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Locally freely productable groups and the primitivity of their group rings

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Let R be a ring with the identity element. R is (right) primitive provided there exists a faithful irreducible (right) R-module. A group G is LFP(locally freely productable) provided for each finitely generated subgroup $H=\langle g_1, \cdots, g_n\rangle$ of G, either H is a non-trivial free products of groups both of which are not isomorphic to $\mathbb{Z}_2$ or there exists an element $x\in G$ with $x\neq 1$ such that $H*\langle x\rangle$ is free product. In this note, we shall introduce the primitivity of group rings of LFP groups. And as a result, we state that every group ring of a one-relator group with torsion is primitive. In order to prove primitivity of group rings, we shall need the graph theoretic approach used in [5] which extends the Formanek's method in [3].

1 Graph theoretic approach

Let KG be the group ring of a group G over a field K, and let $a=\sum_{i=1}^{m}\alpha_i f_i$ and $b=\sum_{i=1}^{n}\beta_i g_i$ be in KG $(\alpha_i\neq 0, \beta_i\neq 0)$. If $ab=0$ then for each f_ig_j, there exists f_pg_q such that $f_ig_j=f_pg_q$. Suppose that the following k equations hold; $f_1g_1=f_2g_2$, $f_3g_2=f_4g_3$, $\cdots$, $f_{2k-3}g_{k-1}=f_{2k-2}g_k$ and $f_{2k-1}g_k=f_{2k}g_1$. Then we can regard the above equations as forming a kind of cycle, and they imply $f_1^{-1}f_2\cdots f_{2k-1}^{-1}f_{2k}=1$. That is, the above equations give us a information on supports of a. We can use this idea for a more general case; $a_1b_1+\cdots+a_nb_n\in K$ for $a_i, b_i\in KG$ with $a_i=\sum\alpha_{ij}f_{ij}$ and $b_i=\sum\beta_{ik}g_{ik}$. In order to do this, regarding the elements $f_{ij}g_{ik}$ appeared in a_ib_i as vertices and the equalities of their elements as edges, we use a graph-theoretic method.

Throughout this section, $\mathcal{G}=(V, E)$ denotes a simple graph; a finite undirected graph which has no multiple edges or loops, where V is the set of vertices and E is the set of edges. A finite sequence $v_0e_1v_1\cdots e_pv_p$ whose terms are alternately elements e_q's in E and v_q's in V is called a path of length p in $\mathcal{G}$ if $v_{q-1}v_q=e_q\in E$ and $v_q\neq v_{q'}$ for any $q, q'\in\{0,1,\cdots,p\}$ with $q\neq q'$; simply denoted by $v_0v_1\cdots v_p$. Two vertices v and w of $\mathcal{G}$ are said to be connected if there exists a path from v to w in $\mathcal{G}$. Connection is an equivalence relation on V, and so there exists a decomposition of V into subsets C_i's $(1\leq i\leq m)$ for some $m>0$ such that $v, w\in V$ are connected if and only if both v and w belong to the same set C_i.

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The subgraph generated by C_i is called a (connected) component of $\mathcal{G}$. Any graph is a disjoint union of components.

Definition 1.1 *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$ be simple graphs with the same vertex set V. For $v \in V$, let $U(v)$ be the set consisting of all neighbours of v in $\mathcal{H}$ and v itself: $U(v) = \{w \in V \mid vw \in F\} \cup \{v\}$. A triple (V, E, F) is an SR-graph (for a sprint relay like graph) if it satisfies the following conditions:*

(i) *$\mathcal{G}$ is a clique graph; thus $uv, vw \in E$ implies $uw \in E$.*
(ii) *If C is a component of $\mathcal{G}$ and $v, w \in C$ with $v \neq w$, then $U(v) \cap U(w) = \emptyset$.*

If $\mathcal{G}$ has no isolated vertices, that is, if $v \in V$ then $vw \in E$ for some $w \in V$, then SR-graph (V, E, F) is called a proper SR-graph.

