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Kyoto University
Superstring Theory and Triple Systems

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1 Introduction

It has been expected that there exists M-theory, which unifies string theories. In M-theory, some structures of 3-algebras were found recently. First, it was found that by using $u(N) \oplus u(N)$ Hermitian 3-algebra, we can describe a low energy effective action of $N$ coincident supermembranes [1-5], which are fundamental objects in M-theory.

Second, recent studies have indicated that there also exist structures of 3-algebras in the Green-Schwartz supermembrane action, which defines full perturbative dynamics of a supermembrane. It had not been clear whether the total supermembrane action including fermions has structures of 3-algebras, whereas the bosonic part of the action can be described by using a tri-linear bracket, called Nambu bracket [6,7], which is a generalization of Poisson bracket. If we fix to a light-cone gauge, the total action can be described by using Poisson bracket, that is, only structures of Lie algebra are left in this gauge [8]. However, it was shown under an approximation that the total action can be described by Nambu bracket if we fix to a semi-light-cone gauge [9]. In this gauge, the eleven dimensional space-time of M-theory is manifest in the supermembrane action, whereas only ten dimensional part is manifest in the light-cone gauge. Moreover, 3-algebra models of M-theory itself were proposed and have been studied in [9-13].

The hermitian $(\epsilon, \delta)$-Freudenthal-Kantor triple systems [14-36] are generalizations of the hermitian 3-algebras [1-9,37-71]. The hermitian 3-algebras are special cases, where $K(a,b) = 0$ or equivalently, $<abc> = -<cba>$, of the hermitian $(-1,-1)$-Freudenthal-Kantor triple systems of second order. And the hermitian 3-algebras are classified into the $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ hermitian 3-algebras [13,43,45,46,52]. Therefore, it is natural to extend these triple systems to more general hermitian $(-1,-1)$-Freudenthal-Kantor triple systems or hermitian generalized Jordan triple systems.

In the following section, we summarize some results concerning with the generalization of the hermitian 3-algebras in M-theory [72,73].

2 Definitions

Let us start with a definition of a $^*-(\epsilon, \delta)$-Freudenthal-Kantor triple system.

Definition. A triple system $U$ is said to be a $^*-(\epsilon, \delta)$-Freudenthal-Kantor triple system if relations (0)-(iv) satisfy;

0) $U$ is a Banach space,
\[ L(a, b), L(c, d) = L(<abc>, d) + \epsilon L(c, <bad>), \]
where \( L(a, b)c = <abc> \) and \( K(a, b)c = <acb> - \delta <bca> \), \( \epsilon = \pm 1 \), \( \delta = \pm 1 \).

\[ K(<abc>, d) + K(c, <abd>) + \delta K(a, K(c, d)b) = 0, \]
where \( <xyz> \) is \( C \)-linear operator on \( x, z \) and \( C \)-anti-linear operator on \( y \).

\[ <abc > \] continuous with respect to a norm \( || \) that is, there exists \( K > 0 \) such that
\[ || <xxx> || \leq K||x||^3 \] for all \( x \in U \).

Furthermore, a \( *-\langle \epsilon, \delta \rangle \)-Freudenthal-Kantor triple system is said to be hermitian if it satisfies the following condition,

v) all operator \( L(x, y) \) is a positive hermitian operator with a hermitian metric
\[ (x, y) = tr L(x, y), \]
that is, \( (L(x, y)z, w) = (z, L^*(x, y)w) \), and \( (x, y) = \overline{(y, x)}. \)

Let \( U \) be a \( *-\langle \epsilon, \delta \rangle \)-Freudenthal-Kantor triple system. Then we may define the notation of tripotent as follows.

Definition. It is said to be a tripotent of \( U \) if
\[ <ccc> = c, c \in U. \]

3 Tripotent basis

In this section, we give decomposition theorems based on the tripotent basis.

Theorem 1.1. Let \( U \) be a hermitian \( (-1, \delta) \)-Freudenthal-Kantor triple system. If \( W \subset U \) is flat (that is, \( L(x, y) = L(y, x) \) for all \( x, y \in W \)), then we have a decomposition,
\[ W = Re_1 \oplus \cdots \oplus Re_n \]
where \( e_i \) are tripotents or bitripotents.

