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Global classical solvability of an interface problem on the motion of two fluids

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Abstract

We deal with the motion of two incompressible fluids in a container. The liquids are separated by an unknown interface on which surface tension is taken into account or ignored. Global existence theorem is proved in anisotropic Hölder classes for small smooth initial data and mass forces. We show in the case of strictly positive surface tension that fluid velocity decreases exponentially as time variable $t \to \infty$ and the interface between the liquids tends to a sphere $S_{R_0}^2(h_{\infty}) = \{|x - h_{\infty}| = R_0\}$ with a center $h_{\infty}$ close to 0, the barycenter of the inner fluid at the initial moment.

The proof is based on a local existence theorem in Hölder spaces and on an exponential estimate of $L_2$-norms of local solutions. We follow to V. A. Solonnikov's scheme for proving global solvability of a problem on the motion of a single drop with free surface [1].

1 Statement of the problem
At the initial moment, let a fluid with the viscosity $\nu^+ > 0$ and the density $\rho^+ > 0$ occupy a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$; we denote $\partial \Omega_0^+$ by $\Gamma$. And let a fluid with the viscosity $\nu^- > 0$ and the density $\rho^- > 0$ fill a domain $\Omega_0^-$ surrounding $\Omega_0^+$. The boundary $S \equiv \partial(\Omega_0^+ \cup \Gamma \cup \Omega_0^-)$ is a given closed surface, $S \cap \Gamma = \emptyset$.

For every $t > 0$ we need to find $\Gamma_t = \partial\Omega_t^+$, velocity vector field $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ and the function $p$, that is the deviation from the hydrostatic pressure $P_0$, which satisfy the following initial–boundary value problem:

$$
\mathcal{D}_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu^\pm \nabla^2 \mathbf{v} + \frac{1}{\rho^\pm} \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega_t^\pm, \quad t > 0,
$$

$$
\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+,
$$

$$
[v]|_{r_t} \equiv \lim_{x \to x_0 \in \Gamma_t} \mathbf{v}(x) - \lim_{x \to -x_0 \in \Gamma_t} \mathbf{v}(x) = 0,
$$

$$
[T(\mathbf{v}, p)n]|_{r_t} = \sigma H n \quad \text{on} \quad \Gamma_t.
$$

Here $\mathcal{D}_t \equiv \frac{\partial}{\partial t}$, $\nabla \equiv \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$, $\mathbf{v}_0$ is the initial velocity, $\nu^\pm, \rho^\pm$ are the step functions of viscosity and density, $T(\mathbf{v}, p)$ is the stress tensor with the elements $T_{ik}(\mathbf{v}, p) = -\delta^k_i p + \mu^\pm S_{ik}(\mathbf{v})$, $i, k = 1, 2, 3$, $\delta^k_i$ is the Kronecker symbol, $S_{ik}(\mathbf{v}) \equiv \partial v_i/\partial x_k + \partial v_k/\partial x_i$, are the components of the doubled strain tensor $\mathbf{S}(\mathbf{v})$; $\mu^\pm = \nu^\pm \rho^\pm$, $\sigma > 0$ is surface tension coefficient, $n$ is the outward normal to $\Omega_t^+$, $H(x, t)$ is twice the mean curvature of $\Gamma_t$ ($H < 0$ at the points where $\Gamma_t$ is convex towards $\Omega^-_t$). A Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The centered dot denotes the Cartesian scalar product. We imply the summation from 1 to 3 with respect to repeated indexes. We mark the vectors and the vector spaces by boldface letters.

If $\sigma > 0$, we suppose the inner domain $\Omega_0^+$ is close to a ball of its volume. In this case, we introduce a new pressure function $p_1 = p$ in $\Omega_t^+$ and $p_1 = p + \sigma \frac{2}{R_0}$ in $\Omega_t^-$. Then only last boundary condition (1.2) changes:

$$
[T(\mathbf{v}, p_1)n]|_{r_t} = \sigma \left( H + \frac{2}{R_0} \right) n \quad \text{on} \quad \Gamma_t.
$$

If $\sigma = 0$, we set $p_1 \equiv p$ in both domains.

We assume the liquids to be immiscible. A condition excluding the mass transportation through $\Gamma_t$ implies that $\Gamma_t$ consists of the points $x(\xi, t)$ which radius vector $x(\xi, t)$ is a solution of the Cauchy problem

$$
\mathcal{D}_t x = \mathbf{v}(x(\xi, t), t), \quad x(\xi, 0) = \xi, \quad \xi \in \Gamma, \quad t > 0.
$$
Hence, \( \Gamma_t = \{ x(\xi, t) | \xi \in \Gamma \} \), \( \Omega_t^\pm = \{ x(\xi, t) | \xi \in \Omega_0^\pm \} \).

Condition (1.4) completes system (1.1), (1.3).

We transform the Eulerian coordinates \( \{ x \} \) into the Lagrangian ones \( \{ \xi \} \) by the formula

\[
x(\xi, t) = \xi + \int_0^t u(\xi, \tau) d\tau \equiv X_u(\xi, t),
\]

where \( u(\xi, t) \) is velocity vector field in the Eulerian coordinates. We separate boundary condition for stress tensor (1.3) onto the tangential and normal components. As a result, we arrive at the problem for \( u, q = p_1(X_u, t) \) with the given interface \( \Gamma \).

If the angle between \( n \) and the exterior normal \( n_0 \) to \( \Gamma \) is acute, this system is equivalent to the following one:

\[
\mathcal{D}_t u - \nu^\pm \nabla_{u}^2 u + \frac{1}{\rho^\pm} \nabla_{u} q = f(X_n, t), \quad \nabla_{u} \cdot u = 0 \quad \text{in} \quad Q^\pm_T = \Omega_0^\pm \times (0, T),
\]

\[
u^\pm \nabla_{u} \cdot u|_{t=0} = v_0 \quad \text{in} \quad \Omega_0^\pm \cup \Omega_0^+, \quad u|_{S_T} = 0,
\]

\[
|u|_{G_T} = 0, \quad [\mu^\pm \Pi_0 \Pi S_{u}(u)n]|_{G_T} = 0 \quad (G_T \equiv \Gamma \times (0, T)),
\]

\[
|n_0 \cdot T_{u}(u, q)n)|_{G_T} = \sigma(H(X_n) + \frac{2}{R_0})n_0 \cdot n \quad \text{on} \quad G_T.
\]

Here \( \nabla_{u} = \mathbb{A} \nabla \), \( \mathbb{A} \) is the matrix of co-factors \( A_{ij} \) to the elements \( a_{ij}(\xi, t) = \delta_i^j + \int_0^t \frac{\partial u}{\partial \xi_j} dt' \) of the Jacobian matrix of transformation (4.2), \( i, j = 1, 2, 3 \);
\( \Pi_0 \omega = \omega - n_0 (n_0 \cdot \omega) \), \( \Pi \omega = \omega - n (n \cdot \omega) \) are projections of a vector \( \omega \) onto the tangent planes to \( \Gamma_t \) and \( \Gamma \), respectively;

\[
T_{u}(w, q)|_{G_T} = -qI + \mu^\pm S_{u}(w),
\]

where \( S_{u}(w) \) is tensor with the elements \( S_{u}(w)_{ij} = A_{ik} \partial w_j / \partial \xi_k + A_{jk} \partial w_i / \partial \xi_k \), the vector \( n \) is connected with \( n_0 \) by the relation: \( n = \frac{An_0}{|An_0|} \).

