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<th>The parents of Weierstrass semigroups and non-Weierstrass semigroups (Algebra and Computer Science)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2014), 1873: 1-6</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-01</td>
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<td>URL</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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Kyoto University
The parents of Weierstrass semigroups and non-Weierstrass semigroups

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Abstract

We consider the map $p$ between the sets of numerical semigroups sending a numerical semigroup to the one whose genus is decreased by 1. We prove that the semigroup $p(H)$, which is called the parent of $H$, of a Weierstrass (resp. non-Weierstrass) numerical semigroup $H$ is Weierstrass (resp. non-Weierstrass) in some cases.

1 Notations and terminologies

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid $H$ of $\mathbb{N}_0$ is called a numerical semigroup if the complement $\mathbb{N}_0\setminus H$ is finite. The cardinality of $\mathbb{N}_0\setminus H$ is called the genus of $H$, denoted by $g(H)$. For a numerical semigroup $H$ we set

$$m(H) = \min\{h \in H \mid h > 0\},$$

which is called the multiplicity of $H$. In this case, the semigroup $H$ is called an $m$-semigroup where we set $m = m(H)$. For any $i$ with $1 \leq i \leq m - 1$ we set

$$s_i = \min\{h \in H \mid h \equiv i \mod m\}.$$

The set $S(H) = \{m, s_1, \ldots, s_{m-1}\}$ is called the standard basis for $H$. We set

$$s_{\text{max}} = \max\{s_i \mid i = 1, \ldots, m-1\}.$$

For a numerical semigroup $H$ we set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteq H\},$$

which is called the conductor of $H$. We note that $c(H) - 1 \not\in H$. We set $p(H) = H \cup \{c(H) - 1\}$, which is a numerical semigroup of genus $g(H) - 1$. The numerical semigroup $p(H)$ is called the parent of $H$.

A curve means a complete non-singular irreducible algebraic curve over an algebraically closed field $k$ of characteristic 0. For a pointed curve $(C, P)$ we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_{\infty} = nP\},$$

where $k(C)$ is the field of rational functions on $C$ and $(f)_{\infty}$ denotes the polar divisor of $f$. A numerical semigroup $H$ is said to be Weierstrass if there exists a pointed curve $(C, P)$ with $H(P) = H$.

1This paper is an extended abstract and the details will appear elsewhere.
2 The parents of non-Weierstrass semigroups

Let $H$ be a numerical semigroup. For any integer $m \geq 2$ we set

$$L_m(H) = \{l_1 + \cdots + l_m \mid l_i \in \mathbb{N}_0 \setminus H, \text{ all } i\}.$$ 

A numerical semigroup $H$ is said to be Buchweitz if there exists an integer $m$ such that $\# L_m(H) \geq (2m - 1)(g(H) - 1) + 1$. Buchweitz [1] showed that every Buchweitz semigroup $H$ is non-Weierstrass. We showed the following in Lemma 4.2 of [5]:

**Remark 2.1** Let $H$ be a primitive $n$-semigroup, i.e., $2n > \max\{l \mid l \notin H\} = c(H) - 1$, with $g(H) \geq n + 5$. Let $\overline{H}$ be a primitive $2n$-semigroup with

$$\mathbb{N}_0 \setminus \overline{H} = \{1, \ldots, 2n - 1\} \cup \{2\ell_n, 2\ell_{n+1}, \ldots, 2\ell_{g(H)}\} \cup \{4n - 3, 4n - 1\}$$

where

$$\mathbb{N}_0 \setminus H = \{1, \ldots, n - 1, \ell_n < \ldots < \ell_{g(H)}\}.$$ 

Assume that $\# L_2(H) \geq 3g(H) - 2$. Then we have

$$\# L_2(\overline{H}) \geq 3g(\overline{H}) - 2 \text{ and } \# L_2(p(\overline{H})) \geq 3g(p(\overline{H})) - 2.$$ 

In Example 4.2 in [5] we give the following example:

**Example 2.1** Let $t$ and $n$ be integers with $t \geq 5$ and $n \geq 4t + 1$. Let $H$ be a primitive $n$-semigroup whose complement $\mathbb{N}_0 \setminus H$ is

$$\{1, \ldots, n - 1\} \cup \{2n - 2t - 1, 2n - 2t - 1 + 2 \cdot 1, \ldots, 2n - 2t - 1 + 2 \cdot (t - 2)\} \cup \{2n - 2, 2n - 1\}.$$ 

Then $H$ satisfies $\# L_2(H) = 3g(H) - 2$. For example, if we set $t = 5$ and $n = 21$, we have

$$\mathbb{N}_0 \setminus H = \{1, \ldots 20\} \cup \{31, 33, 35, 37, 40, 41\}.$$ 

**Example 2.2** Let $H$ be as in the above example with $t = 5$ and $n = 21$. Let $\overline{H}$ be as in Remark 2.1. In fact, we have

$$\overline{H} = \{1 \rightarrow 41\} \cup \{62, 66, 70, 74, 80, 82\} \cup \{81, 83\}$$

and

$$p(\overline{H}) = \{1 \rightarrow 41\} \cup \{62, 66, 70, 74, 80, 82\} \cup \{81\}.$$ 

Then the semigroups $\overline{H}$ and $p(\overline{H})$ are Buchweitz, hence non-Weierstrass.

