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Some questions on the real numbers *

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Friends of main, who are specialists in functional analysis, asked me some questions on the real numbers and real functions. Some are easy to answer and some are not.

1 Queer additive functions

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real function. $f$ is additive, if

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. It is well known that there is a non-linear (thus, discontinuous) additive real function. Now, the first question is

Question 1.1. Is there a discontinuous additive real function $f$ satisfying

$$f(\sqrt{2}x) = \sqrt{2} f(x)$$

for all $x \in \mathbb{R}$?

The following is a general answer to this question.

**Theorem 1.2.** Let $A$ be a set of real numbers such that the field $K = \mathbb{Q}(A)$ generated by $A$ over $\mathbb{Q}$ is not equal to $\mathbb{R}$. Then, there is a discontinuous additive function $f$ satisfying

$$f(\alpha x) = \alpha f(x) \quad (1)$$

for all $\alpha \in A$ and $x \in \mathbb{R}$.

**Proof.** Let $\{e_i\}_{i \in I}$ be a $K$-linear base of $\mathbb{R}$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a $K$-linear mapping such that $f(e_i) = 0$ and $f(e_j) = 1$ for $i, j \in I$ with $i \neq j$. Then, $f$ is a discontinuous additive function satisfying (1).

The next queer question is

*This is a final version and will not appear elsewhere.*
Question 1.3. Is there a nonzero additive real function $f$ satisfying
\[ f(\sqrt{2}x) = \sqrt{3} f(x) \]
for all $x \in \mathbb{R}$?

More generally, for real numbers $\alpha$, $\beta$ and a real function $f$, consider the property
\[ C(\alpha, \beta) : f(\alpha x) = \beta f(x) \quad \text{for all } x \in \mathbb{R}. \]

Question 1.4. Is there a nonzero additive real function $f$ satisfying $C(\alpha, \beta)$ for real numbers $\alpha \neq \beta$?

If both $\alpha$ and $\beta$ are transcendental, or algebraic with the same minimal polynomial over $\mathbb{Q}$, the substitution $\alpha \rightarrow \beta$ induces an isomorphism $\phi_{\alpha\beta} : \mathbb{Q}(\alpha) \rightarrow \mathbb{Q}(\beta)$ of fields. We have
\[ \phi_{\alpha\beta}(\alpha f(\alpha)) = \beta f(\beta) = \beta \phi_{\alpha\beta}(f(\alpha)) \]
for any $f(\alpha) \in K = \mathbb{Q}(\alpha)$. Let $L$ be a $K$-subspace of $\mathbb{R}$ such that $\mathbb{R} = K \oplus L$. Extend $\phi_{\alpha\beta} : K \rightarrow \mathbb{Q}(\beta) \subset \mathbb{R}$ to an additive map $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ by defining $\Phi|_K = \phi_{\alpha\beta}$ and $\Phi|_L = 0$. Then, $\Phi$ satisfies $C(\alpha, \beta)$.

Theorem 1.5. For $\alpha, \beta \in \mathbb{R}$, there is a nonzero additive function $\Phi$ satisfying the condition $C(\alpha, \beta)$, if and only if

(i) both $\alpha$ and $\beta$ are transcendental, or
(ii) $\alpha$ and $\beta$ are algebraic with the same minimal polynomial over $\mathbb{Q}$.

Proof. The above discussion shows the sufficiency of the condition for the existence of $f$ satisfying $C(\alpha, \beta)$.

Conversely, let $f$ be a nonzero additive function with property $C(\alpha, \beta)$. Since the additive function $f$ is $\mathbb{Q}$-linear, for any $p(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Q}[X]$ ($a_i \in \mathbb{Q}$) and for any $x \in \mathbb{R}$, we have
\[
\begin{align*}
f(p(\alpha) \cdot x) &= a_0 \Phi(x) + a_1 f(\alpha \cdot x) + \cdots + a_n f(\alpha^n \cdot x) \\
&= a_0 f(x) + a_1 \beta \cdot f(x) + \cdots + a_n \beta^n \cdot f(x) \\
&= p(\beta) \cdot f(x).
\end{align*}
\]

Hence, if $p(\alpha) = 0$, then $p(\beta) = 0$ because $f(x) \neq 0$ for some $x$. Similarly, $p(\beta) = 0$ implies $p(\alpha) = 0$. Thus, (i) or (ii) in the theorem holds. \qed

This problem has arisen from a research on stability of additive functions (Oda et al. [6]). The problem was already studied in Aczél [2].

2 Continuous semi-(group) structures on $\mathbb{R}_+$

The following question is very naive:
**Question 2.1.** Is the multiplication only the continuous group operation on the space $\mathbb{R}_+$ of positive real numbers?

For a homeomorphism $\phi$ from $\mathbb{R}_+$ onto $\mathbb{R}_+$. define an operation $\star$ on $\mathbb{R}_+$ by

$$x \star y = \phi^{-1}(\phi(x) \cdot \phi(y)) \quad (2)$$

for $x, y \in \mathbb{R}_+$. Then, $(\mathbb{R}_+, \star)$ is a topological group. The following is a positive answer to the question (Aczél [1]).

**Theorem 2.2.** The operation defined as (2) is the only way to make $\mathbb{R}_+$ a topological group.

More generally we have

**Theorem 2.3.** There are exactly three essentially distinct continuous cancellative semigroup operations on $\mathbb{R}_+$. They are the ordinary multiplication $\cdot$, the ordinary addition $+$, and the operation $\star$ defined by

$$x \star y = x + y + 1$$

for $x, y \in \mathbb{R}_+$.