We apply the well known relation for twice the mean curvature:

\[
Hn = \Delta(t)x \equiv \Delta(t)X_u,
\]

where \( \Delta(t) \) is the Beltrami–Laplace operator on \( \Gamma_t \). In local coordinates \( \{ s_1, s_2 \} \) on the surface \( \Gamma \), it has the form:

\[
\Delta(t) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial s_\alpha} g^{\alpha\beta} \sqrt{g} \frac{\partial}{\partial s_\beta} \equiv g^{\alpha\beta} \frac{\partial^2}{\partial s_\alpha \partial s_\beta} + h^\beta \frac{\partial}{\partial s_\beta},
\]

where \( \{ g^{\alpha\beta} \}_{\alpha, \beta=1}^2 \) is the inverse matrix to metric tensor matrix \( \{ g_{\alpha\beta} \}_{\alpha, \beta=1}^2 \),

\[
g_{\alpha\beta} = \frac{\partial X_u(\xi(s), t)}{\partial s_\alpha} \frac{\partial X_u(\xi(s), t)}{\partial s_\beta}, \quad g = \det \{ g_{\alpha\beta} \}_{\alpha, \beta=1}^2, \quad h^\beta = \frac{\partial}{\sqrt{g} \partial s_\alpha} (g^{\alpha\beta} \sqrt{g}).
\]
As a result, instead of (1.7) we arrive in (1.6) at the following interface condition:
\[
[n_0 \cdot T_u(u, q) n]|_{G_T} - \sigma n_0 \cdot \Delta(t) X_u|_{G_T} = \frac{2}{R_0} n_0 \cdot n \quad \text{on } G_T. \quad (1.9)
\]

2 Auxiliary statements

We introduce some Hölder semi-norms.

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$; for $T > 0$ we put $\Omega_T = \Omega \times (0, T)$; finally, let $\alpha \in (0, 1)$.

We need the following semi-norm with $\alpha, \gamma \in (0, 1)$:
\[
|f|_{\Omega_T}^{(\gamma, 1+\alpha)} = \langle f \rangle_{\Omega_T}^{(\gamma, 1+\alpha)} + |f|_{t, \Omega_T}^{(\frac{1+\alpha-\gamma}{2})},
\]
where
\[
\langle f \rangle_{\Omega_T}^{(\gamma, 1+\alpha)} = \max_{t, \tau \in (0, T)} \max_{x, y \in \Omega} \frac{|f(x, t) - f(y, t) - f(x, \tau) + f(y, \tau)|}{|x - y|^\gamma |t - \tau|^{(1+\alpha-\gamma)/2}}.
\]

It is known the estimate
\[
\langle f \rangle_{\Omega_T}^{(\gamma, 1+\alpha)} \leq c_1 \langle f \rangle_{\Omega_T}^{(1+\alpha, \gamma)}.
\]

We consider that $f \in C^{(\gamma, 1+\alpha)}(\Omega_T)$ if $|f|_{\Omega_T} + |f|_{\Omega_T}^{(\gamma, 1+\alpha)} < \infty$.

Finally, if a function $f$ has finite norm
\[
|f|_{\Omega_T}^{(\gamma, \mu)} \equiv \langle f \rangle_{x, \Omega_T}^{(\gamma)} + |f|_{t, \Omega_T}^{(\mu)}, \quad \gamma \in (0, 1), \quad \mu \in [0, 1),
\]
where
\[
|f|_{t, \Omega_T}^{(\mu)} = \begin{cases} |f|_{\Omega_T} + \langle f \rangle_{t, \Omega_T}^{(\mu)} & \text{if } \mu > 0, \\ |f|_{\Omega_T} & \text{if } \mu = 0, \end{cases}
\]
then it belongs to the Hölder space $C^{\gamma, \mu}(\Omega_T)$.

We suppose that a vector valued function is an element of a Hölder space if all its components belong to this space, and its norm is defined as the maximal norm of the components.

Let $T \in (0, \infty]$, $t, \tau > 0$. We set $D_T \equiv \Omega_T^\pm = Q_T^\pm \cup Q_T^\pm$ and $Q_T^\pm = \Omega^\pm \times (0, T)$, $Q_{(t, t+\tau)}^\pm = \Omega_t^\pm \times (t, t+\tau)$; $\Omega = \Omega_0^- \cup \Omega_0^+ \equiv \Omega_t^- \cup \Omega_t^+$, $Q_T = \Omega \times (0, T)$, $Q_{(t, t+\tau)} = \Omega \times (t, t+\tau)$; $G_T = \Gamma \times (0, T)$.

\[
|f|_{D_T}^{(k+\alpha, k+\alpha/2)} \equiv |f|_{Q_T^\pm}^{(k+\alpha, k+\alpha/2)} + |f|_{Q_T^\pm}^{(k+\alpha, k+\alpha/2)} + |f|_{\Omega_T^-}^{(k+\alpha)} + |f|_{\Omega_T^+}^{(k+\alpha)}.
\]
**Theorem 2.1. (Local existence)** Let $\Gamma \in C^{3+\alpha}$, $f, D_x f \in C^{\alpha,1+\frac{\alpha-\gamma}{2}}(Q_T)$, $v_0 \in C^{2+\alpha}(\Omega_0^- \cup \Omega_0^+)$ for some $\alpha, \gamma \in (0,1)$, $T < \infty$. Assume the compatibility conditions hold:

\[
\nabla \cdot v_0 = 0, \quad [v_0]|_{\Gamma} = 0, \quad [\mu^\pm \Pi_0 S(v_0)n_0]|_{\Gamma} = 0, \\
\left[\Pi_0 (\nu^\pm \nabla^2 v_0 - \frac{1}{\rho^\pm} \nabla q_0)\right]|_{\Gamma} = 0, \quad v_0|_S = 0, \\
\Pi_S \left(f(\xi, 0) - \frac{1}{\rho^-} \nabla q_0(\xi) + \nu^- \nabla^2 v_0(\xi)\right)|_{\xi \in S} = 0;
\]

where $q_0(\xi) = p_1(\xi, 0)$ is a solution of the following diffraction problem:

\[
\frac{1}{\rho^\pm} \nabla^2 q_0(\xi) = \nabla \cdot (f(\xi, 0) - D_t \mathbb{B}^*(v_0)v_0(\xi)), \quad \xi \in \Omega_0^- \cup \Omega_0^+, \\
[q_0]|_{\Gamma} = \left[2\mu^\pm \frac{\partial v_0}{\partial n_0} \cdot n_0\right]|_{\Gamma} - \sigma \left(H_0(\xi) + \frac{2}{R_0}\right), \quad \xi \in \Gamma, \\
\left[\frac{1}{\rho^\pm} \frac{\partial q_0}{\partial n_0}\right]|_{\Gamma} = \left[\nu^\pm \nabla^2 v_0\right]|_{\Gamma} \left(\frac{\partial}{\partial n_0} = n_0 \cdot \nabla\right), \\
\frac{1}{\rho^-} \frac{\partial q_0}{\partial n_S}|_S = n_S \cdot (\nu^- \nabla^2 v_0 + f|_{t=0})|_S \left(\frac{\partial}{\partial n_S} = n_S \cdot \nabla\right).
\]

(Here $H_0(\xi) = n_0 \cdot \Delta(0)\xi|_{\Gamma}$ is twice the mean curvature of $\Gamma$; $\mathbb{B}^*$ is the transpose of $\mathbb{B} = \mathbb{A} - I$; $I$ is the identity matrix; $\Pi_S b = b - n_S (n_S \cdot b)$, $n_S$ is the outward normal to $S$.)

