ON MULTIPLICATIVE INDUCTION (Research on finite groups and their representations, vertex operator algebras, and algebraic combinatorics)

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ON MULTIPLICATIVE INDUCTION

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ABSTRACT. Let $G$ be a finite group and $e$ be the proper trivial subgroup of $G$. We compute the value $\text{Jnd}_H^G(\ell[H/e])$ for a subgroup $H$ of $G$ in the Burnside ring $\mathcal{B}(G)$ for an integer $\ell$. Their values induce integer valued polynomials.

1. Notation

Let $G$ be a finite group and $s_G$ be the set of all subgroups of $G$. Denote by $gH$ the conjugate subgroup $gHg^{-1}$ for $H \leq G$ and $g \in G$. Let $[s_G]$ be a set of representatives of $G$-conjugacy classes of $s_G$. If $X$ is a finite $G$-set, write $[X]$ for the isomorphism class of finite $G$-sets containing $X$. Denote by $X^S$ the $S$-fixed points of the $G$-set $X$. If $X$ is a finite set, write $|X|$ for the cardinality of $X$. Denote by $e$ the identity element of $G$. The proper trivial subgroup $\{e\}$ of $G$ is also denoted by $e$. For two subgroups $S, H \leq G$ denote by $[S\backslash G/H]$ a set of representatives of double cosets of $G$ by $S$ and $H$.

2. MULTIPLICATIVE INDUCTIONS FOR BURNSIDE RINGS

Let $\Omega(G)$ be the Burnside ring of $G$. Then $\Omega(G)$ is a free $\mathbb{Z}$-module with basis $\{[G/H]|H \in [s_G]\}$. The multiplication is defined by the Cartesian product. If $S \in s_G$, then there is a unique linear form $\varphi_S^G : \Omega(G) \to \mathbb{Z}$ such that $\varphi_S^G([X]) = |X^H|$ for any finite $G$-set $X$. It is a ring homomorphism. The mark homomorphism is a ring homomorphism $\varphi^G = \prod (S \in [s_G]) \varphi_S^G : \Omega(G) \to \tilde{\Omega}(G)$, where $\tilde{\Omega}(G) = \prod (S \in [s_G]) \mathbb{Z}$ and it is called the ghost ring of $G$.

Lemma 2.1. The ring homomorphism $\varphi^G$ is injective.

We recall some properties for tensor induction of Burnside rings. We refer to [Yo90] for more details. Let set$^G$ be the category of finite $G$-sets. If $H \leq G$, then there is a functor $\text{Jnd}_H^G : \text{set}^H \to \text{set}^G$ which has the values on objects

$$\text{Jnd}_H^G : X \mapsto \text{Map}_H(G, X),$$

where $\text{Map}_H(G, X)$ is the set of $H$-maps $\alpha : G \to X$ such that $\alpha(h \cdot g) = h \cdot \alpha(g)$ for all $h \in H, g \in G$, with the action of $G$ defined by $(k \cdot \alpha)(g) = \alpha(gk)$ for $k \in G$, for an $H$-set $X$.

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Lemma 2.2. Let $H$ be a subgroup of $G$ and $X$ be an $H$-set. If $S$ is a subgroup of $G$, then
\[ \varphi_S^G(\text{Jnd}_H^G(X)) = \prod_{g \in [S \setminus G/H]} \varphi_{H \cap gS}^H(X). \]

Lemma 2.3. Let $H$ be a subgroup of $G$. If $S$ is a subgroup of $G$ and $q \in \mathbb{Z}$, then
\[ \varphi_S^G(\text{Jnd}_e^G(q[e/e])) = q^{[G/S]}. \]

Proof. By Lemma 2.2, we have
\[ \varphi_S^G(\text{Jnd}_e^G(q[e/e])) = \prod_{g \in [S \setminus G/e]} \varphi_{e \cap gS}^e(q[e/e]) = \prod_{g \in [S \setminus e]} q. \]

It has been shown by Gluck ([Gl81]) and independently by Yoshida ([Yo83]) that a formula of primitive idempotent $e_H^G$ of $\mathbb{Q}$-algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ for $H \leq G$ can be expressed as
\begin{equation}
(2.1) \quad e_H^G = \frac{1}{|N_G(H)|} \sum_{K \subseteq H} |K| \mu(K, H) [G/K],
\end{equation}
where $\mu(K, H)$ is the value of the Möbius function of $s_G$.

