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Triangle Constructive Trice 1

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The representative of non-distributive trice is a triangle constructive trice. We would like to introduce basic property on triangle constructive trices.

1 Introduction and Preliminaries

In [4], we proposed the algebra system with three semilattice operations with roundabout-absorption laws. The algebraic structure is called a trice. Here, some terms will be re-introduced.

A semilattice $(S,\ *)$ is a set S with a single binary, idempotent, commutative and associative operation $*$. For A a nonempty set and n a positive integer, let $(A,\ *_1, *_2, ..., *_n)$ be an algebra with n binary operations, and $(A,\ *_i)$ be a semilattice for every $i \in \{1, 2, ..., n\}$. Then, $(A,\ *_1, *_2, ..., *_n)$ is called a **n-semilattice**. We deal mainly with **triple-semilattice**, (i.e. $n = 3$). We denote each order on $(A,\ *_1, *_2, ..., *_n)$ by $a \leq_i b \iff a *_i b = b$. Let S_n be the symmetric group on $\{1, 2, ...n\}$. An algebra $(A, *_1, *_2, ..., *_n)$ has the **n-roundabout-absorption law** if it satisfies the following $n!$ identities:

$$((((a *_{\sigma(1)} b) *_{\sigma(2)} b) *_{\sigma(3)} b) ... *_{\sigma(n)} b) = b \tag{1}$$

for all $a, b \in A$ and for all $\sigma \in S_n$. An algebra $(A, *_1, *_2, *_3)$ which satisfies the 3-roundabout-absorption law is said to be a **trice**. To simplify explanation, we often omit "3" of "3-roundabout-absorption law." For triple-semilattice, we often use "T" as "$(T, *_1, *_2, *_3)$".

Let T be a triple-semilattice, and $a, b, c \in T$. A **sub-triple-semilattice** of T is a subset S of T such that

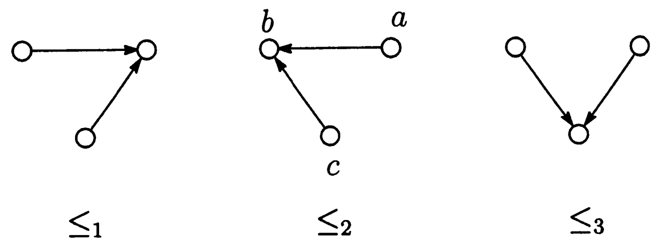
$$a, b \in S \ \text{ imply } \ a *_1 b,\ a *_2 b,\ a *_3 b \in S. \tag{2}$$

If T is a trice, then a sub-triple-semilattice S is a trice. We say that S is a **subtrice** of T.

We defined some important original concepts related to the trice in [4],
Let T be a triple-semilattice. We say that an ordered triple $(a,\ b,\ c)$ is in a **triangular situation** if $(a,\ b,\ c)$ have the following properties:

$$a *_3 b = c \ \text{ and } \ a *_2 c = b \ \text{ and } \ b *_1 c = a. \tag{3}$$

If T is a trice, the set $\{a, b, c\}$ of triangular situation is a subtrice of T (See Fig.1).



Three figures show the same set. A little circle expresses an element. An element in the same position in figures is the same. We depict three orders in the set by arrows of figures. We suppose that arrowhead is larger than the other end.

Figure 1: triangular situation

Let T be a triple-semilattice. We say that T has the **triangle constructive law** if T has the following properties:

$$(d *_1 e) *_3 (d *_2 e) = d *_3 e \tag{4}$$
$$(d *_1 e) *_2 (d *_3 e) = d *_2 e \tag{5}$$
$$(d *_3 e) *_1 (d *_2 e) = d *_1 e \tag{6}$$

for all $d, e \in T$.
The trices with triangle constructive laws are said to be **triangle constructive trices** or **t-c trices.**
We say that T has the **triangle natural law** if T has the following properties:

$$\text{if } x *_1 y = z \text{ and } x *_2 z = y, \quad \text{then } \ y *_3 z = x \tag{7}$$
$$\text{if } x *_3 y = z \text{ and } x *_2 z = y, \quad \text{then } \ y *_1 z = x \tag{8}$$
$$\text{if } x *_1 y = z \text{ and } x *_3 z = y, \quad \text{then } \ y *_2 z = x \tag{9}$$

for all $x, y, z \in T$.
We consider the set $\{x \in T | x \leq_1 a, x \leq_2 b, x \leq_3 c\}$. We will denote it by $[a,\ b,\ c]$, and call it **hyper-interval.**