Fig 1 shows an example of an SR-graph, in which edges in E and F are respectively denoted by solid lines and dotted lines. In what follows, solid lines and dotted lines denote edges in E and F, respectively. In the above definition, the condition (i) means that every component of $\mathcal{G}$ is a complete graph, and (ii) does that each $U(v)$ has at most one vertex from each component of $\mathcal{G}$. Hence, under the assumption (i), (ii) is equivalent to the condition that if $w, u \in U(v)$ then $wv \notin E$. That is, (i) and (ii) implies that there exists no subgraph of types appeared in Fig 2.

We call $U(v)$ the SR-neighbour set of $v \in V$, and set $\mathfrak{U}(V) = \{U(v) \mid v \in V\}$. For $v, w \in V$ with $v \neq w$, it may happen that $U(v) = U(w)$, and so $|\mathfrak{U}(V)| \leq |V|$ generally. Let $\mathcal{S} = (V, E, F)$ be an SR-graph. We say $\mathcal{S}$ is connected if the graph $(V, E \cup F)$ in which there is no distinction between E and F is connected.

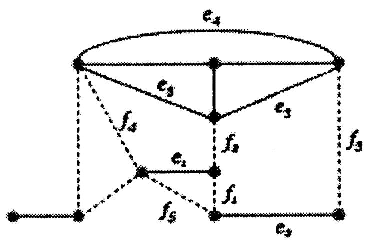

Fig 1. An example of an SR-graph: Solid lines are edges in E and dotted lines are edges in F. Sequences $(e_1, f_1, e_2, f_3, e_4, f_4, e_1)$, $(e_1, f_2, e_3, f_3, e_2, f_5)$ and (e_1, f_2, e_5, f_4) are SR-cycles.

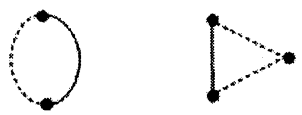

Fig 2. Prohibits : It is not allowed to exist the above two subgraphs in an SR-graph.

Definition 1.2 *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $p > 1$. Then a path $v_1w_1v_2w_2, \cdots, v_pw_pv_{p+1}$ in the graph $(V, E \cup F)$ is called a SR-path of length p in $\mathcal{S}$ if either $v_qw_q \in E$ and $w_qv_{q+1} \in F$ or $v_qw_q \in F$ and $w_qv_{q+1} \in E$ for $1 \leq q \leq p$; simply denoted by $(e_1, f_1, \cdots, e_p, f_p)$ or $(f_1, e_1, \cdots, f_p, e_p)$, respectively, where $e_q \in E$ and $f_q \in F$. If, in addition, it is a cycle in $(V, E \cup F)$, that is, $v_{p+1} = v_1$, then it is an SR-cycle of length p in $\mathcal{S}$.*

That is, for $e_q \in E$ and $f_q \in F$, an SR-cycle $(e_1, f_1, \cdots, e_p, f_p)$ means that it is a cycle in $(V, E \cup F)$ which consists alternately solid lines and dotted lines (see Fig1).

In what follows, let $\mathcal{S} = (V, E, F)$ be an SR-graph with $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$. $\mathfrak{C}(V)$ denotes the set of components of V on $\mathcal{H} = (V, F)$. In addition, we set $\mathfrak{N}(\mathcal{S}) = \{U \in \mathfrak{U}(V) \mid |U| = 1\}$, $\mathfrak{M}(\mathcal{S}) = \{U \in \mathfrak{U}(V) \mid |U| = 2\}$ and $\mathfrak{L}(\mathcal{S}) = \{U \in \mathfrak{U}(V) \mid |U| > 2\}$.

We would like to know when $\mathcal{S}$ has an SR-cycle. We first consider the somewhat trivial case of $\mathcal{S}$ in which $\mathcal{H} = (V, F)$ is also a clique graph. In this case, $\mathfrak{U}(V)$ coincides with $\mathfrak{C}(V)$. We have the next theorem:

Theorem 1.3 *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and let ω_E and ω_F be, respectively, the number of components of $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$. Suppose that $\mathcal{H} = (V, F)$ is a clique graph and $\mathcal{S}$ is connected. Then $\mathcal{S}$ has an SR-cycle if and only if $\omega_E + \omega_F < |V| + 1$.*

In particular, if $\mathcal{S}$ is proper and $|\mathfrak{N}(\mathcal{S})| \leq |\mathfrak{L}(\mathcal{S})|$ then $\mathcal{S}$ has an SR-cycle.