Then for $\forall T < \infty \exists$ such $\epsilon(T)$ that problem (1.6), (1.9) has a unique solution $(u, q): u \in C^{2+\alpha,1+\alpha/2}(D_T)$, $q \in C^{(\gamma,1+\alpha)}(D_T)$, $\nabla q \in C^{\alpha,\alpha/2}(D_T)$ provided that

\[
|f|^{(\alpha,1+\frac{\alpha-\gamma}{2})}_{Q_T} + |D_x f|^{(\alpha,1+\frac{\alpha-\gamma}{2})}_{Q_T} + |v_0|^{(2+\alpha)}_{\cup \Omega^\pm} + \sigma |H_0 + \frac{2}{R_0}|^{(1+\alpha)}_\Gamma \leq \epsilon(T),
\]

$q$ being unique up to a function of time. The interface $\Gamma_t$ is a surface of $C^{3+\alpha}$. Moreover, there holds the estimate

\[
|u|^{(2+\alpha,1+\alpha/2)}_{D_T} + |\nabla q|^{(\alpha,\alpha/2)}_{D_T} + |q|^{(\gamma,1+\alpha)}_{D_T} \\
\leq c \left\{ |f|^{(\alpha,1+\frac{\alpha-\gamma}{2})}_{Q_T} + |D_x f|^{(\alpha,1+\frac{\alpha-\gamma}{2})}_{Q_T} + |v_0|^{(2+\alpha)}_{\cup \Omega^\pm} + \sigma |H_0 + \frac{2}{R_0}|^{(1+\alpha)}_\Gamma \right\}.
\]

We note that local and global solvability of the problem governing the motion of two fluids without surface tension was obtained by the author in
[2]. In the case of strictly positive $\sigma$, Theorem 2.1 was proven in [3] for the case when $S$ was absent and $\overline{\Omega^+} \cup \Omega^-$ coincided with the whole space $\mathbb{R}^3$. This result was obtained in Hölder spaces with power-like weights at infinity but it is also valid in our case because the weighted spaces are equivalent to the ordinary Hölder spaces in bounded domains.

**Remark 2.1.** If $\sigma = 0$, it is sufficiently to assume $\Gamma \in C^{2+\alpha}$, the interface $\Gamma_t$ also will be of class $C^{2+\alpha}$.

The proof of Theorem 2.1 is based on the solvability of the following linearized problem:

$$
\mathcal{D}_t w - \nu^\pm \nabla u w + \frac{1}{\rho^\pm} \nabla s = f, \quad \nabla u \cdot w = r \quad \text{in} \quad Q^\pm_T,
$$

$$
[w]_{G_T} = 0, \quad w|_S = 0, \quad [\mu^\pm \Pi_0 \Pi \mathbb{S}_u(w)n]|_{G_T} = \Pi_0 d,
$$

$$
[n_0 \cdot \mathbb{T}_u(w, s)n]|_{G_T} - \sigma n_0 \cdot \Delta(t) \int_0^t w|_\Gamma d\tau = b + \sigma \int_0^t B d\tau \quad \text{on} \quad G_T.
$$

Unique solvability for problem (2.5) was proved in any finite time interval.

**Theorem 2.2.** Let $\alpha, \gamma \in (0, 1)$, $\gamma < \alpha$, $T < \infty$. Assume that $\Gamma, S \in C^{2+\alpha}$ and that for $u \in C^{2+\alpha,1+\alpha/2}(D_T)$, $[u]|_{G_T} = 0$, we have

$$(T + T^{\gamma/2}) |u|^{(2+\alpha,1+\alpha/2)}_{D_T} \leq \delta$$

for some sufficiently small $\delta > 0$. Moreover, we assume that the following four groups of conditions are fulfilled:

1) $\exists$ a vector $g \in C^{\alpha/2}(D_T)$ and a tensor $\mathbb{G} = \{G_{ik}\}_{i,k=1}^3$ with $G_{ik} \in C^{(\gamma,1+\alpha)}(D_T) \cap C^{\gamma,0}(D_T)$ such that

$$
\mathcal{D}_t r - \nabla u \cdot f = \nabla \cdot g, \quad g_i = \partial G_{ik}/\partial \xi_k, \quad i = 1, 2, 3,
$$

(these equalities are understood in a weak sense) and, moreover,

$$
[(g + A^\ast f) \cdot n_0]|_{G_T} = 0;
$$

2) $f \in C^{\alpha/2}(D_T)$, $r \in C^{1+\alpha,1+\alpha/2}(D_T)$, $w_0 \in C^{2+\alpha}(\Omega_0^- \cup \Omega_0^+)$, $d \in C^{1+\alpha,1+\alpha/2}(G_T)$, $b \in C^{(\gamma,1+\alpha)}(G_T)$;
\[ \nabla \cdot w_0(\xi) = r(\xi, 0) = 0, \quad \xi \in \Omega_0^- \cup \Omega_0^+, \quad [w_0]_{\Gamma} = 0, \quad w_0|_S = 0, \]

\[ [\Pi_0 \nabla s(\xi, 0) + \nu^+ \nabla^2 w_0(\xi)] |_{\xi \in \Gamma} = [\Pi_0 d(\xi, 0)] |_{\xi \in \Gamma}, \]

\[ \Pi_0 (f(\xi, 0) - \frac{1}{p^\pm} \nabla s(\xi, 0) + \nu^\pm \nabla^2 w_0(\xi)) |_{\xi \in \Gamma} = 0; \]

4) \( s_0(\xi) = s(\xi, 0) \) is a solution of the problem

\[ \frac{1}{\rho^\pm} \nabla^2 s_0(\xi) = \nabla \cdot (\mathcal{D}_t \mathbb{B}^* \big|_{t=0} w_0(\xi) - g(\xi, 0)) \text{ in } \Omega_0^- \cup \Omega_0^+, \]

\[ [s_0]|_{\Gamma} = [2 \mu^\pm \frac{\partial w_0}{\partial n_0} \cdot n_0]|_{\Gamma} - b|_{t=0}, \]

\[ \frac{1}{\rho^\pm} \frac{\partial s_0}{\partial n_0}|_{S} = n_0 \cdot (\nu^{-} \nabla^2 v_0 + f|_{t=0})|_{S}. \]

