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Superstring Theory and Triple Systems

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1 Introduction

It has been expected that there exists M-theory, which unifies string theories. In M-theory, some structures of 3-algebras were found recently. First, it was found that by using $u(N) \oplus u(N)$ Hermitian 3-algebra, we can describe a low energy effective action of $N$ coincident supermembranes [1-5], which are fundamental objects in M-theory.

Second, recent studies have indicated that there also exist structures of 3-algebras in the Green-Schwartz supermembrane action, which defines full perturbative dynamics of a supermembrane. It had not been clear whether the total supermembrane action including fermions has structures of 3-algebras, whereas the bosonic part of the action can be described by using a tri-linear bracket, called Nambu bracket [6,7], which is a generalization of Poisson bracket. If we fix to a light-cone gauge, the total action can be described by using Poisson bracket, that is, only structures of Lie algebra are left in this gauge [8]. However, it was shown under an approximation that the total action can be described by Nambu bracket if we fix to a semi-light-cone gauge [9]. In this gauge, the eleven dimensional space-time of M-theory is manifest in the supermembrane action, whereas only ten dimensional part is manifest in the light-cone gauge. Moreover, 3-algebra models of M-theory itself were proposed and have been studied in [9-13].

The hermitian $(\epsilon, \delta)$-Freudenthal-Kantor triple systems [14-36] are generalizations of the hermitian 3-algebras [1-9,37-71]. The hermitian 3-algebras are special cases, where $K(a, b) = 0$ or equivalently, $\langle abc \rangle = -\langle cba \rangle$, of the hermitian $(-1, -1)$-Freudenthal-Kantor triple systems of second order. And the hermitian 3-algebras are classified into the $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ hermitian 3-algebras [13,43,45,46,52]. Therefore, it is natural to extend these triple systems to more general hermitian $(-1, -1)$-Freudenthal-Kantor triple systems or hermitian generalized Jordan triple systems.

In the following section, we summarize some results concerning with the generalization of the hermitian 3-algebras in M-theory [72,73].

2 Definitions

Let us start with a definition of a $^{*}(\epsilon, \delta)$-Freudenthal-Kantor triple system.

**Definition.** A triple system $U$ is said to be a $^{*}(\epsilon, \delta)$-Freudenthal-Kantor triple system if relations (0)-(iv) satisfy;

0) $U$ is a Banach space,
i) \[ [L(a, b), L(c, d)] = L(<abc>, d) + \varepsilon L(c, <bad>) \]

where \( L(a, b)c = <abc> \) and \( K(a, b)c = <acb> - \delta <bca> \), \( \varepsilon = \pm 1 \), \( \delta = \pm 1 \).

ii) \[ K(<abc>, d) + K(c, <abd>) + \delta K(a, K(c, d)b) = 0, \]

where \( \langle x, y \rangle \) is \( \mathbb{C} \)-linear operator on \( x, z \) and \( \mathbb{C} \)-anti-linear operator on \( y, \)

iii) \[ \langle xyz \rangle \] is \( \mathbb{C} \)-linear operator on \( x, z \) and \( \mathbb{C} \)-anti-linear operator on \( y, \)

iv) \[ \langle abc \rangle \] continuous with respect to a norm \( || \cdot || \) that is, there exists \( K > 0 \) such that

\[ ||<xxx>|| \leq K||x||^3 \text{ for all } x \in U. \]

Furthermore, a \( *-(\varepsilon, \delta) \)-Freudenthal-Kantor triple system is said to be hermitian if it satisfies the following condition,

v) all operator \( L(x, y) \) is a positive hermitian operator with a hermitian metric

\[ (x, y) = tr L(x, y), \]

that is, \( (L(x, y)z, w) = (z, L^*(x, y)w), \) and \( (x, y) = (y, x) \).

Let \( U \) be a \( *-(\varepsilon, \delta) \)-Freudenthal-Kantor triple system. Then we may define the notation of tripotent as follows.

**Definition.** It is said to be a tripotent of \( U \) if

\[ \langle ccc \rangle = c, \quad c \in U. \]

3 Tripotent basis

In this section, we give decomposition theorems based on the tripotent basis.

