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Some questions on the real numbers *

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Friends of main, who are specialists in functional analysis, asked me some questions on the real numbers and real functions. Some are easy to answer and some are not.

1 Queer additive functions

Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a real function. f is *additive*, if

$$f(x+y)=f(x)+f(y)$$

for all $x, y\in\mathbb{R}$. It is well known that there is a non-linear (thus, discontinuous) additive real function. Now, the first question is

Question 1.1. Is there a discontinuous additive real function f satisfying

$$f(\sqrt{2}x)=\sqrt{2}f(x)$$

for all $x\in\mathbb{R}$?

The following is a general answer to this question.

Theorem 1.2. *Let A be a set of real numbers such that the field $K=\mathbb{Q}(A)$ generated by A over $\mathbb{Q}$ is not equal to $\mathbb{R}$. Then, there is a discontinuous additive function f satisfying*

$$f(\alpha x)=\alpha f(x) \tag{1}$$

for all $\alpha\in A$ and $x\in\mathbb{R}$.

Proof. Let $\{e_{i}\}_{i\in I}$ be a K-linear base of $\mathbb{R}$. Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a K-linear mapping such that $f(e_{i})=0$ and $f(e_{j})=1$ for $i, j\in I$ with $i\neq j$. Then, f is a discontinuous additive function satisfying (1). □

The next queer question is

*This is a final version and will not appear elsewhere.

Question 1.3. Is there a nonzero additive real function f satisfying

$$f(\sqrt{2}x) = \sqrt{3}\, f(x)$$

for all $x \in \mathbb{R}$?

More generally, for real numbers α, β and a real function f, consider the property

$$C(\alpha,\beta): \quad f(\alpha x) = \beta\, f(x) \quad \text{for all } x \in \mathbb{R}.$$

Question 1.4. Is there a nonzero additive real function f satisfying $C(\alpha,\beta)$ for real numbers $\alpha \neq \beta$?

If both α and β are transcendental, or algebraic with the same minimal polynomial over $\mathbb{Q}$, the substitution $\alpha \to \beta$ induces an isomorphism $\phi_{\alpha\beta}$: $\mathbb{Q}(\alpha) \to \mathbb{Q}(\beta)$ of fields. We have

$$\phi_{\alpha\beta}(\alpha f(\alpha)) = \beta f(\beta) = \beta\, \phi_{\alpha\beta}(f(\alpha))$$

for any $f(\alpha) \in K = \mathbb{Q}(\alpha)$. Let L be a K-subspace of $\mathbb{R}$ such that $\mathbb{R} = K \oplus L$. Extend $\phi_{\alpha\beta} : K \to \mathbb{Q}(\beta) \subset \mathbb{R}$ to an additive map $\Phi : \mathbb{R} \to \mathbb{R}$ by defining $\Phi|_K = \phi_{\alpha\beta}$ and $\Phi|_L = 0$. Then, Φ satisfies $C(\alpha,\beta)$.

Theorem 1.5. *For $\alpha, \beta \in \mathbb{R}$. there is a nonzero additive function Φ satisfying the condition $C(\alpha,\beta)$, if and only if*
(i) both α and β are transcendental, or
(ii) α and β are algebraic with the same minimal polynomial over $\mathbb{Q}$.

Proof. The above discussion shows the sufficiency of the condition for the existence of f satisfying $C(\alpha,\beta)$.

Conversely, let f be a nonzero additive function with property $C(\alpha,\beta)$. Since the additive function f is $\mathbb{Q}$-linear, for any $p(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{Q}[X]$ ($a_i \in \mathbb{Q}$) and for any $x \in \mathbb{R}$, we have

$$\begin{aligned} f(p(\alpha)\cdot x) &= a_0\, \Phi(x) + a_1\, f(\alpha \cdot x) + \cdots + a_n\, f(\alpha^n \cdot x) \\ &= a_0\, f(x) + a_1\, \beta \cdot f(x) + \cdots + a_n\, \beta^n \cdot f(x) \\ &= p(\beta)\cdot f(x). \end{aligned}$$

Hence, If $p(\alpha) = 0$, then $p(\beta) = 0$ because $f(x) \neq 0$ for some x. Similarly, $p(\beta) = 0$ implies $p(\alpha) = 0$. Thus, (i) or (ii) in the theorem holds. □

This problem has arisen from a research on stability of additive functions (Oda et al. [6]). The problem was already studied in Aczél [2].

2 Continuous semi-(group) structures on $\mathbb{R}_+$

The following question is very naive:

Question 2.1. Is the multiplication only the continuous group operation on the space $\mathbb{R}_+$ of positive real numbers?

For a homeomorphism ϕ from $\mathbb{R}_+$ onto $\mathbb{R}_+$. define an operation $*$ on $\mathbb{R}_+$ by

$$x * y = \phi^{-1}(\phi(x) \cdot \phi(y)) \tag{2}$$

for $x, y \in \mathbb{R}_+$. Then, $(\mathbb{R}_+, *)$ is a topological group. The following is a positive answer to the question (Aczél [1]).