Under all these assumptions, problem (2.5) has a unique solution \((w, s)\) with the properties:

\[ w \in C^{2+\alpha,1+\alpha/2}(D_T), \quad s \in C^{(\gamma,1+\alpha)}(D_T), \quad \nabla s \in C^{\alpha,(\alpha/2)}(D_T), \]

moreover, this solution satisfies the inequality

\[ |w|_{D_T}^{(2+\alpha,1+\alpha/2)} + |\nabla s|_{D_T'}^{(\alpha,\alpha/2)} + [s]_{D_T'}^{(\gamma,1+\alpha)} \leq c_1(t') \left\{ |f|_{D_T'}^{(\alpha,\alpha/2)} + |r|_{D_T'}^{(1+\alpha,\frac{1+\alpha}{2})} + |w_0|_{\cup \Omega_0^\pm}^{(2+\alpha)} + |g|_{D_T'}^{(\alpha,\alpha/2)} + |\mathcal{G}|_{D_T'}^{(\gamma,1+\alpha)} + |d|_{G_T'}^{(1+\alpha,\frac{1+\alpha}{2})} + |b|_{G_T'}^{(\alpha,\alpha/2)} + \langle b \rangle_{G_T'}^{(\gamma,1+\alpha)} + |\nabla r b|_{G_T'}^{(\alpha,\alpha/2)} + \sigma |B|_{G_T'}^{(\alpha,\alpha/2)} + P_t[u]|w_0|_{\cup \Omega_0^\pm}^{(1)} \right\}, \]

where \( c_1(t') \) is a monotone nondecreasing function of \( t' \leq T \), and

\[ P_t[u] = t^{\frac{1-\alpha}{2}} |\nabla u|_{D_t} + |\nabla u|_{D_t'}^{(\alpha,\alpha/2)}. \]

3 Global solvability of problem (1.1), (1.3), (1.4) without surface tension

**Theorem 3.1.** (Global existence theorem) Let \( \alpha, \gamma \in (0,1) \). Assume that \( \Gamma \in C^{2+\alpha} \) and \( v_0 \in C^{2+\alpha}(\Omega_0^- \cup \Omega_0^+), \quad f, \nabla f \in C^{\alpha,\frac{1+\alpha-\gamma}{2}}(Q_\infty) \) satisfy compatibility conditions (2.1), where \( q_0 = p_0(\xi) \equiv p_1(\xi, 0) \) being a solution of diffraction problem (2.2) with \( \sigma = 0 \).
Moreover, we suppose that the data are small enough, i.e.

\[ |v_0|_{\cup\Omega_{0}^\pm}^{(2+\alpha)} + |e^{bt}f|_{Q_{\infty}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |e^{bt}\nabla f|_{Q_{\infty}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + \int_0^\infty \|e^{bt}f\|_{2,\Omega} dt \leq \epsilon \ll 1, \quad (3.1) \]

where \( b = \min\{v^+, v^-\}/(2c_0) \) with the constant \( c_0 \) from inequality (3.3).

Then problem (1.1), (1.3), (1.4) with \( \sigma = 0 \) is uniquely solvable on infinite time interval \( t > 0 \). The solution \( (v, p) \) has the properties: \( v \in C^{2+\alpha,1+\alpha/2} \), \( p \in C^{(\gamma,1+\alpha)} \), \( \nabla p \in C^{\alpha,\alpha/2} \), the surface \( \Gamma_t \) being from \( C^{2+\alpha} \)-class. It means that for every \( t_0 \in (0, \infty) \) the solution \( (u, q) \) and its derivatives in Lagrangian coordinates belong to the corresponding spaces over \( D_T \equiv \cup Q_{(t_0,t_0+\tau)}^\pm \) for a small enough time interval \( (t_0,t_0+\tau) \) and

\[ N_{(t_0-\tau_0,t_0)}[v, p] \leq c(\delta, \tau_0) e^{-bt_0} \left\{ |v_0|_{\cup\Omega_{0}^\pm}^{(2+\alpha)} + \int_0^\infty e^{b\tau} \|f(\cdot, \tau)\|_{2,\Omega} d\tau \right\} (3.2) \]

We have introduced the notation: \( \|\cdot\|_{2,\Omega} = \|\cdot\|_{L_2(\Omega)} \). We note that \( \partial\Omega = S \) and for \( \nu \) in \( \Omega \), the Korn inequality

\[ \|\nu\|_{W_2^1(\Omega)} \leq c_0 \|S(\nu)\|_{2,\Omega} \quad (3.3) \]

holds because \( \nu|_S = 0 \) (see [4]), \( \|\nu\|_{W_2^1(\Omega)} \) coinciding with \( \|\nu\|_{W_2^1(\Omega^-\cup\Omega^+)} \) due to \( [\nu]|_{\Gamma_t} = 0 \).

**Proposition 3.1.** Assume that a classical solution of problem (1.1), (1.2) with \( \sigma = 0 \) is defined in \( [0, T] \) and \( v_0 \) satisfies compatibility conditions (2.1). Let \( f(\cdot, \tau) \in L_2(\Omega) \) and \( \int_0^\infty \|e^{\tau f}(\cdot, \tau)\|_{2,\Omega} d\tau < \infty \).

Then

\[ \|v(\cdot, t)\|_{2,\Omega} \leq e^{-bt} \left\{ \|v_0\|_{2,\Omega} + \int_0^t \|e^{b\tau}f(\cdot, \tau)\|_{2,\Omega} d\tau \right\}, \quad t \in (0, T], \quad (3.4) \]

\[ \int_0^\infty \|v(\cdot, \tau)\|_{2,\Omega} d\tau \leq c\left\{ \|v_0\|_{2,\Omega} + \int_0^\infty \|f(\cdot, \tau)\|_{2,\Omega} d\tau \right\}. \quad (3.5) \]

**Proof.** We multiply the 1st equation in (1.6) by \( \nu \) and integrate by parts over \( \Omega^- \cup \Omega^+ \).

\[ \frac{1}{2} \frac{d}{dt} \|v\|_{2,\Omega}^2 + \|\nu S(v)\|_{2,\Omega}^2 = \int_\Omega f \cdot \nu \, dx. \]
If we make use of the Korn and Hölder inequalities and divide by $\|v\|_{2,\Omega}$, we arrive at
\[
\frac{d}{dt}\|v\|_{2,\Omega} + b\|v\|_{2,\Omega} \leq \|f\|_{2,\Omega}
\]
with $b = \min\{\nu^+, \nu^-\}/(2c_0)$. By the Gronwall lemma,
\[
\|v\|_{2,\Omega} \leq e^{-bt}\|v_0\|_{2,\Omega} + \int_0^t e^{-b(t-\tau)}\|f(\cdot, \tau)\|_{2,\Omega} d\tau \leq e^{-bt}\varepsilon
\]
which implies (3.4). Next, we integrate over $t \in (0, \infty)$ and apply the Fubini theorem to obtain (3.5):
\[
\int_0^\infty \|v\|_{2,\Omega} dt \leq \frac{1}{b}\|v_0\|_{2,\Omega} + \int_0^\infty \int_0^t e^{-b(t-\tau)}\|f(\cdot, \tau)\|_{2,\Omega} d\tau dt
\]
\[
\leq \frac{1}{b}\|v_0\|_{2,\Omega} + \int_0^\infty \int_0^\tau e^{-b(t-\tau)}\|f(\cdot, \tau)\|_{2,\Omega} dt d\tau
\]
\[
\leq \frac{1}{b}\left\{\|v_0\|_{2,\Omega} + \int_0^\infty \|f\|_{2,\Omega} d\tau\right\}.
\]

\[\square\]

We cite now two lemmas from [2].