We investigated distributive trices in [5].
A **distributive trice** is a trice satisfying the following six distributive laws:

$$a *_n (b *_m c) = (a *_n b) *_m (a *_n c) \tag{10}$$

for all $a, b, c \in T$ and $m, n \in \{1, 2, 3\}$ $(m \neq n)$.
One of basic subtrice is the set $\{a, b\}$ such that $b \leq_1 a$, $b \leq_2 a$ and $a \leq_3 b$ (see Fig.2).

A triple semilattice T is called **bounded** if and only if T has "**1**" as maximum of $\leq_1$, "**2**" as maximum of $\leq_2$ and "**3**" as maximum of $\leq_3$.

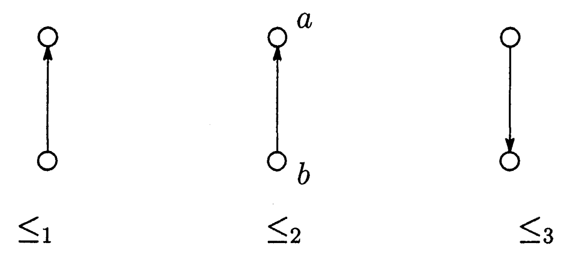


Figure 2: two elements trice

2 Three fundamental theorems

In [8], we prove three fundamental theorems of triangle constructive trices. The representative of non-distributive trice is a triangle constructive trice. The first theorem is that two elements trice which composes the distributive trice doesn't exist in a triangle constructive trice at all. The second is a theorem to compose the order reversing operation on triangle constructive trices. The third is a theorem concerning homomorphism of triangle constructive trices. The homomorphism of two operations leads the homomorphism of another operation in triangle constructive trices.

In a sense, the trice is enhancing of the dimension of the lattice. However, the character is greatly different from the lattice. The most important lattice is a Boolean algebra. A Boolean algebra is complemented distributive lattice. And, if the distributive law is satisfied, the calculation of lattice is easy. Hence, it is natural that the distributivity is an important concept in lattices. On the other hand, non-distributive trices are interesting. The reason why is that, if a triangular situation exists in a trice, the distributivity is not satisfied. That is, the triangular situation which is an essential example of trice is non-distributive. And the triangular situation is one of t-c trices. The trices with triangle constructive law do not positively satisfy distributivity.

Theorem 1 *Let T be a t-c trice. For $x, y \in T$ and $m, n \in \{1,2,3\}$ $(m \neq n)$,*

$$if\ \ x \leq_m y\ \ and\ \ x \leq_n y\ \ \ then\ \ x = y.$$

We compare this theorem with theorem 1 and theorem 2 in [5]. On the one hand, every distributive trice is a subtrice of the direct product of two element trices. On the other hand, every t-c trice do not have two element trice as subtrice. The concept of t-c trice is in directly opposite position to the concept of distributive trice.

In [7], negation operation ${}^c(not)$ on L_I could be constructed by combining the operations $*_1$, $*_2$ and $*_3$. The negation of fuzzy set is a concept that is weaker than complement on Boolean algebra. The negation is order-reversing (i.e. if $a \leq b$, then $b^c \leq a^c$) and involution (i.e. $(a^c)^c = a$). We will show that order-reversing operation can be constructed on t-c trices.

