<table>
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<th>Title</th>
<th>ALGEBRAIC INDEPENDENCE OF VALUES OF EXPONENTIAL TYPE POWER SERIES (Analytic Number Theory: Number Theory through Approximation and Asymptotics)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>Elsner, C.; Nesterenko, Yu. V.; Shiokawa, Iekata</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2014), 1874: 112-114</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195538">http://hdl.handle.net/2433/195538</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
ALGEBRAIC INDEPENDENCE OF VALUES OF EXPONENTIAL TYPE POWER SERIES

by C. Elsner, Yu.V. Nesterenko, and I. Shiokawa

In this article we announce our results in [1] without proof.

1 Exponential type power series with periodic coefficients

Let $q \geq 2$ be an integer and let $\xi = e^{2\pi i/q}$. We consider the power series

$$e_r(z) = e_{q,r}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (r = 0, 1, \ldots, q-1). \quad (1)$$

Trivially the relation

$$e_0(z) + e_1(z) + \cdots + e_{q-1}(z) = e^z$$

holds. Using the formula

$$\frac{1}{q} \sum_{k=0}^{q-1} \xi^{kn-r} = \begin{cases} 1 & \text{if } n \equiv r \pmod{q}, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$e_r(z) = \frac{1}{q} \sum_{k=0}^{q-1} \xi^{kr} \sum_{n=0}^{\infty} \frac{\xi^{kn} z^n}{n!}$$

$$= \frac{1}{q} \left( e^z + \xi^{-r} e^{\xi z} + \xi^{-2r} e^{\xi^2 z} + \cdots + \xi^{-(q-1)r} e^{\xi^{q-1} z} \right),$$

or

$$\begin{pmatrix} e_0(z) \\ e_1(z) \\ e_2(z) \\ \vdots \\ e_{q-1}(z) \end{pmatrix} = C \begin{pmatrix} e^z \\ e^{\xi z} \\ e^{\xi^2 z} \\ \vdots \\ e^{\xi^{q-1} z} \end{pmatrix}, \quad (2)$$
where
\[
C = \frac{1}{q} \begin{pmatrix}
  c_{1,1} & c_{1,2} & \cdots & c_{1,q} \\
  c_{2,1} & c_{2,2} & \cdots & c_{2,q} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{q,1} & c_{q,2} & \cdots & c_{q,q}
\end{pmatrix}
= \frac{1}{q} \begin{pmatrix}
  1 & 1 & 1 & \cdots & 1 \\
  1 & \xi^{-1} & \xi^{-2} & \cdots & \xi^{-(q-1)} \\
  1 & \xi^{-2} & \xi^{-2} & \cdots & \xi^{-(q-1)-2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & \xi^{-(q-1)} & \xi^{-2(q-1)} & \cdots & \xi^{-(q-1)(q-1)}
\end{pmatrix}.
\] (3)

Since \(\xi\) is a root of the \(q\)-th cyclotomic polynomial, \(1, \xi, \xi^2, \ldots, \xi^{\varphi(q)-1}\) are linearly independent over \(\mathbb{Q}\) and every \(\xi^k\) can be written as a linear combination of these \(\varphi(q)\) numbers over \(\mathbb{Z}\). If \(\alpha\) is a nonzero algebraic number, \(e^\alpha, e^{\xi\alpha}, \ldots, e^{\xi^{\varphi(q)-1}\alpha}\) are algebraically independent over \(\mathbb{Q}\) by the Lindemann-Weierstrass theorem and in view of (2) each of the numbers \(e_0(\alpha), e_1(\alpha), \ldots, e_{q-1}(\alpha)\) is transcendental.

**Theorem 1.** Let \(q \geq 3\) be an integer. If \(\alpha\) is a nonzero algebraic number, then among \(q\) numbers
\[
e_0(\alpha), e_1(\alpha), \ldots, e_{q-1}(\alpha)
\]
any \(\varphi(q)\) are algebraically independent over \(\mathbb{Q}\). Moreover, any \(\varphi(q) + 1\) of the \(q\) functions \(e_0(z), e_1(z), \ldots, e_{q-1}(z)\) are algebraically dependent over \(\mathbb{Q}\).

