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A certain double series in aerodynamic interference calculations
(空力干渉計算における2重級数について)

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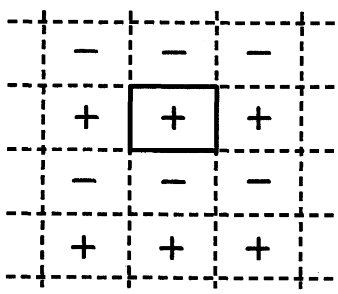
1 Abstract

In 1949, F. Olver [4] established a transformation formula which converts a certain slowly convergent series into a rapidly convergent and easily computable form. The original (double) series occurs in aerodynamic interference calculations, and its numerical estimates have some practical importance. In this report, we revisit this double series and show their transformation properties by using Mellin-Barnes type integrals (cf. [3]).

2 Classical results

We consider the correction interference of wind tunnel testing for the aerofoil in the low-speed case, and its mathematical model. It is known that the constraint of the tunnel walls on the flow due to the vorticity is equivalent to a distribution of up wash in the neighbourhood of the wing and be evaluated from a system of images of the vorticity (cf. figure below).

Here we set the tunnel section is rectangular. Let s be the span of the aerofoil and $\mu = h/b$ be the ratio of tunnel height h to tunnel breadth b.



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According to H. Glauert [2], we introduce the double series for the effect of the wall to cause an upward inclination of the stream as follows:

$$\begin{aligned}\varepsilon_0 &= \frac{s}{\pi b}\cdot \sum_{\substack{n=\infty \\ (m,n)\neq(0,0)}}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^n \frac{m^2-\mu^2 n^2}{(m^2+\mu^2 n^2)^2} \\ &= \frac{s}{\pi b}\cdot\left\{\frac{\pi^2}{3} + 8\pi^2 \sum_{n=1}^{\infty} \frac{n}{1+\exp(2\mu\pi n)}\right\}.\end{aligned}$$

The second equality was deduced by Glauert using the partial fraction decomposition of $\cot z$.

Remark. Above result can be interpreted in terms of arithmetic functions. Let $\mathscr{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the complex upper half plane. For $z \in \mathscr{H}$, we define a certain kind of the holomorphic Eisenstein series by

$$\mathscr{G}_k(z) = \sum_{\substack{n=\infty \\ (m,n)\neq(0,0)}}^{\infty} \left\{\sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(m+nz)^k}\right\},$$

for $k \geqq 2$. Then the Fourier series expansion

$$\begin{aligned}\mathscr{G}_k(z) &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{l=1}^{\infty} \widehat{\sigma_{k-1}}(l)\exp(2\pi i l z) \\ &= 2\zeta(k) - \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \frac{n^{k-1}}{1+\exp(-2\pi i n z)}\end{aligned}$$

holds.

Here we denote $\widehat{\sigma_{k-1}}(l) = \sum_{n|l}(-1)^{l/n} n^{k-1}$. As a consequence, we obtain

$$\varepsilon_0 = \frac{s}{\pi b}\cdot \mathscr{G}_2(\mu\sqrt{-1}).$$

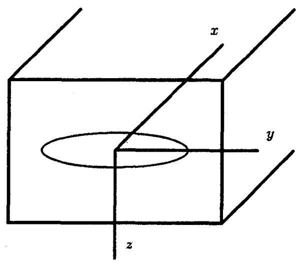
3 Olver's result

Let $(\alpha,\beta) \in \mathbb{R}^2$ and μ be a positive parameter. In the study of the upwash angle at the tail due to tunnel interference, the double series $F(\alpha,\beta)$ is approximately evaluated by L. W. Bryant and H. C. Garner (cf. [1]), which is defined as follows:

$$F(\alpha,\beta) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^n [f_n(\alpha-m) - f_n(\beta-m)]$$

with

$$f_n(y) = \frac{y(y^2+2\mu^2 n^2)}{n^2(y^2+\mu^2 n^2)^{3/2}}.$$



The elliptical shape in the figure above shows the tail, x-axis extends forward in the flow direction, y-axis is along the span of the aerofoil and z-axis indicates the downwash direction.