Theorem 2 *Let T be a t-c trice. For $m, n, k \in \{1, 2, 3\}$ $(m \neq n \neq k \neq m)$, if $b \leq_m c \leq_m d$ and $d \leq_n c \leq_n b$, then $b \leq_k b *_k c \leq_k b *_k d$ and $b *_k d \leq_n b *_k c \leq_n b$.*

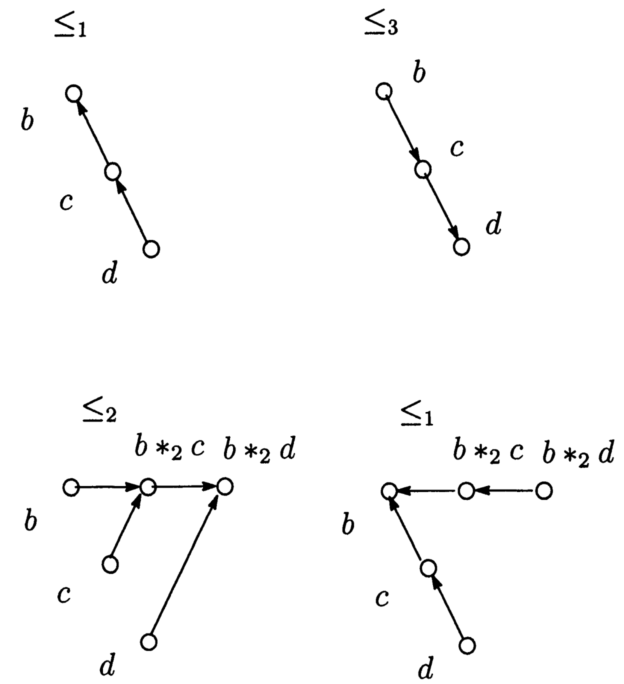


Figure 3: In case of $m = 3$, $n = 1$ and $k = 2$

This is a theorem to compose the order reversing operation. On bounded t-c trice, the set $\{t \in T : \mathbf{2} \leq_1 t \leq_1 \mathbf{1}$ and $\mathbf{1} \leq_2 t \leq_2 \mathbf{2}\}$, $\{t \in T : \mathbf{3} \leq_2 t \leq_2 \mathbf{2}$ and $\mathbf{2} \leq_3 t \leq_3 \mathbf{3}\}$ and $\{t \in T : \mathbf{1} \leq_3 t \leq_3 \mathbf{3}$ and $\mathbf{3} \leq_1 t \leq_1 \mathbf{1}\}$ are isomorphic. If we develop this theorem, it may leads the symmetry of t-c trices. Next theorem indicate that the homomorphism of two operations leads the homomorphism of another operation in t-c trices.

Theorem 3 *Let T and T' be t-c trices. And let f be a function T to T'. For $m, n, k \in \{1, 2, 3\}$ $(m \neq n \neq k \neq m)$, if $f(a *_m b) = f(a) *'_m f(b)$ for any $a, b \in T$ and $f(a *_n b) = f(a) *'_n f(b)$ for any $a, b \in T$, then $f(a *_k b) = f(a) *'_k f(b)$ for any $a, b \in T$.*

The third theorem is a theorem concerning homomorphism of t-c trices, hence we think that this is an especially important theorem as algebra.

3 Concrete composition method of t-c trices

The three points triple-semilattice $\{a, b, c\}$ of triangular situation (Fig.1) is the most basic bounded t-c trice. Next six-points triple-semilattice (Fig.4) is a typical example of bounded t-c trice. More generally, the triangular number points triple-semilattice

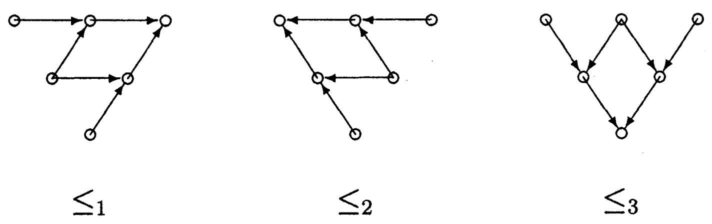


Figure 4: six-points triple-semilattice

of the type of Fig.5 is bounded t-c trice.