In the above theorem, every component is a complete graph. We next consider the case that every component $\mathcal{G}_i = (V_i, E_i)$ is a complete k-partite graph $K_{m_1, \cdots, m_k}$. Let $\mu(V_i)$ be the maximum number in $\{m_1, \cdots, m_k\}$. For $v \in V$, let $d_{\mathcal{G}}(v)$ be the degree of v in $\mathcal{G}$; thus the number of edges of $\mathcal{G}$ incident with v. $I(V)$ denotes the set of isolated vertices in $\mathcal{G}$; thus $I(V) = \{v \in V \mid d_{\mathcal{G}}(v) = 0\}$. Then we have

Theorem 1.4 *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $\mathfrak{C}(V) = \{V_1, \cdots, V_n\}$ with $n > 1$. Suppose that every component $\mathcal{G}_i = (V_i, E_i)$ of $\mathcal{G}$ is a complete k-partite graph. If $|V_i| > 2\mu(V_i)$ for each $i \in \{1, \cdots, n\}$ and $|I(V)| \leq n$ then $\mathcal{S}$ has an SR-cycle.*

We can prove two theorems above by a similar argument in [5].

2 LFP groups

Definition 2.1 *A group G is LFP provided for each finitely generated subgroup $H = \langle g_1, \cdots, g_n \rangle$ of G, either H is a non-trivial free products of groups both of which are not isomorphic to $\mathbb{Z}_2$ or there exists an element $x \in G$ with $x \neq 1$ such that $H * \langle x \rangle$ is free product.*

It is obvious that a locally free group is LFP and so is free group. Moreover, by the Kurosh Subgroup Theorem for free products, we can see that the non-trivial free product $A * B$ of groups A and B is LFP provided both of A and B are not isomorphic to $\mathbb{Z}_2$.

By making use of theorems in the previous section, we can state the following theorem:

Theorem 2.2 *If G is LFP, then the group ring KG is primitive for any field K.*

3 Primitivity of group rings of one-relator groups with torsion

Let $\langle X\rangle$ be the free group with the base X. For a word R in $\langle X\rangle$, $G = \langle X \mid R\rangle$ denotes the one-relator group with a generating set X of G and a defining relation $R = 1$. If W is a cyclically reduced word in $\langle X\rangle$ and $R = W^n$ $(n > 1)$, then G is called a one-relator group with torsion. The class of one-relator groups with torsion has been well studied, in particular, on residual finiteness (for instance, [2], [7], [8], [1]).

In this section, by making use of the Theorem 2.2, we shall show the next theorem:

Theorem 3.1 *The group ring KG of $G = \langle X \mid W^n\rangle$ over a field K is primitive provided $n > 1$ and $|X| > 1$, where W is a cyclically reduced word in $\langle X\rangle$.*

In what follows, let $F = \langle X\rangle$ be the free group with the base $X = \{x_1, \cdots, x_m\}$. $\langle g_1, \cdots g_m\rangle_G$ denotes the subgroup of a group G generated by $g_1, \cdots, g_m \in G$. If $W \in F$, then $\mathcal{N}_F(W)$ denotes the normal closure of W in F. For a cyclically reduced word W, $\mathcal{W}_F(W)$ denotes the set of all cyclically reduced conjugates of both W and W^{-1}. If $W_i, \cdots, W_t$ are reduced words in F and $W = W_i \cdots W_t$ is also reduced, that is, there is no cancellation in forming the product $W_i \cdots W_t$, then we write $W \equiv W_i \cdots W_t$.