In [4], Olver proved the following theorem after the collaboration of Garner who supplied him the aerodynamic information.
Theorem (F. Olver 1949) *The double series $F(\alpha,\beta)$ attached to aerodynamic calculation is able to transformed as follows:*

$$F(\alpha,\beta) = \chi(\beta) - \chi(\alpha),$$

where

$$\chi(u) = \frac{1}{6}\pi^2 u + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin 2\pi m u \Big\{ 8\pi\mu^2 m(-1)^{n-1} \times K_0(2\pi\mu m n) + 4\mu \frac{(-1)^{n-1}}{n} K_1(2\pi\mu m n) \Big\},$$

and $K_\nu(z)$ denotes the modified Bessel function.

4 Main Theorem

Our main purpose is to interpret Olver's result in terms of some arithmetic functions. For $z \in \mathscr{H}$, real parameter $u \in \mathbb{R} \setminus \mathbb{Z}$ and complex parameter $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$, we define

$$f_{m,n}(u;s;z) = \frac{(-1)^n}{n^2} \cdot \frac{\mathrm{Re}(m-u+nz)}{|(m-u)+nz|} \left\{ 1 + \frac{n^2|z|^2}{|(m-u)+nz|^s} \right\},$$

and define a certain kind of the non-holomorphic Eisenstein series (or the Epstein zeta-function) by

$$\mathscr{T}(u;s;z) = \sum_{\substack{n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \left\{ \lim_{M\to\infty} \sum_{m=-M}^{M} f_{m,n}(u;s;z) \right\}.$$

Following the method of the Mellin-Barnes integral transformation in the study of the asymptotic expansion of the Epstein zeta function developed by M. Katsurada [3], we are able to obtain the transformation formula for $\mathscr{T}(u;s;z)$ as below.

First, let $U(s_1, s_2; z)$ be the confluent hypergeometric function of the second kind defined by

$$U(s_1; s_2; z) = \frac{1}{\Gamma(s_1)} \int_0^\infty e^{-zu} u^{s_1-1} (1+u)^{s_2-s_1-1} du,$$

for $\mathrm{Re}(s_1) > 0$ and $|\arg(z)| < \pi/2$. Then we have

Theorem 1 (N. 2012) *Let* $\mathrm{Re}s > 1$ *for* $s \in \mathbb{C}$ *and put* $\alpha = u - nx$. *Assume* $\alpha \notin \mathbb{Z}$ *for any* $n \in \mathbb{Z}$. *Then the transformation formula*

$$\begin{aligned}
\mathscr{T}(u; s; z) &= 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}\alpha}{n^2} \\
&+ \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}(4\pi y)^2}{\sqrt{\pi}} \\
&\qquad \times m \cdot \sin(2\pi m\alpha) e^{-2Z} U(\tfrac{3}{2}; 3; 4Z) \\
&+ \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}(2\pi)^{s+1}|z|^2}{(s-1)\pi\Gamma(\frac{s}{2}-\frac{1}{2})} \\
&\qquad \times m^{s-3} \cdot \sin(2\pi m\alpha) e^{-2Z} U(\tfrac{s}{2}-\tfrac{1}{2}; s-1; 4Z)
\end{aligned}$$

holds. Here we have used the notation $Z = \pi y m|n|$.

Remark. From Theorem 1, Olver's transformation formula is proved as a corollary. Furthermore, it is possible to obtain asymptotic expansion of $\mathscr{T}(u; s; z)$ as $\mathrm{Im} z \to +\infty$.

5 Acknowledgment

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References

[1] L. W. Bryant and H. C. Garner *Control Testing in Wind Tunnels* Aeronautical Research Council Reports and Memoranda, 2881, Her Majesty's Stationery Office, (1951)

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