Proof. See [72,73].

We define the odd power of \( x \) inductively as follows;
\[ x^{(3)} := <xxx>, \]
\[ x^{(2n+1)} := <xx^{(2n-1)}x>. \]

By using this theorem, we have

Theorem 1.2. Let \( U \) be a hermitian \( (-1, \delta) \)-Freudenthal-Kantor triple system. Then every \( x \in U \) can be written uniquely
\[ x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_ne_n, \]
where the \( e_i \) are tripotents or bitripotents, which are linear combinations of power of \( x \), and the \( \lambda_i \) satisfy
\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \]

Proof. See [72,73].
4 Peirce decomposition

In this section, we give a theorem on a Peirce decomposition of a $^*\!\!\!\!\!\!\!\!(−1, −1)$-Freudenthal-Kantor triple system equipped with the tripotent $<ccc >= c$.

**Theorem 2.1.** Let $U$ be a $^*\!\!\!\!\!\!\!\!(−1, −1)$-Freudenthal-Kantor triple system. Then, we have a decomposition with respect to a tripotent $c$ (i.e., $<ccc >= c$) as follows

$$U = U_1(c) \oplus U_{\frac{1}{2}}(c) \oplus U_0(c),$$

where

$$U_1(c) = \{x | (L(c, c) + R(c, c))x = 0, (R(c, c) - Id)x \neq 0\},$$

$$U_{\frac{1}{2}}(c) = \{x | (L(c, c) + R(c, c))x \neq 0, (R(c, c) - Id)x = 0\},$$

$$U_0(c) = \{x | (L(c, c) + R(c, c))x = 0, (R(c, c) - Id)x = 0\}.$$

**Proof.** See [72, 73].

5 Generalized hermitian 3-algebra

In this section, we extend the $u(N) \oplus u(M)$ 3-algebras to a hermitian $(-1, -1)$-Freudenthal-Kantor triple system.

Let $D_{N,M}^*$ be the set of all $N \times M$ matrices with operation

$$<xyz> = x\overline{y}^Tz - z\overline{y}^Tx + zx^T\overline{y},$$

where $x^T$ and $\overline{x}$ mean transpose and conjugation of $x$, respectively.

Then $D_{N,M}^*$ is a hermitian $(-1, -1)$-Freudenthal-Kantor triple system. In fact, it satisfies the conditions (0), (i),(ii),(iii),(iv) and (v). This is an extension of the $u(N) \oplus u(M)$ hermitian 3-algebra, $<xyz> = x\overline{y}^Tz - z\overline{y}^Tx$, which is a basis for the effective action of the multiple membranes in M-theory.

One of the tripotents is given by

$$c = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix},$$

where $Id$ is a $n \times n$ identity matrix ($n \leq N, M$). Because any element is decomposed as

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left( \frac{1}{2}(A - A^T) & B \\ 0 & D \right) + \left( \frac{1}{2}(A + A^T) & 0 \\ C & 0 \right),$$

the Peirce decomposition is given by

$$U_1(c) = \left\{ \left( \frac{1}{2}(A - A^T) & B \\ 0 & D \right) \right\},$$

$$U_{\frac{1}{2}}(c) = \left\{ \left( \frac{1}{2}(A + A^T) & 0 \\ C & 0 \right) \right\},$$

$$U_0(c) = \{0\}.$$

As in Theorem 1.1, we can expand any element as $x = \Sigma(\lambda_{ij}E_{ij} + \mu_{ij}\sqrt{-1}E_{ij})$, where $E_{ij}$ means that $(i, j)$ element is 1 and other element is zero, and $E_{ij}$ and $\sqrt{-1}E_{ij}$ are tripotents, i.e., $<E_{ij}E_{ij}E_{ij}> = E_{ij}$, and $<\sqrt{-1}E_{ij}\sqrt{-1}E_{ij}\sqrt{-1}E_{ij}> = \sqrt{-1}E_{ij}$.
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