Lemma 3.2 *Let $m, n > 1$ and $W_0 = W_0(x_1, \cdots, x_m)$ be a cyclically reduced word in F which involves all x_i's in X. Suppose that $V \in \mathcal{N}_F(R_0)$, where $R_0 = W_0^n$. If $V \equiv V_1V_2$, then every generator in X appears either in V_1 or in V_2.*

Proof. By the well-known the Newman-Gurevich Spelling Theorem([6], cf. [4]), V contains a subword $S^{n-1}S_0$, where $S \equiv S_0S_1 \in \mathcal{W}_F(W_0)$ and every generator in X appears in S_0. Hence either V_1 or V_2 contains the subword S_0, and the assertion follows.

Lemma 3.3 *For $m > 1$, $n > 1$ and $X = \{x_1, \cdots, x_m\}$, let $G = \langle X \mid R\rangle$, where $R = W^n$ and W is a cyclically reduced words in the free group $\langle X\rangle$ with the base X. If $S, T \subseteq X$, then $\langle S\rangle_G \cap \langle T\rangle_G = \langle S \cap T\rangle_G$.*

Proof. It is obvious that $\langle S\rangle_G \cap \langle T\rangle_G \supseteq \langle S\cap T\rangle_G$. Suppose, to the contrary, that $\langle S\rangle_G \cap \langle T\rangle_G \neq \langle S\cap T\rangle_G$. Then there exist reduced words $u = u(s, a, \cdots, b)$ in $\langle S\rangle \setminus \langle S\cap T\rangle$ and $v = v(t, c, \cdots, d)$ in $\langle T\rangle \setminus \langle S\cap T\rangle$ such that $uv \in \mathcal{N}_F(R)$, where $a, \cdots, b \in S$, $c, \cdots, d \in T$, $s \in S \setminus (S\cap T)$ and $t \in T \setminus (S\cap T)$. Let w be the reduced word for uv, say $w \equiv u_1v_1$, where $u \equiv u_1u_2$ and $v \equiv u_2^{-1}v_1$. Then $w \in \mathcal{N}_F(R)$, however, u_1 involves s but not t, and v_1 involves t but not s, which cntradicts the assertion of Lemma 3.2.

Let $X = \{a_i, b_i, \cdots \mid i \in \mathbb{Z}\}$ and W_i $(i \in \mathbb{Z})$ cyclically reduced words in the free group $\langle X\rangle$ with the base X such that

$$W_i = W_i(a_{j_{a1}+i}, \cdots, a_{j_{as}+i}, b_{j_{b1}+i}, \cdots, b_{j_{bt}+i}, \cdots),$$

where $j_{a1} < j_{a2} < \cdots < j_{as}$ and $j_{b1} < j_{b2} < \cdots < j_{bt}$ and $\cdots$. Let α_*, $\beta_*, \cdots$ be the minimum subscripts on a, b, $\cdots$ occurring in W_0, respectively, and α^*, β^*, $\cdots$ be the maximum subscript on a, b, $\cdots$ occurring in W_0, respectively. That is, $\alpha_* = j_{a1}$, $\alpha^* = j_{as}$ and $\beta_* = j_{b1}$, $\beta^* = j_{bt}$ and $\cdots$. We set $A = \{a_i \mid i \in \mathbb{Z}\}, B = \{b_i \mid i \in \mathbb{Z}\}, \cdots$; in this case, $X = A \cup B \cup \cdots$.

Let

$$G_\infty = \langle X \mid R_i(i \in \mathbb{Z})\rangle \text{ with } R_i = W_i^n(n > 1). \tag{1}$$

In G_∞, we set subgroups Q_t and P_t of G_∞ for all $t \in \mathbb{Z}$, as follows:

$$\begin{cases} \text{For } N \neq 0, \\ Q_t \;= \langle a_{i+t}, b_{j+t}, \cdots \mid \alpha_* \leq i \leq \alpha^*,\ \beta_* \leq j \leq \beta^*, \cdots\rangle_{G_\infty}, \\ P_t \;= \langle a_{i+t}, b_{j+t}, \cdots \mid \alpha_* \leq i \leq \alpha^* - 1,\ \beta_* \leq j \leq \beta^* - 1, \cdots\rangle_{G_\infty}. \\ \text{For } N = 0, \\ Q_t \;= \langle a_{\alpha^*+t}, b_{\beta^*+t}, \cdots\rangle_{G_\infty}, \\ P_t \;= 1. \end{cases} \tag{2}$$

where N is the maximum number in $\{\alpha^* - \alpha_*, \beta^* - \beta_*, \cdots\}$.