**Lemma 3.1.** Let $u \in C^{0,1+\alpha/2}(\cup Q^\pm_{T_0}), T_0 > 0, 0 < r < \sqrt{T_0}$.

Then $u$ satisfies the inequality
\[
\langle u \rangle^{(1+\alpha, \frac{\alpha}{2})}_{\cup Q^\pm_{T_0}} \leq 2r^{\gamma}\langle u \rangle^{(1+\alpha, \frac{\alpha}{2})}_{\cup Q^\pm_{T_0}} + cr^{\alpha-\frac{9}{2}}\int_0^{T_0} \|u\|_{2,\Omega} d\tau. \quad (3.6)
\]

**Lemma 3.2.** For an arbitrary $u \in C^{2+\alpha,1+\alpha/2}(\cup Q^\pm_{T_0}), 0 < r < \text{diam}\{\Omega\}$, the inequality
\[
\langle u \rangle^{(\alpha, \frac{\alpha}{2})}_{\cup Q^\pm_{T_0}} \leq 2r^{2}\langle u \rangle^{(2+\alpha,1+\alpha/2)}_{\cup Q^\pm_{T_0}} + cr^{-\alpha-\frac{7}{2}}\int_0^{T_0} \|u\|_{2,\Omega} d\tau \quad (3.7)
\]
holds.

**Proposition 3.2.** Let a solution $(v, p)$ of problem (1.1), (1.2) with $\sigma = 0$ exist on the interval $(0, T]$ and
\[
N_{(0,T)}[v, p] \equiv |u^0|^{(2+\alpha,1+\alpha/2)}_{\cup Q^\pm_{1}} + |\nabla q^0|^{(\alpha, \alpha/2)}_{\cup Q^\pm_{1}} + [q^0]^{(\alpha,1+\alpha)}_{\cup Q^\pm_{1}} \leq \mu \quad (3.8)
\]
hold. Here $(u^0, q^0)$ is the solution of problem (1.1), (1.2) in Lagrangian coordinates.
Then
\[ N_{(t_{0}-\tau_{0},t_{0})}[v,p] \leq c(\delta, \tau_{0}) \left\{ \left| f \right|_{\cup Q_{\delta}^{(\alpha,\frac{1+\sigma-\gamma}{2})}} + \left| \nabla f \right|_{\cup Q_{\delta}^{(\alpha,\frac{1+\sigma-\gamma}{2})}} + \int_{t_{0}-\tau_{0}}^{t_{0}} \|v(\cdot,\tau)\|_{2,\Omega} d\tau \right\}, \]
\[ \forall t_{0} \in (0, T], \tag{3.9} \]
where \( \tau_{0} \in (0, t_{0}/2) \), \( \tau_{0} \) depends on \( \mu, \delta \) is the value from (3.13), \( c(\delta, \tau_{0}) \) is a non-decreasing function. Here \( \cup Q'_{\beta} = \cup Q_{(t_{0}-2\tau_{0}+\beta,t_{0})}^{\pm} \).

Proof. We fixe arbitrary \( t_{0} \in (0, T] \). Let \( \tau_{0} \in (0, t_{0}/2) \) and \( \eta_{\lambda}(t) \) be a smooth monotone function of \( t \) with \( \lambda \in (0, \tau_{0}] \) such that
\[ \eta_{\lambda}(t) = \begin{cases} 0 & \text{if } t \leq t_{0}-2\tau_{0}+\lambda/2, \\ 1 & \text{if } t \geq t_{0}-2\tau_{0}+\lambda, \end{cases} \]
and for \( \dot{\eta}_{\lambda}(t) \equiv \frac{d\eta_{\lambda}(t)}{dt} \) the inequalities
\[ \left| \dot{\eta}_{\lambda}(t) \right|_{\mathbb{R}} \leq c\lambda^{-1}, \quad \left\langle \dot{\eta}_{\lambda}(t) \right\rangle_{\mathbb{R}}^{(\alpha/2)} \leq c\lambda^{-1-\alpha/2} \]
hold.

We consider the functions \( w = v\eta_{\lambda}, s = p\eta_{\lambda} \) which satisfy the system
\[ D_{t}w + (v \cdot \nabla)w - \nu^{\pm} \nabla^{2}w + \frac{1}{\rho^{\pm}} \nabla s = f\eta_{\lambda} + v\dot{\eta}_{\lambda}, \]
\[ \nabla \cdot w = 0 \text{ in } \Omega_{t}^{-} \cup \Omega_{t}^{+}, \quad t > t_{0}-2\tau_{0}, \]
\[ w|_{t=t_{0}-2\tau_{0}} = 0 \text{ in } \Omega_{t_{0}-2\tau_{0}}^{-} \cup \Omega_{t_{0}-2\tau_{0}}^{+}, \tag{3.10} \]
\[ [w]|_{\Gamma} = 0, \quad [w]|_{S_{T}} = 0, \]
\[ [\Pi_{0} \Pi T_{u}(w)n]|_{\Gamma_{r}} = 0, \quad [n_{0} \cdot \Pi T_{u}(w,s)n]|_{\Gamma_{r}} = 0. \]

Introduce the Lagrangian coordinates by the formula
\[ x = \xi + \int_{t_{0}-2\tau_{0}}^{t} u(\xi, \tau) d\tau \equiv X(\xi, t), \quad t > t_{0}-2\tau_{0}, \tag{3.11} \]
here \( u(\xi, t) = v(X(\xi, t), t) \). Then we can rewrite (3.10) in the form
\[ D_{t}w - \nu^{\pm} \nabla^{2}w + \frac{1}{\rho^{\pm}} \nabla s = f(X, t)\eta_{\lambda} + u\dot{\eta}_{\lambda}, \]
\[ \nabla \cdot w = 0 \text{ in } \cup \Omega', \quad t > t_{0}-2\tau_{0}, \]
\[ w|_{t=t_{0}-2\tau_{0}} = 0 \text{ in } \cup \Omega', \tag{3.12} \]
\[ [w]|_{\Gamma} = 0, \quad w|_{S_{T}} = 0, \]
\[ [\Pi_{0} \Pi T_{u}(w)n]|_{\Gamma_{r}} = 0, \quad [n_{0} \cdot \Pi T_{u}(w,s)n]|_{\Gamma_{r}} = 0. \]
Here $\cup \Omega' = \Omega_{t_0 - 2\tau_0}^- \cup \Omega_{t_0 - 2\tau_0}^+, \Gamma' = \Gamma_{t_0 - 2\tau_0}$, $n_0$ is the outward normal to $\Gamma'$, $\Pi_0$ and $\Pi$ are the projections onto the tangential planes to $\Gamma'$ and to $\Gamma_t$, respectively. Other notation, for example $\nabla u$, also corresponds the transform (3.11). The functions $w$, $s$, $f$ in the Lagrangian coordinates are denoted by the same letters.