Denote by $NH$ (resp. $WH$) $N_G(H)$ (resp. $H_G(H)/H$) for a subgroup $H$ of $G$. Put $q^G = \text{Jnd}_H^G(q[e/e])$ for $q \in \mathbb{Z}$.

Lemma 2.4. If $G$ is a finite group and $q$ is an integer, then
\[ q^G = \sum_{(D) \in [s_G]} |WD|^{-1} \sum_{S \leq G} \mu(D, S) q^{[G/S]} [G/D] \]

Proof. By Lemma 2.3 and idempotent formula (2.1), we have that
\[ q^G = \sum_{S \in [s_G]} \varphi_S^G(q^G) e_S^G \]
\[ = \sum_{S \in [s_G]} q^{[G/S]} |NS|^{-1} \sum_{D \leq S} |D| \mu(D, S) [G/D] \]
\[ = \sum_{S \leq G} (G : NS)^{-1} q^{[G/S]} |NS|^{-1} \sum_{D \leq G} |D| \mu(D, S) [G/D] \]
\[ = |G|^{-1} \sum_{D \leq G} |D| \left( \sum_{S \leq G} \mu(D, S) q^{[G/S]} \right) [G/D] \]
\[ = |G|^{-1} \sum_{D \in [s_G]} (G : ND) |D| \left( \sum_{S \leq G} \mu(D, S) q^{[G/S]} \right) [G/D] \]
\[ = \sum_{D \in [s_G]} |WD|^{-1} \left( \sum_{S \leq G} \mu(D, S) q^{[G/S]} \right) [G/D]. \]

\[\square\]
In particular, coefficients of $[G/D]$ in $q^G$ as above are integers.

**Proposition 2.5.** If $G$ is a finite group and $q$ is an integer, then

$$|WD|^{-1} \sum_{S \leq G} \mu(D, S)q^{|G/S|}$$

is an integer for a subgroup $D$ of $G$.

Substituting $x$ for $q$ we obtain integer-valued polynomials $f_D^G(x)$ as follows.

**Theorem 2.6.** Let $G$ be a finite group and put

$$f_D^G(x) = \frac{1}{|WD|} \sum_{S \leq G} \mu(D, S)x^{|G/S|}$$

for subgroup $D$ of $G$. Then $f_D^G(x)$ is an integer-valued polynomial.

3. TAMBARA FUNCTORS

In this section, we recall some notes on Tambara functors. For a $G$-map $f : X \rightarrow Y$ we consider a set

$$\Pi_f(A) = \left\{ (y, \sigma) \mid y \in Y, \sigma : f^{-1}(y) \rightarrow A : \text{map} \right\}$$

with $G$-action defined by

$$g(y, \sigma) := (gy, g\sigma), \quad g\sigma(x) := g\sigma(g^{-1}x)$$

and denote by $\Pi_f\alpha$ the projection $(y, \sigma) \mapsto y$. For a $G$-map $\alpha : A \rightarrow X$ the pullback functor

$$f^* : \text{set}^G/Y \rightarrow \text{set}^G/X,$$

$$(B \rightarrow Y) \mapsto (X \times_Y B \overset{pr}{\rightarrow} X)$$

has a left adjoint functor

$$\Sigma_f : \text{set}^G/X \rightarrow \text{set}^G/Y,$$

$$(A \overset{\alpha}{\rightarrow} X) \mapsto (A \overset{\alpha}{\rightarrow} X \overset{f}{\rightarrow} Y)$$

and a right adjoint functor

$$\Pi_f : \text{set}^G/X \rightarrow \text{set}^G/Y,$$

$$(A \overset{\alpha}{\rightarrow} X) \mapsto (\Pi_f(A) \overset{\Pi_f\alpha}{\rightarrow} Y).$$

Two natural transformations

$$\Sigma_f \xrightarrow{\Sigma_f\epsilon'} \Sigma_f f^* \Pi_f \xrightarrow{\epsilon \Sigma_f} \Pi_f,$$
give a commutative diagram

\[
\begin{array}{c}
A \\
\downarrow \\
X \\
\downarrow \\
Y \\
\downarrow \\
\Pi_f A
\end{array}
\begin{array}{c}
{\alpha} \\
\leftarrow \\
\beta \\
\leftarrow \\
\gamma \\
\leftarrow
\end{array}
\begin{array}{c}
X \times Y \\
\downarrow \\
\Pi_f A
\end{array}
\]

where \( e : X \times Y \Pi_f A \ni (x, (y, \sigma)) \mapsto \sigma(x) \in A \) and \( f' \) is projection. In order to discuss the TNR-functors, this diagram is introduced by Tambara in [Ta93]. Brun called it Tambara functor in [Br05]. There are some works concerning about Tambara functors ([Na12a], [Na12b], [Na13], [OY11]).