**Theorem 1.1.** Let \( U \) be a hermitian \( (-1, \delta) \)-Freudenthal-Kantor triple system. If \( W \subset U \) is flat (that is, \( L(x, y) = L(y, x) \) for all \( x, y \in W \)), then we have a decomposition,

\[ W = Re_1 \oplus \cdots \oplus Re_n \]

where \( e_i \) are tripotents or bitripotents.

**Proof.** See [72, 73].

We define the odd power of \( x \) inductively as follows;

\[ x^{(3)} := \langle xxx \rangle, \]

\[ x^{(2n+1)} := \langle xx^{(2n-1)}x \rangle. \]

By using this theorem, we have

**Theorem 1.2.** Let \( U \) be a hermitian \( (-1, \delta) \)-Freudenthal-Kantor triple system.

Then every \( x \in U \) can be written uniquely

\[ x = \lambda_1e_1 + \lambda_2e_2 + \cdots + \lambda_ne_n, \]

where the \( e_i \) are tripotents or bitripotents, which are linear combinations of power of \( x \), and the \( \lambda_i \) satisfy

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \]

**Proof.** See [72, 73].
4 Peirce decomposition

In this section, we give a theorem on a Peirce decomposition of a $^*\cdot(-1,-1)$-Freudenthal-Kantor triple system equipped with the tripotent $<ccc>=c$.

**Theorem 2.1.** Let $U$ be a $^*\cdot(-1,-1)$-Freudenthal-Kantor triple system. Then, we have a decomposition with respect to a tripotent $c$ (i.e., $<ccc>=c$) as follows

$$U = U_1(c) \oplus U_\frac{1}{2}(c) \oplus U_0(c),$$

where

$$U_1(c) = \{x| (L(c,c)+R(c,c))x=0, (R(c,c)-Id)x \neq 0\},$$

$$U_\frac{1}{2}(c) = \{x| (L(c,c)+R(c,c))x \neq 0, (R(c,c)-Id)x = 0\},$$

$$U_0(c) = \{x| (L(c,c)+R(c,c))x=0, (R(c,c)-Id)x = 0\}.$$

**Proof.** See [72,73].

5 Generalized hermitian 3-algebra

In this section, we extend the $u(N)\oplus u(M)$ 3-algebras to a hermitian $(-1,-1)$-Freudenthal-Kantor triple system.

Let $D_{N,M}^*$ be the set of all $N \times M$ matrices with operation

$$<xyz>:= x\overline{y}^Tz - z\overline{y}^Tx + zx^T\overline{y},$$

where $x^T$ and $\overline{x}$ mean transpose and conjugation of $x$, respectively.

Then $D_{N,M}^*$ is a hermitian $(-1,-1)$-Freudenthal-Kantor triple system. In fact, it satisfies the conditions (0), (i),(ii),(iii),(iv) and (v). This is an extension of the $u(N) \oplus u(M)$ hermitian 3-algebra, $<xyz>:= x\overline{y}^Tz - z\overline{y}^Tx$, which is a basis for the effective action of the multiple membranes in M-theory.

One of the tripotents is given by

$$c = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix},$$

where $Id$ is a $n \times n$ identity matrix ($n \leq N,M$). Because any element is decomposed as

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(A-A^T) & B \\ 0 & D \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(A+A^T) & 0 \\ C & 0 \end{pmatrix},$$

the Peirce decomposition is given by

$$U_1(c) = \{ \begin{pmatrix} \frac{1}{2}(A-A^T) & B \\ 0 & D \end{pmatrix} \},$$

$$U_\frac{1}{2}(c) = \{ \begin{pmatrix} \frac{1}{2}(A+A^T) & 0 \\ C & 0 \end{pmatrix} \},$$

$$U_0(c) = 0.$$
References


K. Meyberg,;Lecture on algebras and triple systems, Lecture Notes, the Univ. of Virginia, 1972.


[57] A. Gustavsson, S-J. Rey, Enhanced N=8 Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2), arXiv:0906.3568 [hep-th].


[73] N. Kamiya and M. Sato, “Hermitian $(\epsilon, \delta)$-Freudenthal-Kantor triple systems and certain applications of $^\ast$-generalized Jordan triple systems to field theory,” preprint