In order to apply Theorem 2.2 (existence theorem for the linearized problem) to problem (3.12), we need to verify the hypotheses of it. To this end, we choose $\tau_0$ so small that inequality (2.6) holds. It is enough to take $\tau_0$ such that

$$(2\tau_0 + (2\tau_0)^{\gamma/2})\mu \leq \delta. \tag{3.13}$$

Since $\nabla u \cdot w = 0$, hypothesis 1)

$$-\nabla u \cdot (f(X, t)\eta + u\dot{\eta}) = \nabla \cdot g$$

is satisfied with $g = -A^*(f(X, t)\eta + u\dot{\eta})$ due to $\nabla u \equiv A\nabla = \nabla A^*$. It is evident that

$$[(g + A^*(f(X, t)\eta + u\dot{\eta})) \cdot n_0]|_{\Gamma'} = 0.$$

As tensor $\mathbb{G} = \{G_{ik}\}_{i,k=1}^3$, we take the potential

$$\mathbb{G}(\xi, t) = \nabla \int_{\mathbb{R}^3} \mathcal{E}(\xi, y) A^*(f(X(y, t), t)\eta + u\dot{\eta})dy,$$

where $\mathcal{E}(x, y) = \frac{-1}{4\pi|x-y|}$ is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, $f$ and $u$ are extended with preservation of class in the whole space $\mathbb{R}^3$ and vanish at infinity.

Condition 2) is fulfilled because $u \in C^{\alpha,\alpha/2}(D_T)$.

As to 3), 4), all the functions in (2.7), (2.8) with $\Omega^\pm = \Omega_{t_0 - 2\tau_0}$ equal zero at $t = t_0 - 2\tau_0$. Hence, by (2.9)

$$N(t_0 - 2\tau_0 + \lambda, t_0)[u, q] \leq N(t_0 - 2\tau_0, t_0)[w, s] \leq c_1(2\tau_0) \left\{ |f(X, t)\eta|_{\cup Q_0'}^{(\alpha, \alpha/2)} + |u\dot{\eta}|_{\cup Q_0'}^{(\alpha, \frac{\alpha}{2})} + |g|_{\cup Q_0'}^{(\alpha, \frac{\alpha}{2})} + |\mathbb{G}|_{\cup Q_0'}^{(\gamma, 1+\alpha)} + |\mathbb{G}|_{\cup Q_0'}^{(\gamma, 0)} \right\}.$$  

We can estimate the Hölder norm of the composite function $f(X(\xi, t), t)$ as follows:

$$|f(X, t)|_{\cup Q_0'}^{(\alpha, \alpha/2)} \leq |f|_{\cup Q_0'}^{(\alpha, \alpha/2)} + |\nabla f|_{\cup Q_0'}(2\tau_0|u|_{x, \cup Q_0'}^{(\alpha)} + (2\tau_0)^{1-\alpha/2}|u|_{\cup Q_0'}).$$
since
\[ f(X_u, t) - f(X_{u'}, t) = \sum_{k=1}^{3} \int_{0}^{1} f_{x_k}(X_{u_z}, t) \, dz \int_{t_{0}-2\tau_{0}}^{t} (u_k - u_k') \, d\tau. \]

For \( \lambda < 1 \), we conclude
\[ N_{(t_{0}-2\tau_{0}+\lambda, t_{0})}[u, q] \leq c_2(1 + \delta) \left\{ \frac{1}{\lambda^\frac{\alpha}{2}} |f|_{x, \cup Q_{0}'}^{(\alpha)} + \frac{1}{\lambda^\frac{1+\alpha-\gamma}{2}} (|f|_{t, \cup Q_{0}'} + |\nabla f|_{\cup Q_{0}'}) + \frac{1}{\lambda} \langle u \rangle_{t, \cup Q_{\lambda/2}} + \frac{1}{\lambda^\frac{3+\alpha-\gamma}{2}} |u|_{\cup Q_{\lambda/2}'} \right\}. \quad (3.14) \]

We take now \( r_1 = (\epsilon \lambda)^{1/\gamma} \) and \( r_2 = (\epsilon \lambda)^{1/2} \) in estimates (3.6), (3.7), respectively, and evaluate \( |u|_{\cup Q_{\lambda/2}} \) in (3.14) by a way similar to Lemmas 3.1, 3.2. As a result, we obtain
\[ N_{(t_{0}-2\tau_{0}+\lambda, t_{0})}[u, q] \leq c_3(\delta) \left\{ \epsilon N_{(t_{0}-2\tau_{0}+\lambda/2, t_{0})}[u, q] + \frac{1}{\lambda^\frac{\alpha}{2}} |f|_{x, \cup Q_{0}'}^{(\alpha)} + \frac{1}{\lambda^\frac{1+\alpha-\gamma}{2}} (|f|_{t, \cup Q_{0}'} + |\nabla f|_{\cup Q_{0}'}) + c(\epsilon) \lambda^{\gamma - \gamma} \int_{t_{0}-2\tau_{0}}^{t_{0}} \|u(\cdot, \tau)\|_{2, \Omega} \, d\tau \right\}, \quad (3.15) \]

where \( \gamma = \max \left\{ \frac{9}{2\gamma} + \frac{\alpha}{\gamma}, \frac{11}{4} + \frac{\alpha}{2}, \frac{3+\alpha-\gamma}{2} (1 + \frac{7}{2(1+\alpha)}) \right\} = \frac{9}{2\gamma} + \frac{\alpha}{\gamma}. \)

Let us introduce the function
\[ \Phi(\lambda) = \lambda^\kappa N_{(t_{0}-2\tau_{0}+\lambda, t_{0})}[u, q]. \]

Then we can rewrite (3.15) as follows:
\[ \Phi(\lambda) \leq c_4\epsilon \Phi(\lambda/2) + K, \quad (3.16) \]

where \( c_4 = c_3(\delta)2^\kappa, \)
\[ K = c_3(\delta) \left\{ |f|_{\cup Q_{0}'}^{(\alpha, \frac{1+\alpha-\gamma}{2})} + |\nabla f|_{\cup Q_{0}'} + c(\epsilon) \int_{t_{0}-2\tau_{0}}^{t_{0}} \|u(\cdot, \tau)\|_{2, \Omega} \, d\tau \right\}. \]

We set \( \epsilon = \frac{1}{2c_4} \) in (3.16). By iterations with \( \lambda/2, \ldots, \lambda/2^k \), we deduce from inequality (3.16) in the limit \( k \to \infty \) that
\[ \Phi(\lambda) \leq 2K. \]

This inequality with \( \lambda = \tau_0 \) implies (3.9).
Now we can prove Theorem 3.1 (global existence theorem for $\sigma = 0$).