Denote by Set the category of sets and maps and by \( \text{set}^G \) the category of finite \( G \)-sets and \( G \)-maps. For any \( G \)-sets \( X \) and \( Y \) we denote by \( X + Y \) the disjoint union of them.

For any \( G \)-map \( f : X \to Y \) we consider the triplet of functors

\[
T = (T_!, T^*, T_*) : \text{set}^G \to \text{Set},
\]

consisting of a contravariant functor \( T^* : \text{set}^G \to \text{Set} \) and two covariant functors \( T_!, T_* : \text{set}^G \to \text{Set} \) which coincide on the objects, and so we write

\[
T(X) := T_!(X) = T^*(X) = T_*(X),
\]

\[
f_! := T_!(f), f_* := T_*(f) : T(X) \to T(Y), f^* : T(Y) \to T(X).
\]

for any \( G \)-sets \( X, Y \) and any \( G \)-map \( f : X \to Y \). A triplet \( T = (T_!, T^*, T_*) \) is called a semi-Tambara functor if these functors satisfy the following axioms:

(T.1) (Additivity) If

\[
X \xrightarrow{j} X + Y \xleftarrow{i} Y
\]

is a coproduct diagram of finite \( G \)-sets, then

\[
T(X) \xleftarrow{i^*} T(X + Y) \xrightarrow{j^*} T(Y)
\]

is a product diagram of sets; and \( T(\emptyset) = 0(:= \{0\}) \).

(T.2) (Pullback formula)

\[
\begin{array}{c}
X \xrightarrow{a} Y \\
\downarrow \\
Z \xrightarrow{d} W
\end{array}
\begin{array}{c}
T(X) \xrightarrow{a^*} T(Y) \\
\uparrow \\
\circ \\
\uparrow \\
T(Z) \xrightarrow{d_*} T(W)
\end{array}
\]

(T.3) (Distributive law)
The axioms (T.1) and (T.2) mean that both of pairs \((T^*, T_1)\) and \((T^*, T_*)\) form semi-Mackey functors (see 3.3 of [OY04]). If all \(T(X)\) are commutative ring and \(f_!, f^*, f_*\) are homomorphisms of additive groups, rings, multiplicative monoids, respectively, then \(T\) is called a Tambara functor.

For any finite \(G\)-set \(X\), let \(\Omega_+(X)\) be the set of isomorphism classes \([A \rightarrow X]\) of finite \(G\)-sets over \(X\). Then \(\Omega_+(X)\) is a semiring by coproducts and products in the comma category \(\text{set}^G/X\). A \(G\)-map \(f: X \rightarrow Y\) induces three maps:

\[
\begin{align*}
fi &: \Omega_+(X) \rightarrow \Omega_+(Y); [A \rightarrow^\alpha X] \mapsto [A \rightarrow^\alpha X \rightarrow^f Y], \\
f^* &: \Omega_+(Y) \rightarrow \Omega_+(X); [B \rightarrow Y] \mapsto [X \times_Y B \rightarrow^p X], \\
f_* &: \Omega_+(X) \rightarrow \Omega_+(Y); [A \rightarrow^\alpha X] \mapsto [\Pi_f(A) \rightarrow^J X].
\end{align*}
\]

Then the family \(\Omega_+(X), fi, f^*, f_*\) form a semi-Tambara functor \(\Omega_+\). By the Grothendieck ring construction, we have the Burnside ring functor \(\Omega\), which is a Tambara functor.