Cartesian products of trices is thought as a method of composing new trice. Cartesian products of t-c trices are a t-c trice. However, it is difficult to compose new trice by direct sum or quotient of trices. We introduce concrete composition method of t-c trice.

Suppose $(T_1,\ *_1, *_2, *_3)$ and $(T_2,\ *_1, *_2, *_3)$ are bounded triple-semilattices. Let $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ be the maximum of $\leq_1$, $\leq_1 2$ and $\leq_3$ of $(T_1,\ *_1, *_2, *_3)$. And let $\mathbf{1}'$, $\mathbf{2}'$ and $\mathbf{3}'$ be the maximum of $\leq_1$, $\leq_1 2$ and $\leq_3$ of $(T_2,\ *_1, *_2, *_3)$. On the set $T_1 \cup T_2$, we assume next equivalence relation

$$\mathbf{1} \equiv \mathbf{1}' \quad \mathbf{2} \equiv \mathbf{2}' \quad \mathbf{3} \equiv \mathbf{3}' \tag{11}$$

Let $T_3 = (T_1 \cup T_2)/\equiv$. Then, the $(T_3,\ *_1, *_2, *_3)$ is new bounded triple-semilattice. We call this method **maximum-identified**, And $(T_3,\ *_1, *_2, *_3)$ is called **maximum-identified** of $(T_1,\ *_1, *_2, *_3)$ and $(T_2,\ *_1, *_2, *_3)$ (see Fig.6).

Even if T_1 and T_2 are trice, T_3 is not always trice. We prepare the following theorem.

Theorem 4 *Suppose* $(T,\ *_1, *_2, *_3)$ *is a bounded t-c trice. Let* $\mathbf{1}$, $\mathbf{2}$ *and* $\mathbf{3}$ *be the maximum of* $\leq_1$, $\leq_2$ *and* $\leq_3$, *respectively. Then,*