Then $P_t \leq Q_t$ and $Q_t \simeq \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t}, b_{\beta_*+t}, \cdots, b_{\beta^*+t}, \cdots \mid R_t\rangle$. By the Magnus' method for Freiheitssatz, we may identify G_∞ as the union of the chain of the following G_i's:

$$\begin{aligned} &G_\infty = \textstyle\bigcup_{i=0}^\infty G_i, \text{ where} \\ &G_0 = Q_0, \quad G_{2i} = Q_{-i} *_{P_{-i+1}} G_{2i-1} \text{ and } G_{2i+1} = G_{2i} *_{P_{i+1}} Q_{i+1}. \end{aligned} \tag{3}$$

Generally, for each $k \in \mathbb{Z}$, set

$$G_0 = Q_k,\ G_{2i} = Q_{-i+k} *_{P_{-i+k+1}} G_{2i-1} \text{ and } G_{2i+1} = G_{2i} *_{P_{i+k+1}} Q_{i+k+1}, \tag{4}$$

and we can also identify G_∞ as $\bigcup_{i=0}^\infty G_i$. Then we have

$$\begin{aligned} G_0 \quad &= Q_k = \langle a_{\alpha_*+k}, \cdots, a_{\alpha^*+k}, b_{\beta_*+k}, \cdots, b_{\beta^*+k}, \cdots\rangle_{G_\infty} \\ G_{2i} \quad &= \langle a_{\alpha_*+k-i}, \cdots, a_{\alpha^*+k+i}, b_{\beta_*+k-i}, \cdots, b_{\beta^*+k+i}, \cdots\rangle_{G_\infty} \\ G_{2i+1} \quad &= \langle a_{\alpha_*+k-i}, \cdots, a_{\alpha^*+k+i+1}, b_{\beta_*+k-i}, \cdots, b_{\beta^*+k+i+1}, \cdots\rangle_{G_\infty} \end{aligned} \tag{5}$$

Lemma 3.4 *Let H be a subgroup of G_∞ generated by a finite subset Y of X; thus $H = \langle Y \rangle_{G_\infty}$. Set $I = \{i \in \mathbb{Z} \mid a_i \in A \cap Y$ or $\cdots$ or $b_i \in B \cap Y\}$, and let i^* (resp. i_*) be the maximum number (resp. the minimum number) in I and M_* (resp. m^*) the maximum number (resp. the minimum number) in $\{\alpha_*, \beta_*, \cdots\}$ (resp. $\{\alpha^*, \beta^*, \cdots\}$).*

If $N < t$ and $N + i^* - i_* + M_* - m^* < t$, then $H \cap P_t = 1$.

Proof. If $N = 0$ then the assertion of the Lemma is trivial, and so we suppose $N \neq 0$, and also suppose, to the contrary, there exists $t \in \mathbb{Z}$ such that

$$N < t, \ N + i^* - i_* + M_* - m^* < t \text{ and } H \cap P_t \neq 1.$$

If we set $k = \mu = i_* - M_*$ in (4) just above this lemma, then

$$G_0 = Q_\mu, \text{ and } G_{2i} = Q_{-i+\mu} *_{P_{-i+\mu+1}} G_{2i-1}.$$

Moreover, let τ be the largest number between 0 and $i^* - \mu - m^*$. If we set $i = \tau$ in the above, then we can see that $G_{2\tau} \supseteq H$ and $\alpha^* + \tau < \alpha_* + t$, $\beta^* + \tau < \beta_* + t$, $\cdots$.