**Lemma 3.1.** Let \(f: G/H \rightarrow G/G\) be the canonical surjection for a subgroup \(H \leq G\). If \(\alpha: A \rightarrow G/H\) is a \(G\)-map to transitive \(G\)-set \(G/H\), then there exists a \(G\)-isomorphism

\[
\Pi_f(A) \cong \text{Map}_H(G, \alpha^{-1}(eH)).
\]

**Proof.** Since \(G/G\) is a set of cardinality 1 and \(f\) is surjective, we may identify

\[
\Pi_f(A) = \{\sigma: G/H \rightarrow A | \sigma: \text{map, } \alpha \circ \sigma = \text{id}_{G/H}\}.
\]

Then we see that the map \(\varphi: \Pi_f(A) \rightarrow \text{Map}_H(G, \alpha^{-1}(eH)), \varphi: s \mapsto \varphi(s): G \rightarrow \alpha^{-1}(eH): g \mapsto gs(g^{-1}H)\)

gives the isomorphism. \(\square\)

Let \(f: G/H \rightarrow G/G\) be the canonical surjection and \(\Omega\) be the Burnside Tambara functor. Then by Lemma 3.1, we see that the image \(\Omega_*(f)([A \rightarrow G/H])\) for the map \(\Omega_*(f): \Omega(G/H) \rightarrow \Omega(G/G)\) is

\[
\Omega_*(f)([A \rightarrow G/H]) = [\text{Map}_H(G, \alpha^{-1}(eH)) \rightarrow G/G].
\]

By Lemma 2.4, we have the following result.

**Proposition 3.2.** If \(f: G/e \rightarrow G/G\) is the canonical surjection, \(q\) is an integer, and \(\Omega\) is the Burnside Tambara functor. Then we have

\[
\Omega_*(f)(q[G/e \rightarrow G/e]) = \sum_{(D) \in [G]} |WD|^{-1} \sum_{S \leq G} \mu(D, S)q^{[G/S]}[G/D \rightarrow G/G].
\]
4. NECKLACE POLYNOMIALS

In this section, we show that the polynomial $f_{D}^{G}(x)$ is a generalization of necklace polynomials. It is well known that the number $M(\alpha, n)$ of primitive necklaces of length $n$ that can be constructed using a set of beads with $\alpha$-colors is computed by a formula

$$M(\alpha, n) = \frac{1}{n} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \alpha^{d} = \frac{1}{n} \sum_{d \mid n} \mu(d) \alpha^{\frac{n}{d}},$$

where $\mu$ is the classical Möbius function (see [MR83] for instance). It is called necklace polynomial. In this section, we show that there is a relationship between the equation of Theorem 2.6 and the necklace polynomials. Denote by $C_n$ the cyclic group of order $n$. Denote by $\mathcal{S}_G$ the poset $(s_G, \leq)$ of the subgroups of $G$ ordered by inclusion. Denote by $\mathcal{D}(n)$ the divisor poset of a positive integer $n$ ordered by divisibility relation. If $m$ is a divisor of $n$, then there exists an isomorphism of posets from the closed interval $[C_m, C_n]_{\mathcal{S}_G}$ to $\mathcal{D}\left(\frac{n}{m}\right)$. The following lemma is well known.

**Lemma 4.1.** If $C_d$ is an element of $[C_m, C_n]_{\mathcal{S}_G}$, then

$$\mu_{\mathcal{S}_G}(C_m, C_d) = \mu_{\mathcal{D}(\frac{n}{m})}(1, \frac{d}{m}).$$

In particular, $\mu_{\mathcal{S}_G}(C_m, C_d) = \mu\left(\frac{d}{m}\right)$.

**Theorem 4.2.** If $G$ is a cyclic group of order $n$, then $f_{C_m}^{G}(x) = M\left(x, \frac{n}{m}\right)$ for any divisor $m$ of $n$.

**Proof.** By the definition of $f_{C_m}^{G}(x)$ and Lemma 4.1,

$$f_{C_m}^{G}(x) = |WC_m|^{-1} \sum_{S \leq C_n} \mu(C_m, S)x^{[G/S]}$$

$$= \left|\frac{n}{m}\right|^{-1} \sum_{C_d \leq C_n} \mu(C_m, C_d)x^{[C_n/C_d]}$$

$$= \left|\frac{n}{m}\right|^{-1} \sum_{\frac{d}{m} \mid \frac{n}{m}} \mu\left(\frac{d}{m}\right) x^{\frac{n}{m}/\frac{d}{m}}$$

$$= M\left(x, \frac{n}{m}\right).$$

$\square$

Theorem 4.2 and Theorem 2.6 show the following.

**Corollary 4.3.** If $G$ is a cyclic group of order $n$ and $\ell$ is a positive integer, then

$$\ell^{G} = \sum_{m \mid n} M\left(\ell, \frac{n}{m}\right) [G/C_m].$$
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