**Corollary 1.** Let \(q \geq 3\) be an integer and let \(\alpha\) be a nonzero algebraic number. Then any \(\varphi(q)\) of the numbers
\[
\sum_{n=0}^{\infty} \frac{\alpha^n}{(qn+r)!} \quad (r = 0, 1, \ldots, q-1)
\]
are algebraically independent over \(\mathbb{Q}\).

**Example 1.** In the case of \(q = 2\), we have
\[
e_{2,0}^2(z) - e_{2,1}^2(z) = \cosh^2(z) - \sinh^2(z) = 1,
\]
and for \(q = 3\)
\[
e_0^3(z) + e_1^3(z) + e_2^3(z) - 3e_0(z)e_1(z)e_2(z) = 1.
\]

2 Series involving fractional parts of polynomials

For any real number \(\alpha\) we denote by \([\alpha]\) and \(\{\alpha\}\) the integer and the fractional parts of \(\alpha\) respectively.

**Theorem 2.** Let \(f(x) \in \mathbb{Q}[x]\), \(\alpha\) be a nonzero algebraic number, and
\[
S = \sum_{n=0}^{\infty} \frac{\{f(n)\}}{n!} \alpha^n \neq 0.
\]

Then \(S\) is a transcendental number.
Corollary 2. Let \( f(x) \in \mathbb{Q}[x] \), \( \alpha \) be a nonzero algebraic number, and
\[
S = \sum_{n=0}^{\infty} \frac{[f(n)]}{n!} \alpha^n 
eq 0.
\]
Then \( S \) is a transcendental number.

In the case of linear polynomials, we obtain the following results.

Theorem 3. Let \( q \) and \( a \) are coprime integers with \( q \geq 3 \) and \( 0 < a < q \). Let
\[
f_b(z) = \sum_{n=0}^{\infty} \left( \frac{an+b}{q} \right) \frac{z^n}{n!} \quad (b = 0, 1, \ldots, q - 1).
\]
If \( \alpha \) is a nonzero algebraic number, then among \( q \) numbers \( f_0(\alpha), \ldots, f_{q-1}(\alpha) \) any \( \varphi(q) \) are algebraically independent over \( \mathbb{Q} \). Moreover, any \( \varphi(q) + 1 \) of the functions \( f_0(z), \ldots, f_{q-1}(z) \) are algebraically dependent over \( \mathbb{Q} \).

3 Series involving Fibonacci numbers

In this section we set \( \rho := (1 + \sqrt{5})/2 \). Let
\[
F_n = \frac{1}{\sqrt{5}} \left( \rho^n - \left( -\frac{1}{\rho} \right)^n \right), \quad L_n = \rho^n + \left( -\frac{1}{\rho} \right)^n
\]
denote the Fibonacci numbers and the Lucas numbers, respectively.

Theorem 4. Let \( f_s(\alpha) \) and \( g_s(\alpha) \) be power series defined by
\[
f_s(z) = \sum_{n=0}^{\infty} F^n \frac{z^n}{n!}, \quad g_s(z) = \sum_{n=0}^{\infty} L^n \frac{z^n}{n!}.
\]
If \( \alpha \) is a nonzero algebraic number, then all the numbers in the set \( \{f_s(\alpha) \mid s \in \mathbb{N}\} \cup \{g_s(\alpha) \mid s \in \mathbb{N}\} \) are distinct and any two are algebraically independent over \( \mathbb{Q} \). Moreover, any three functions in the set \( \{f_s(z) \mid s \in \mathbb{N}\} \cup \{g_s(z) \mid s \in \mathbb{N}\} \) are algebraically dependent over \( \mathbb{Q} \).

Theorem 5. Let \( f_{a,b}(z) \) and \( g_{a,b}(z) \) be power series defined by
\[
f_{a,b}(z) = \sum_{n=0}^{\infty} F_{an+b} \frac{z^n}{n!}, \quad g_{a,b}(z) = \sum_{n=0}^{\infty} L_{an+b} \frac{z^n}{n!}.
\]
If \( \alpha \) is a nonzero algebraic number, then any two numbers in the set \( \{f_{a,b}(\alpha) \mid a \in \mathbb{N}, b \in \mathbb{N}_0\} \) are algebraically independent over \( \mathbb{Q} \). Moreover, any three functions in the set \( \{f_{a,b}(z) \mid a \in \mathbb{N}, b \in \mathbb{N}_0\} \) are algebraically dependent over \( \mathbb{Q} \). The same statements hold also for the power series \( g_{a,b}(z) \).

References