In fact, if $\tau = 0$, then $\alpha^* + \tau = \alpha^* \leq \alpha_* + N < \alpha_* + t$, because of $N < t$. On the other hand, if $\tau \neq 0$, then $\tau = i^* - (i_* - M_*) - m^*$, and so,

$$\alpha^* + \tau \leq \alpha_* + N + \tau = \alpha_* + N + i^* - i_* + M_* - m^* < \alpha_* + t,$$

because of $N + i^* - i_* + M_* - m^* < t$. We similarly obtain that $\beta^* + \tau < \beta_* + t$, $\cdots$.

Next, we shall show $G_{2\tau} \supseteq H$. To see this, since

$$G_{2\tau} = \langle a_{\alpha_*+\mu-\tau}, \cdots, a_{\alpha^*+\mu+\tau}, b_{\beta_*+\mu-\tau}, \cdots, b_{\beta^*+\mu+\tau}, \cdots \rangle_{G_\infty},$$

it sufficies to show that $\alpha_* + \mu - \tau \leq i_*$, $\beta_* + \mu - \tau \leq i_*$, $\cdots$, and $\alpha^* + \mu + \tau \geq i^*$, $\beta^* + \mu + \tau \geq i_*$, $\cdots$. Note that $\mu + \tau = i^* - m^*$ if $\tau \neq 0$ and $\mu \geq i^* - m^*$ if $\tau = 0$. In fact, if $\tau \neq 0$, then $\mu + \tau = \mu + i^* - \mu - m^* = i^* - m^*$, and if $\tau = 0$, then $i^* - \mu - m^* \leq 0$ and so $i^* - m^* \leq \mu$.

Since $\tau \geq 0$ and $\alpha_* - M_* \leq 0$ by definitions, we have

$$\alpha_* + \mu - \tau \leq \alpha_* + \mu = i_* + \alpha_* - M_* \leq i_*.$$

We similarly obtain that $\beta_* + \mu - \tau \leq i_*$, $\cdots$. Moreover, as mentioned above, if $\tau = 0$, then $\mu \geq i^* - m^*$, and so we have that

$$\alpha^* + \mu + \tau \geq \alpha^* + i^* - m^* \geq \alpha^* + i^* - \alpha^* = i^*$$

because $m^* \leq \alpha^*$. If $\tau \neq 0$, since $\mu + \tau = i^* - m^*$, we also have

$$\alpha^* + \mu + \tau = \alpha^* + i^* - m^* \geq \alpha^* + i^* - \alpha^* = i^*.$$

We have thus seen $\alpha^*+\mu+\tau \geq i^*$ for either cases, and similarly we have $\beta^*+\mu+\tau \geq i^*$, $\cdots$, as desired.

In the above, replacing $\alpha_* + \mu$ with α_*, $\alpha^* + \mu$ with α^*, $\beta_* + \mu$ with β_*, $\cdots$, and τ with k, we may assume that $G_\infty = \bigcup_{i=0}^{\infty} G_i$ with the presentation (4) and there exists $k \geq 0$ such that $G_{2k} \supseteq H$ and

$$\alpha^* + k < \alpha_* + t,\ \beta^* + k < \beta_* + t,\ \cdots. \tag{6}$$

Now, let $n = \beta^* - \beta_*$, and we may here assume $N = \alpha^* - \alpha_* \geq \cdots \geq \beta^* - \beta_*$. For $j \in \{0, 1, \cdots, N\}$, we define $P_t^{(j)}$'s so as to satisfy

$$P_t = P_t^{(N)} \supset P_t^{(1)} \supset \cdots \supset P_t^{(0)} = 1$$

as follows:

$$\begin{aligned}
P_t = \ & P_t^{(N)} && = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-1}, b_{\beta_*+t}, \cdots, b_{\beta^*+t-1}, \cdots \rangle_{G_\infty} \\
& P_t^{(N-1)} && = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-2}, b_{\beta_*+t}, \cdots, b_{\beta^*+t-2}, \cdots \rangle_{G_\infty}, \\
& \vdots && \vdots \\
& P_t^{(N-n+1)} && = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-n}, b_{\beta_*+t}, \cdots \rangle_{G_\infty}, \\
& P_t^{(N-n)} && = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-n-1}, \cdots \rangle_{G_\infty}, \\
& \vdots && \vdots \\
& P_t^{(1)} && = \langle a_{\alpha_*+t} \rangle_{G_\infty}, \\
& P_t^{(0)} && = 1.
\end{aligned}$$