$$\mathbf{1} *_3 \mathbf{2} = \mathbf{3} \tag{12}$$

$$\mathbf{2} *_1 \mathbf{3} = \mathbf{1} \tag{13}$$

$$\mathbf{3} *_2 \mathbf{1} = \mathbf{2} \tag{14}$$

that is, $(\mathbf{1}, \mathbf{2}, \mathbf{3})$ *is a triangular situation.*

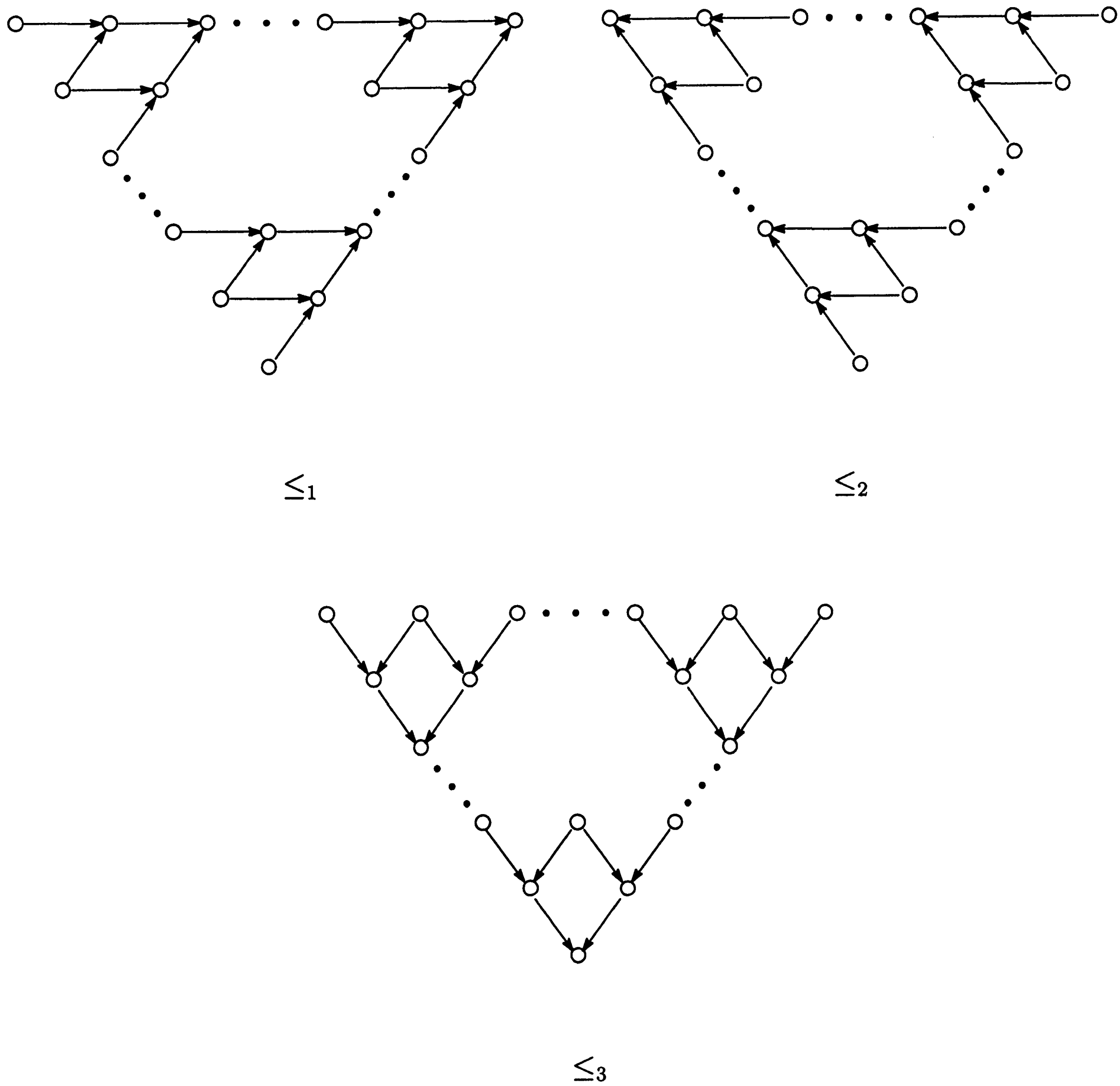
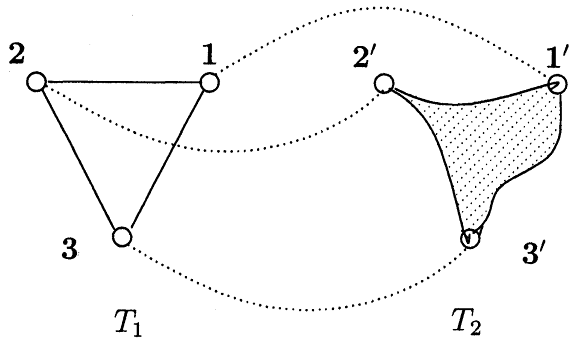
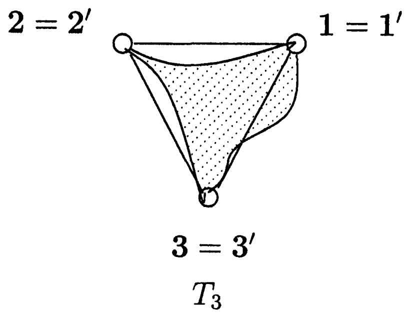


Figure 5: triangular number points triple-semilattice

T_1 and T_2 are bounded triple-semilattices.





T_3 is a new bounded triple-semilattice.

Figure 6: image of maximum-identified

Theorem 4 yields

Theorem 5 *If* $(T_1, *_1, *_2, *_3)$ *and* $(T_2, *_1, *_2, *_3)$ *are bounded t-c trices, then* $(T_3, *_1, *_2, *_3)$ *is a bounded t-c trice.*

The following result can be obtained by actually composing.

Theorem 6 *There exists a t-c trice with any number points, excluding 2, 4, 5, 7, 8, 11, 14.*

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