**Proof.** By Theorem 2.1 (local existence theorem), we have a solution $(v, p)$ on an interval $(0, T_0]$. We can take so small data in estimate (2.3) that value $T_0$ will be greater than 1. The solution norm satisfies the inequality

$$N_{(0,T_0)}[v, p] \leq \mu$$

with some $\mu > 0$. Then by Proposition 3.2, there exists $\tau_0 < T_0/2$ such that (3.13) is satisfied and for $(v, p)$, $T_0$ estimate (3.9) holds. Together with (3.4), it implies that

$$N_{(0,T_0)}[v, p] \leq c_5(\delta, \tau_0)e^{-b\tau_0}\left\{\|v_0\|_{2,\Omega} + \int_{0}^{\infty} e^{b\tau}\|f(\cdot, \tau)\|_{2,\Omega} d\tau + |e^{bt}f|_{Q_\infty}^{(a,\frac{1+a-\gamma}{2})} + |e^{bt}\nabla f|_{Q_\infty}^{(a,\frac{1+a-\gamma}{2})}\right\}$$

$$\leq c_5(\delta, \tau_0)e^{-b\tau_0}(1 + |\Omega|^{1/2})\epsilon,$$

where $|\Omega|$ is the measure of $\Omega$, $t_0 \in (2\tau_0, T_0]$.

Thus,

$$|v(\cdot, T_0)|_{L^2(\Omega_{T_0})}^{(2+\alpha)} \leq \mu.$$

Now we apply Theorem 2.1 (local existence theorem) again and obtain a solution in an interval $(T_0, T_0 + T_1]$ corresponding to the initial data $v(\cdot, T_0)$.

Note that

$$N_{(T_0,T_0+T_1)}[v, p] \leq \mu_1.$$

There exists $\tau_1 < T_1/2$ such that (3.13) holds for $\tau_1, \mu_1$, hence,

$$N_{(T_0+T_1-\tau_1, T_0+T_1)}[v, p] \leq c(\delta, \tau_1)\left\{|f|_{Q_{(T_0+T_1-2\tau_1,T_0+T_1)}}^{(a,\frac{1+a-\gamma}{2})} + |\nabla f|_{Q_{(T_0+T_1-2\tau_1,T_0+T_1)}}^{(a,\frac{1+a-\gamma}{2})}\right\}$$

$$+ \int_{T_0+T_1-2\tau_1}^{T_0+T_1} \|v(\cdot, \tau)\|_{2,\Omega} d\tau$$

$$\leq c_6(\delta, \tau_1)e^{-b(T_0+T_1-2\tau_1)}\left\{\|v_0\|_{2,\Omega} + \int_{0}^{\infty} e^{b\tau}\|f(\cdot, \tau)\|_{2,\Omega} d\tau + |e^{bt}f|_{Q_\infty}^{(a,\frac{1+a-\gamma}{2})} + |e^{bt}\nabla f|_{Q_\infty}^{(a,\frac{1+a-\gamma}{2})}\right\}$$

$$\leq c_6(\delta, \tau_1)e^{-b(T_0+T_1)}(1 + |\Omega|^{1/2})\epsilon.$$

(3.18)
We take in (3.1) \( \varepsilon \) so small that \( c_6(\delta, \tau_1)(1 + |\Omega|^{1/2})\varepsilon < \mu \). Hence, we have again

\[
|v(\cdot, T_0 + T_1)|_{\cup \Omega_{T_0+T_1}}^{(2+\alpha)} \leq \mu e^{-b(T_0+T_1)} \leq \mu.
\]

Since the norms of the data have not increased, a solution of problem (1.6), (1.9) exists in \( (T_0 + T_1, T_0 + 2T_1] \) and

\[
N(T_0+T_1, T_0+2T_1)[v, p] \leq \mu_1.
\]

Inequality (3.18) is valid for the interval \( (T_0 + 2T_1 - \tau_1, T_0 + 2T_1) \). Thus, the solution of (1.1), (1.3), (1.4) with \( \sigma = 0 \) can be extended as far as one likes.

Solution uniqueness follows from the uniqueness of local solutions at every moment of time (Theorem 2.1).

\[\square\]

4 Problem (1.1), (1.3), (1.4) with positive surface tension. Energy estimate

We state global existence theorem for \( \sigma > 0 \).

**Theorem 4.1.** Let the hypotheses of Theorem 2.1 with \( q_0 = p_1(x, 0) \) hold, and for \( t = 0 \) let \( \Gamma \) be given by the equation

\[
|x| = R\left(\frac{x}{|x|}, 0\right)
\]

on the unit sphere \( S_1 \). Suppose, in addition, that the initial data are small enough, i.e.

\[
|e^{bt}f|_{Q_\infty}^{(\alpha, \frac{1+\alpha-\gamma}{2})} + |e^{bt}\nabla f|_{Q_\infty}^{(\alpha, \frac{1+\alpha-\gamma}{2})} + \|e^{bt}f\|_{2,Q_\infty} +
+ |v_0|_{\cup \Omega_0^\pm}^{(2+\alpha)} + |r_0|_{S_1}^{(3+\alpha)} \leq \varepsilon \ll 1, \quad (4.1)
\]

where \( r_0(x/|x|) = R(x/|x|, 0) - R_0 \), \( R_0 \) is the radius of the ball \( B_{R_0} \):

\[
|\Omega_0^+| = 4\pi R_0^3/3.
\]

Then problem (1.1), (1.3), (1.4) is uniquely solvable in the whole positive half-axis \( t > 0 \), and solution \( (v, p_1) \) possesses the properties:

\[v \in C^{2+\alpha, 1+\alpha/2}, \quad p_1 \in C(\gamma, 1+\alpha), \quad \nabla p_1 \in C^{\alpha, \alpha/2}, \]

the boundary \( \Gamma_t \) being given for every \( t \) by a function \( R(\cdot, t) \) of the class \( C^{3+\alpha} \):
(where $h(t)$ is a position of the barycenter of $\Omega_t^+$ at the moment $t$), and tending to a sphere of the radius $R_0$ with center in a certain point $h_\infty$, and the pressure being defined up to a bounded function of time. It means that for any $t_0 \in (0, \infty)$, the solution $(u, q)$ and its derivatives in Lagrangian coordinates belong to respective Hölder spaces over $D_{(t_0, t_0+\tau)} \equiv \cup Q_{(t_0, t_0+\tau)}^\pm$ for a sufficiently small time interval $(t_0, t_0+\tau)$. Moreover, there holds the estimate

$$
|u|_{D_{(t_0, t_0+\tau)}}^{(2+\alpha, 1+\alpha/2)} + |\nabla q|_{D_{(t_0, t_0+\tau)}}^{(\alpha, 1+\alpha)} + |q|_{D_{(t_0, t_0+\tau)}}^{(\gamma, 1+\alpha)} + \sup_{t \in (t_0, t_0+\tau)} |r(\cdot, t)|_{S_1}^{(3+\alpha)} \leq c e^{-b_1 t_0} \left( |e^{b_1} tf|_{Q_{\infty}}^{(\alpha, \frac{1+\alpha-\gamma}{2})} + |e^{b_1} t \nabla f|_{Q_{\infty}}^{(\alpha, \frac{1+\alpha-\gamma}{2})} + \|e^{b_1} tf\|_{2, Q_{\infty}} + |v_0|_{\cup \Omega_0^\pm}^{(2+\alpha)} + |r_0|_{S_1}^{(3+\alpha)} \right),
$$

(4.2)

where $r(\omega, t) = R(\omega, t) - R_0$, $\omega \in S_1$.

This theorem for $f = 0$ was proven in [5].