By our assumption, $H \cap P_t \neq 1$, that is, there exists $u \in H \cap P_t$ such that $u \neq 1$. Then there exists $l \in \{0, 1, \cdots, N-1\}$ such that $u \in P_t^{(N-l)}$ and $u \notin P_t^{(N-l-1)}$. We shall show that this is impossible. In fact, we shall show that $u \in P_t^{(N-l)}$ implies $u \in P_t^{(N-l-1)}$, and this completes the proof of the Lemma.

By (6), $\alpha^* + k \leq \alpha_* + t - 1$, and so $k \leq -N + t - 1 \leq -l + t - 2$, which implies

$$H \subseteq G_{2(t-l-2)} \tag{7}$$

because $H \subseteq G_{2k} \subseteq G_{2(t-l-2)}$. By way of construction of $P_t^{(N-l)}$, we have

$$P_t^{(N-l)} = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-l-1}, b_{\beta_*+t}, \cdots, b'_{\beta^*+t-l-1}, \cdots \rangle_{G_\infty},$$

where $b'_{\beta^*+t-l-1} = b_{\beta^*+t-l-1}$ if $l < n$ and $b'_{\beta^*+t-l-1} = 1$ if $l \geq n$. By (2), we also have

$$Q_{t-l-1} = \langle a_{\alpha_*+t-l-1}, \cdots, a_{\alpha^*+t-l-1}, b_{\beta_*+t-l-1}, \cdots, b_{\beta^*+t-l-1}, \cdots \rangle_{G_\infty},$$

and therefore we see that $P_t^{(N-l)} \subseteq Q_{t-l-1}$. Combining this with (7), it follows that $u \in G_{2(t-l-2)} \cap Q_{t-l-1}$. Since $G_{2(t-l-2)} \cap Q_{t-l-1} = P_{t-l-1}$, we have $u \in P_{t-l-1}$, and thus $u \in P_{t-l-1} \cap P_t^{(N-l)}$.

On the other hand, $P_{t-l-1} = \langle S \rangle_{Q_{t-l-1}}$ and $P_t^{(N-l)} = \langle T \rangle_{Q_{t-l-1}}$ in Q_{t-l-1}, where

$$\begin{aligned} S &= \{a_{\alpha_*+t-l-1}, \cdots, a_{\alpha^*+t-l-2}, b_{\beta_*+t-l-1}, \cdots, b_{\beta^*+t-l-2}, \cdots\} \\ \text{and} \quad T &= \{a_{\alpha_*+t}, \cdots, a_{\alpha^*+t-l-1}, b_{\beta_*+t}, \cdots, b'_{\beta^*+t-l-1}, \cdots\}. \end{aligned}$$

Then it is easily seen that $\langle S \cap T \rangle_{Q_{t-l-1}} = P_t^{(N-l-1)}$. We can here identify Q_{t-l-1} as the one-relator group with torsion, and therefore it follows form Lemma 3.3 that

$$u \in P_{t-l-1} \cap P_t^{(N-l)} = \langle S \rangle_{Q_{t-l-1}} \cap \langle T \rangle_{Q_{t-l-1}} = \langle S \cap T \rangle_{Q_{t-l-1}} = P_t^{(N-l-1)};$$

thus $u \in P_t^{(N-l-1)}$, as desired.

By the proof of the above Lemma, we have

Corollary 3.5 *If H be a subgroup of G_∞ generated by a finite subset Y of X, then there exists a positive integer t such that $H \subseteq G_{2(t-1)}$ and $H \cap P_t = 1$.*

Lemma 3.6 *If G_∞ and W_i are as in (1), then for each finite elements $g_1, \cdots, g_m$ in G_∞, there exists an integer i such that $\langle g_1, \cdots, g_m, W_i \rangle_{G_\infty}$ is the free product $\langle g_1, \cdots, g_m \rangle_{G_\infty} * \langle W_i \rangle_{G_\infty}$.*

Proof. Let G_∞ be as in (3) and Y the set of generators which appear in g_i's. By virtue of Corollary 3.5, for $H = \langle Y \rangle_{G_\infty}$, there exists $t > 0$ such that $H \subseteq G_{2(t-1)}$ and $H \cap P_t = 1$.

