next aim to replace representability in the right-hand side by familial representability.

9. Discrete fibration

Recall that a functor \(F: B \to C\) is called a discrete fibration if the following condition holds.

\[
\forall g: F(X) \to Y' \text{ morphism of } C, \quad \exists! f: X \to X' \text{ morphism of } B, \quad F(f) = g.
\]

If \(F: B \to C\) is a discrete fibration, then
\[F^*(h_Y) \cong \coprod_{X \in F^{-1}(Y)} h_X\]
for every \(Y \in C\).

**Proposition.** Let \(F: B \to C\) be a functor. The following are equivalent.

(i) \(F^*(h_Y)\) are familially representable for all \(Y \in C\).

(ii) There exists a factorization
\[
\begin{array}{ccc}
C' & \xrightarrow{F'} & C \\
\nearrow & \searrow^\pi & \\
B & \xrightarrow{F} & C
\end{array}
\]
such that \(F'\) has a right adjoint and \(\pi\) is a discrete fibration.

10. Condition (G)

Here we aim to replace representability in the definition of adjoint by being isomorphic to a sum of quotients of hom-functors.

Let \(F: B \to C\) be a functor. We introduce the condition (G) for \(F\). It consists of the following:

(i)
\[
g: F(X) \to Y' \quad \implies \exists f: X \to X', \; F(f) = g.
\]
(ii)
\[ f_1: X \to X'_1, \ f_2: X \to X'_2, \ F(f_1) = F(f_2) \]
\[ \Rightarrow \exists u: X'_1 \to X'_2, \ F(u) = 1, \ f_2 = uf_1. \]

If condition (G) holds, then \( F^*(h_Y) \) is isomorphic to a sum of quotients of hom-functors for every \( Y \in C \).

**Theorem 4.** Let \( F: B \to C \) be a functor. Assume that \( C \) is finite. The following are equivalent.

(i) \( F^*(h_Y) \) are isomorphic to sums of quotients of hom-functors for all \( Y \in C \).

(ii) There exists a commutative diagram

\[
\begin{array}{ccc}
B' & \xrightarrow{F'} & C' \\
\downarrow{\nu} & & \downarrow{\pi} \\
B & \xrightarrow{F} & C
\end{array}
\]

such that \( F' \) has a right adjoint, \( \nu \) is full and dense, and \( \pi \) satisfies condition (G).

**References**