Let $\tilde{H}$ be a non-Weierstrass numerical semigroup. We consider the sequence

$$\tilde{H} \rightarrow p(\tilde{H}) \rightarrow p^2(\tilde{H}) \rightarrow \cdots \rightarrow p^{g(\tilde{H})-8}(\tilde{H}).$$
Since $g(p^{g(	ilde{H})-8}(	ilde{H})) = 8$, $p^{g(	ilde{H})-8}(	ilde{H})$ is Weierstrass (see [8]). Hence, there exists $i$ with $0 \leq i \leq g(\tilde{H}) - 7$ such that $p^{i}(\tilde{H}) = H$ is non-Weierstrass and $p^{i+1}(\tilde{H}) = p(H)$ is Weierstrass. In fact, we have the following example with $i = 0$:

**Example 2.3** The numerical semigroup $H = \langle 8, 12, 8\ell + 2, 8\ell + 6, n, n + 4 \rangle$ with $\ell \geq 2$ and odd $n \geq 16\ell + 19$ is non-Weierstrass (see [6]). The parent $p(H) = H + (n + 8\ell - 2)\mathbb{N}_0$ is Weierstrass (See [7]).

### 3 The parents of Weierstrass semigroups

**Problem 3.1** Let $H$ be a numerical semigroup. When are the numerical semigroups $H$ and $p(H)$ Weierstrass?

Let $\mathbb{N}_0 \backslash H = \{l_1, \ldots, l_{g(H)}\}$. We set $w(H) = \sum_{i=1}^{g(H)} (l_i - i)$, which is called the weight of $H$. Then it is well-known that $0 \leq w(H) \leq \frac{(g(H)-1)g(H)}{2}$.

**Proposition 3.1** If $w(H) = \frac{(g(H)-1)g(H)}{2}$, then $H$ and $p(H)$ are Weierstrass. In fact, we have $H = \langle 2, 2g(H)+1 \rangle$ and $p(H) = \langle 2, 2(g(H)-1)+1 \rangle$, which are Weierstrass.

We have the following:

**Remark 3.2**

i) If $H$ is primitive and $w(H) \leq g(H) - 2$, then $H$ is Weierstrass (see [2]).

ii) If $H$ is primitive and $w(H) = g(H) - 1$, then $H$ is Weierstrass (see [3]).

Moreover, we see the following:

**Lemma 3.3**

i) If $0 < w(H) \leq g - 1$, then we have $w(p(H)) \leq w(H) - 1$.

ii) If $w(H) \geq g$, then we have $w(p(H)) \leq w(H) - 2$.

By Lemma 3.3 and Remark 3.2 we get the following:

**Proposition 3.4**

i) If $w(H) \leq \frac{g(H)}{2}$, then $H$ and $p(H)$ are Weierstrass.

ii) If $w(H) \leq g(H) - 1$ and $H$ is primitive, then $H$ and $p(H)$ are Weierstrass.

iii) If $w(H) = g(H)$ and $H$ is primitive, then $p(H)$ is Weierstrass.

We note the following:

**Remark 3.5** We have $g(H) + 1 \leq c(H) \leq 2g(H)$. 
If $c(H) = g(H) + 1$, then we obtain
\[ H = \langle g(H) + 1 \to 2g(H) + 1 \rangle \] and
\[ p(H) = \langle g(H) \to 2g(H) - 1 \rangle, \]
which are Weierstrass. Hence, we get the following:

**Proposition 3.6** If $c(H) = g(H) + 1$, then $H$ and $p(H)$ are Weierstrass.

Moreover, we can prove the following:

**Theorem 3.7** If we have $c(H) = g(H) + 2$, then $H$ and $p(H)$ are Weierstrass.

**Proof.** Since $c(H) = g(H) + 2$, we have $\mathbb{N}_0 \setminus H \subseteq \{1 \to g(H) + 1\}$. Assume that $2m(H) \leq g(H) + 1$. Since we have $m(H), 2m(H) \notin \mathbb{N}_0 \setminus H$, we get
\[ \mathbb{N}_0 \setminus H \subseteq \{1 \to g(H) + 1\} \setminus \{m(H), 2m(H)\} \]
which is a contradiction. Hence, we get $2m(H) > g(H) + 1$, i.e., $H$ is primitive. We may assume that $g(H) \geq 3$. Hence, we have some $i \geq 3$ such that $i \in H$. In this case, we obtain
\[ \mathbb{N}_0 \setminus H = \{1, \ldots, i - 1, i + 1, \ldots, g(H) + 1\}. \]
We have $w(H) = g(H) + 1 - i \leq g(H) - 2$. By Remark 3.2 i), $H$ is Weierstrass. Moreover, we have
\[ \mathbb{N}_0 \setminus p(H) = \{1, \ldots, i - 1, i + 1, \ldots, g(H)\}. \]
By the same method as in the above we can show that $p(H)$ is Weierstrass. \qed

**Problem 3.2** Let $H$ be a Weierstrass numerical semigroup. Then is the numerical semigroup $p(H)$ also Weierstrass?

Using the standard method constructing a double covering we can show the following theorem:

**Theorem 3.8** Let $c(H) = 2g(H)$, i.e., $H$ is symmetric. If $g(H) \geq 6g(d_2(H)) + 4$ and $H$ is Weierstrass, then $p(H)$ is also Weierstrass.