Theorem 2.2. *The operation defined as (2) is the only way to make $\mathbb{R}_+$ a topological group.*

More generally we have

Theorem 2.3. *There are exactly three essentially distinct continuous cancellative semigroup operations on $\mathbb{R}_+$. They are the ordinary multiplication $\cdot$, the ordinary addition $+$, and the operation $\star$ defined by*

$$x \star y = x + y + 1$$

for $x, y \in \mathbb{R}_+$.

Let $S = (\mathbb{R}_+, *)$ be a topological semigroup, that is, $*$ is a continuous with respect to the ordinary topology of $\mathbb{R}_+$. Suppose that S is cancellative. Then, for any $x \in S$ the left transformation L_x ($L_x(y) = x * y$ for $y \in S$) and the right transformation R_x ($R_x(y) = y * x$) are monotone. L_x cannot be decreasing, otherwise, $L_x(y) = y$ for some $y \in S$, which implies $x = e$ (the identity element) and L_x is strictly increasing. Similarly R_x is strictly increasing. This discussion implies that S is an ordered semigroup.

An element $x \in S$ is *positive* (resp. *negative*) if $x * x > x$ (resp. $x * x < x$) Let P (resp. Q) be the set of positive (resp. negative) elements of S. P and Q are open subsets of S because $x * x - x$ is a continuous function.

If S has no idempotent, then $S = P \cup Q$, but since $\mathbb{R}_+$ is connected, either $S = P$ or $S = Q$ holds. Suppose that $S = P$. Then any $x \in S$ is positive and we have an increasing sequence $\{x^{n*}\}$ in S, where x^{n*} is the n-th power of x with respect $*$. If $\lim_{n\to\infty} x^{n*} = \hat{x} \in S$, then $\hat{x} * \hat{x} = \lim_{n\to\infty} x^{2n*} = \hat{x}$. But this cannot happen because S has no idempotent. Hence, $\lim_{n\to\infty} x^{n*} = +\infty$. Thus, for another $y \in S$, there is $n > 0$ such that $x^{n*} > y$. So, S is a *positively Archimedean* semigroup.

For a positively Archimedean semigroup S and a fixed element $a \in S$, we define a function $\phi_a : S \to \mathbb{R}$ by

$$\phi_a(x) = \inf\{m/n \,|\, m, n > 0, a^{m*} > x^{n*}\}$$

for $x \in S$. Then, ϕ is a ordered homomorphism from S to the additive semigroup of positive real numbers (see Fuchs [4], Hölder [5]). Moreover, it is continuous and injective (Craigen & Pales [3]).

When S is negatively Archimedean, the function ϕ'_a defined by

$$\phi'_a(x) = \inf\{m/n \mid m, n > 0, a^{m*} < x^{n*}\}$$

for $x \in S$ is an injective order-reversing continuous homomorphism from S to $(\mathbb{R}_+, +)$.

Let $\mu_a = \inf\{\phi_a(x) \mid x \in S\}$. If $\mu_a = 0$, then ϕ_a is an isomorphism from S to $(\mathbb{R}_+, +)$ of ordered topological semigroups. If $\mu_a > 0$, then $\phi_a/\mu_a - 1$ is an isomorphism from S to $(\mathbb{R}_+, \star)$,

If S has an idempotent e, it is the identity element. Then we have $S = P \cup \{e\} \cup Q$, where $P = \{x \in S \mid x > e\}$ is a positively Archimedean semigroup and $Q = \{x \in S \mid x < e\}$ is a negatively Archimedean semigroup. Let $x \in S$. Because $\lim_{n\to\infty} x * a^{n*} = \lim_{n\to\infty} a^{n*} = +\infty$ for $a \in P$ and $\lim_{n\to\infty} x * b^{n*} = \lim_{n\to\infty} b^{n*} = -\infty$ for $b \in Q$, L_x and R_x are unbounded above and below. It follows that S is a group.

Let $a \in P$ and a^{-*} be the inverse of a. Define a function $\Phi : S \to \mathbb{R}$ by

$$\Phi(x) = \begin{cases} \phi_a(x) & \text{if } x \in P \\ 0 & \text{if } x = e \\ -\phi'_{a^{-*}}(x) & \text{if } x \in Q. \end{cases}$$

Then, Φ is an isomorphism from S to $(\mathbb{R}, +)$, and $\exp \circ \Phi$ is an isomorphism from S to $(\mathbb{R}_+, \cdot)$, where exp: $\mathbb{R} \to \mathbb{R}_+$ is the exponential map.

In this way we get Theorems 2.2 and 2.3.

3 Subfields of $\mathbb{R}$

Question 3.1. Is there a subfield K of $\mathbb{R}$ such that $\mathbb{R}$ is finite dimensional over K?

Proposition 3.2. *There is no subfield K of $\mathbb{R}$ such that* $\dim_K \mathbb{R} = 2$.

Proof. Suppose that $a\,(> 0) \in K$, $\sqrt{a} \notin K$ and $\mathbb{R} = K(\sqrt{a})$. Then,

$$\sqrt[4]{a} = x + y\sqrt{a}$$

for some $x, y \in K$. Hence,

$$\sqrt{a} = x^2 + 2xy\sqrt{a} + y^2 a.$$

It follows that

$$x^2 + y^2 a = 0,\ 2xy = 1.$$

But, this is impossible in $\mathbb{R}$ because $a > 0$. □

In view of this calculation, I suspect that the answer to the question might be negative.

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