Let $S = (\mathbb{R}_+, \star)$ be a topological semigroup, that is, $\star$ is a continuous with respect to the ordinary topology of $\mathbb{R}_+$. Suppose that $S$ is cancellative. Then, for any $x \in S$ the left transformation $L_x$ ($L_x(y) = x \star y$ for $y \in S$) and the right transformation $R_x$ ($R_x(y) = y \star x$) are monotone. $L_x$ cannot be decreasing, otherwise, $L_x(y) = y$ for some $y \in S$, which implies $x = e$ (the identity element) and $L_x$ is strictly increasing. Similarly $R_x$ is strictly increasing. This discussion implies that $S$ is an ordered semigroup.

An element $x \in S$ is positive (resp. negative) if $x \star x > x$ (resp. $x \star x < x$) Let $P$ (resp. $Q$) be the set of positive (resp. negative) elements of $S$. $P$ and $Q$ are open subsets of $S$ because $x \star x - x$ is a continuous function.

If $S$ has no idempotent, then $S = P \cup Q$, but since $\mathbb{R}_+$ is connected, either $S = P$ or $S = Q$ holds. Suppose that $S = P$. Then any $x \in S$ is positive and we have an increasing sequence $\{x^n\}$ in $S$, where $x^n$ is the $n$-th power of $x$ with respect $\star$. If $\lim_{n \to \infty} x^n = \hat{x} \in S$, then $\hat{x} \star \hat{x} = \lim_{n \to \infty} x^{2n} = \hat{x}$. But this cannot happen because $S$ has no idempotent. Hence, $\lim_{n \to \infty} x^n = +\infty$. Thus, for another $y \in S$, there is $n > 0$ such that $x^n > y$. So, $S$ is a positively Archimedean semigroup.

For a positively Archimedean semigroup $S$ and a fixed element $a \in S$, we define a function $\phi_a : S \to \mathbb{R}$ by

$$\phi_a(x) = \inf\{m/n \mid m, n > 0, a^m > x^n\}$$

for $x \in S$. Then, $\phi$ is a ordered homomorphism from $S$ to the additive semigroup of positive real numbers (see Fuchs [4], Hölder [5]). Moreover, it is continuous and injective (Craigen & Pales [3]).
When $S$ is negatively Archimedean, the function $\phi_\alpha'$ defined by
\[
\phi_\alpha'(x) = \inf\{m/n \mid m, n > 0, a^{m*} < x^{n*}\}
\]
for $x \in S$ is an injective order-reversing continuous homomorphism from $S$ to $(\mathbb{R}_+, +)$.

Let $\mu_\alpha = \inf\{\phi_\alpha(x) \mid x \in S\}$. If $\mu_\alpha = 0$, then $\phi_\alpha$ is an isomorphism from $S$ to $(\mathbb{R}_+, +)$ of ordered topological semigroups. If $\mu_\alpha > 0$, then $\phi_\alpha/\mu_\alpha - 1$ is an isomorphism from $S$ to $(\mathbb{R}_+, *)$.

If $S$ has an idempotent $e$, it is the identity element. Then we have $S = P \cup \{e\} \cup Q$, where $P = \{x \in S \mid x > e\}$ is a positively Archimedean semigroup and $Q = \{x \in S \mid x < e\}$ is a negatively Archimedean semigroup. Let $x \in S$. Because $\lim_{n \to \infty} x \ast a^{n*} = \lim_{n \to \infty} a^{n*} = +\infty$ for $a \in P$ and $\lim_{n \to \infty} x \ast b^{n*} = \lim_{n \to \infty} b^{n*} = -\infty$ for $b \in Q$, $L_x$ and $R_x$ are unbounded above and below. It follows that $S$ is a group.

Let $a \in P$ and $a^{-*}$ be the inverse of $a$. Define a function $\Phi : S \to \mathbb{R}$ by
\[
\Phi(x) = \begin{cases} 
\phi_\alpha(x) & \text{if } x \in P \\
0 & \text{if } x = e \\
-\phi_\alpha^{-*}(x) & \text{if } x \in Q.
\end{cases}
\]

Then, $\Phi$ is an isomorphism from $S$ to $(\mathbb{R}, +)$, and $\exp \circ \Phi$ is an isomorphism from $S$ to $(\mathbb{R}_+, \cdot)$, where $\exp : \mathbb{R} \to \mathbb{R}_+$ is the exponential map.

In this way we get Theorems 2.2 and 2.3.

3 Subfields of $\mathbb{R}$

**Question 3.1.** Is there a subfield $K$ of $\mathbb{R}$ such that $\mathbb{R}$ is finite dimensional over $K$?

**Proposition 3.2.** There is no subfield $K$ of $\mathbb{R}$ such that $\dim_K \mathbb{R} = 2$.

**Proof.** Suppose that $a (> 0) \in K$, $\sqrt{a} \notin K$ and $\mathbb{R} = K(\sqrt{a})$. Then,
\[
\sqrt{a} = x + y\sqrt{a}
\]
for some $x, y \in K$. Hence,
\[
\sqrt{a} = x^2 + 2xy\sqrt{a} + y^2a.
\]
It follows that
\[
x^2 + y^2a = 0, \; 2xy = 1.
\]
But, this is impossible in $\mathbb{R}$ because $a > 0$. \qed

In view of this calculation, I suspect that the answer to the question might be negative.
References