Now, by (3), $G_{2t-1} = G_{2(t-1)} *_{P_t} Q_t$, where

$$Q_t = \langle a_{\alpha_*+t}, \cdots, a_{\alpha^*+t}, b_{\beta_*+t}, \cdots, b_{\beta^*+t}, \cdots \mid R_t \rangle,$$

and either $P_t = \langle a_{i+t}, b_{j+t}, \cdots \mid \alpha_* \le i \le \alpha^* - 1,\ \beta_* \le j \le \beta^* - 1, \cdots \rangle_{G_\infty}$ or $P_t = 1$. We see then that $W_t \in Q_t$. As is well known, $W_t^m \neq 1$ if $1 \le m < n$ because $R_t = W_t^n$ and $n > 1$. Moreover, if $W_t^m \in P_t$, then $(W_t^m)^n \neq 1$ becasuse P_t is a free subgroup in Q_t by Freiheitssatz, which implies contradiction. Hence we have that $\langle W_t \rangle \cap P_t = 1$. Combining this with $H \cap P_t = 1$, we see that $\langle Y, W_t \rangle_{G_{2t-1}} = \langle Y \rangle_{G_{2t-1}} * \langle W_t \rangle_{G_{2t-1}} = H * \langle W_t \rangle_{G_\infty}$. Since $\langle g_1, \cdots, g_m \rangle_{G_\infty} \subseteq H$, we have that $\langle g_1, \cdots, g_m, W_t \rangle_{G_\infty} = \langle g_1, \cdots, g_m \rangle_{G_\infty} * \langle W_t \rangle_{G_\infty}$.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1 If there exists $x \in X$ such that W contains none of x or x^{-1}, then G is a non-trivial free product of groups both of which are not isomorphic to $\mathbb{Z}_2$. Hence we may assume that $X = \{x_1, \cdots, x_m\}$ $(m > 1)$ and W contains either x_i or x_i^{-1} for all $i \in \{1, \cdots, m\}$.

If W has no zero exponent sum $\sigma_x(W)$ on x for all $x \in X$, say $\sigma_{x_1}(W) = \alpha$ and $\sigma_{x_2}(W) = \beta$, then $G \simeq \langle a^\beta, x_2, \cdots, x_m \mid R \rangle \subset E$, by the Magnus' method for Freiheitssatz, where $R = W^n(a^\beta, x_2, \cdots, x_m)$ and $E = \langle a, x_2, \cdots, x_m \mid R\rangle$. Let $N = \mathcal{N}_F(x_2a^\alpha, x_3\cdots, x_m)$, where $F = \langle x_1, \cdots, x_m\rangle$. Then we have that $N \supset \mathcal{N}_F(R)$ and $N/\mathcal{N}_F(R) \simeq G_\infty$, where G_∞ is as in (1), and so we may let $G_\infty = N/\mathcal{N}_F(R)$.

Let $F_G = \langle a^\beta, x_2, \cdots, x_m\rangle$ and $H = (N \cap F_G)/\mathcal{N}_{F_G}(R)$. Then we can easily see that H can be isomorphically embedded in G_∞ and that G is a cyclic extension of H. Since $W_i \in H$, it follows from Lemma 3.6 that H is LFP. Hence KH is primitive for any field K by Theorem 2.2. Since G/H is cyclic, by [9, Theorem 1], we have that KG is also primitive.

If W has a zero exponent sum $\sigma_x(W)$ on x for some $x \in X$, say $\sigma_{x_1}(W) = 0$, then we set $N = \mathcal{N}_F(x_2, x_3\cdots, x_m)$. Since $N/\mathcal{N}_F(R) \simeq G_\infty$ and G is a cyclic extension of $N/\mathcal{N}_F(R)$, the result is similarly obtained as above. This completes the proof of the theorem.

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