We set
\[ d_2(H) = \left\{ \frac{h}{2} \mid h \in H \text{ which is even} \right\}, \]
which is also a numerical semigroup. If $\pi : C \to C'$ is a double covering with a ramification point $P$, then we have $H(\pi(P)) = d_2(H(P))$. We set
\[ n(H) = \min\{h \in H \mid h \text{ is odd}\}. \]

**Remark 3.9** Assume that $g(H) \geq 6g(d_2(H)) + 4$.

i) We have
\[ g' + \frac{n - 1}{2} \leq g(H) \leq 2g' + \frac{n - 1}{2} \]
where we set $g' = g(d_2(H))$ and $n = n(H)$ (see [4]).

ii) If $H$ is Weierstrass, then so is $d_2(H)$ (see [9]).
Theorem 3.10 Let $g(H) \geqq 6g(d_2(H)) + 4$. Assume that $g(H) = 2g(d_2(H)) + \frac{n-1}{2}$ where we set $n = n(H)$. In this case, $H = 2d_2(H) + n\mathbb{N}_0$. If $H$ is Weierstrass, then so is $p(H)$.

Proof. We have $p(H) = 2d_2(H) + n\mathbb{N}_0 + (n + 2(s_{\text{max}} - m))\mathbb{N}_0$. Since $d_2(p(H)) = d_2(H)$ is Weierstrass by Remark 3.9 ii), $p(H)$ is Weierstrass (see Proposition 2.4 in [6]). □

By a similar method to the proof of Proposition 2.4 in [6] we can prove the following:

Theorem 3.11 Let $g(H) \geqq 6g(d_2(H)) + 4$. Assume that $H \not\supset n + 2(s_{\text{max}} - m)$ where we set $n = n(H)$. If $H$ is Weierstrass, then so is $p(H)$.

Moreover, we get the following:

Theorem 3.12 We set $\mathbb{N}_0 \setminus d_2(H) = \{l_1 < \cdots < l_{g'}\}$ where $g' = g(d_2(H))$. Let $H_i = 2d_2(H) + \langle n, n + 2l_{g'}, n + 2l_{g'-1}, \ldots, n + 2l_{g'-i}\rangle$ where we set $n = n(H)$. Assume that $g(H) \geqq 6g(d_2(H)) + 4$. If $H = 2d_2(H) + n\mathbb{N}_0$ is Weierstrass, then so is $H_i$ for any $i$ with $0 \leqq i \leqq g' - 1$.

Using Theorems 3.11 and 3.12 we get the following:

Corollary 3.13 Let $g(H) \geqq 6g(d_2(H)) + 4$. Assume that $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$. If $H$ is Weierstrass, then so is $p(H)$.

Proof. By the assumption $H = 2d_2(H) + \langle n, n + 2(s_i - m)\rangle$ for some $i$ with $s_i + s_j \not\in S(d_2(H))$, all $j$ (see [6]). If $s_i \neq s_{\text{max}}$, then by Theorem 3.11 we get the result. If $s_i = s_{\text{max}}$, then by Theorem 3.12 we get the result. □

By Proposition 2.4 in [4] we have the following:

Remark 3.14 Let $n \geqq 4g(d_2(H)) + 1$ where we set $n = n(H)$. Assume that $g(H) = g(d_2(H)) + \frac{n-1}{2}$. In this case, $H = 2d_2(H) + \langle n, n + 2, \ldots, n + 2(m(d_2(H)) - 1)\rangle$. If $d_2(H)$ is Weierstrass, then so is $H$.

By Remarks 3.14 and 3.9 ii) we get the following:

Proposition 3.15 Let $g(H) \geqq 6g(d_2(H)) + 4$. Assume that $g(H) = g(d_2(H)) + \frac{n-1}{2} + 1$ where we set $n = n(H)$. If $H$ is Weierstrass, then so is $p(H)$.

Proposition 3.16 Let $g(H) \geqq 6g(d_2(H)) + 4$. Assume that $g(H) = g(d_2(H)) + \frac{n-1}{2}$ where we set $n = n(H)$. If $H$ is Weierstrass, then so is $p(H)$.

Proof. We have $n(p(H)) = n - 1$. Hence, by Remarks 3.14 and 3.9 ii) we get the result. □
References