One can conclude from Theorem 4.1 that the trivial solution is unique when the initial velocity and mass forces are absent and the initial interface coincides with a sphere. The stability of this solution takes place in the sense that the solution differs a little from zero under a small deviation of the initial data and mass forces from zero ones, the interface tending to a sphere $S_{R_0}(h_\infty)$. However, the center $h_\infty$ of this limit sphere may be displaced with respect to the initial barycenter of $\Omega_0^+$ for no matter how small an initial velocity $v_0$ is. This displacement is evaluated by inequality (5.5) at the end of the paper. There we also give a necessary estimate with above of initial distance between the outer boundary and fluid interface.

We need again an exponential estimate for the solution of the nonlinear problem (1.1), (1.3), (1.4) in $L_2$. Now we prove it by using the notion of generalized energy $\mathcal{E}$ introduced in [7, 8].

Assume the solution exists in the interval $[0, T]$. It is guaranteed by local existence theorem (Theorem 2.1). Thus, we have also barycenter trajectory of the drop $\Omega_t^+$: $h(t) = \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} x \, dx$.

Remind that $r(\omega, t)$ is the deviation function of $\Gamma_t$ from the sphere $S_{R_0}(t) \equiv S_{R_0}(h(t)) = \{|x - h(t)| = R_0\}$. Incompressibility of the fluids implies that the domains $\Omega_t^+$ conserve their volumes for all $t > 0$:

$$
\int_{\Omega_t^+} dx = \int_{\Omega_0^+} dx \Rightarrow \int_{S_1} (R^3 - R_0^3) \, d\omega = 0.
$$
Since \( r = R - R_0 \), the equality

\[
\int_{S_1} r \, d\omega = -\frac{1}{R_0} \int_{S_1} r^2 \, d\omega - \frac{1}{3R_0^2} \int_{S_1} r^3 \, d\omega
\]

(4.3) holds. We are going to use it later on.

Assume (without restriction of generality) that \( h(0) = 0 \). Moreover, assume that \( \Gamma \) is defined by the equation

\[
x = y + N(y) r_0 \left( \frac{y}{|y|} \right), \quad y \in S_{R_0}.
\]

(4.4)

(\( N(y) \) is the exterior normal to \( S_{R_0} = \{|y| = R_0\} \), i.e., \( N(y) = y/|y| \).) We assume that also for \( 0 < t \leq T \) \( \Gamma_t \) can be defined by

\[
x = y + N(y) r_0 \left( \frac{y}{|y|}, t \right) + h(t), \quad y \in S_{R_0}.
\]

(4.5)

Equations (4.4) and (4.5) are equivalent to the relations

\[
|x| = R_0 + r_0 \left( \frac{x}{|x|} \right) \quad \text{and} \quad |x-h(t)| = R_0 + r \left( \frac{x-h}{|x-h|}, t \right),
\]

respectively.

From the kinematic condition (1.4), it follows that

\[
\mathcal{D}_\tau x \cdot n = v \cdot n|_{\Gamma_t}.
\]

Hence

\[
n_0N(y) \mathcal{D}_\tau r|_{t=0} = v_0 \cdot n_0 - h'(t) \cdot n_0,
\]

\[
n_N(y) \mathcal{D}_\tau r = v \cdot n - h'_t \cdot n, \quad y \in S_{R_0},
\]

(4.6)

\( n_0N = n_0 \cdot N \) is a radial part of \( n_0 \) in the coordinate system with the origin at 0, and \( n_N = n \cdot N \) is a radial part of \( n \) in the coordinate system with the origin at \( h \).

We can rewrite \( h(t) \) in the form:

\[
h(t) = \frac{1}{|\Omega_t^+|} \int_0^t \int_{\Omega_t^+} u(\xi, \tau) \, d\xi \, d\tau \quad \text{and} \quad h'_t(t) = \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} v(x, t) \, dx.
\]

(4.7)

Note that the barycenter of \( \Omega_t^+ \) always coincides with the origin in the coordinates \( \{y\} \), it means that

\[
\int_{S_1} (R_1^4 - R_0^4) \omega_i \, d\omega = 0,
\]

(4.8)

where \( \omega_i = y_i/|y| \).
Proposition 4.1. Assume that the classical solution of the problem (1.1), (1.3), (1.4) is defined in $[0, T]$ and $\mathbf{v}_0$ satisfies compatibility conditions (2.1). In addition, let $r$ be such that

$$|r(\omega, t)|_{S_{1} \times (0, T)} + |\nabla_{S_{1}} r(\omega, t)|_{S_{1} \times (0, T)} \leq \delta_1 R_0 \ll 1. \quad (4.9)$$

Then for $\forall t \in (0, T]$

$$\|\mathbf{v}(\cdot, t)\|_{2, \Omega}^2 + \|r(\cdot, t)\|_{W_{2}^{1}(S_{1})}^2 \leq c e^{-2b_{1}t} \left\{ \|e^{b_{1}\tau} f\|_{2,Q_{\infty}}^2 + \|\mathbf{v}_0\|_{2,\Omega}^2 + \|r_0\|_{W_{2}^{1}(S_{1})}^2 \right\}, \quad (4.10)$$

$$\int_0^T \left( \|\mathbf{v}(\cdot, \tau)\|_{2, \Omega} + \|r(\cdot, \tau)\|_{W_{2}^{1}(S_{1})} \right) d\tau \leq c \left\{ \|e^{b_{1}\tau} f\|_{2,Q_{\infty}} + \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{W_{2}^{1}(S_{1})} \right\}, \quad (4.11)$$

with the constants $b_1$ and $c$ independent of $T$.

This proposition with $f = 0$ was proven in [5].

Proof. We multiply the first equation in (1.1) by $\rho^\pm \mathbf{v}$ and integrate by parts in $\Omega^-_t \cup \Omega^+_t$. Then

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho^\pm} \mathbf{v}\|_{2, \Omega}^2 + \|\mu^\pm S(\mathbf{v})\|_{2, \Omega}^2 = \int_{\Omega} f \cdot \mathbf{v} \, dx + \int_{\Gamma_{i}} \left( H + \frac{2}{R_0} \right) \mathbf{n} \cdot \mathbf{v} \, d\Gamma.$$ 

In view of the formula (1.8) and of the fact that $\int_{\Gamma_{i}} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = 0$, the integral in the right hand side has the form $\int_{\Gamma_{i}} \mathbf{v} \cdot \Delta(t) \mathbf{x} \, d\Gamma$. In [6] it is proved that

$$\int_{\Gamma_{i}} \mathbf{v} \cdot \Delta(t) \mathbf{x} \, d\Gamma = -\sigma \frac{d}{dt} |\Gamma_{i}|.$$ 

Therefore we can write

$$\frac{d}{dt} \left\{ \frac{1}{2} \|\sqrt{\rho^\pm} \mathbf{v}\|_{2, \Omega}^2 + \sigma (|\Gamma_{i}| - 4\pi R_0^2) \right\} + \frac{1}{2} \|\mu^\pm S(\mathbf{v})\|_{2, \Omega}^2 = \int_{\Omega} f \cdot \mathbf{v} \, dx.$$ 

Since $\mathbf{v}|_{S} = 0$, the vector field $\mathbf{v}$ satisfies the Korn inequality (3.3) in $\Omega$. Consequently,

$$\frac{d}{dt} \left\{ \frac{1}{2} \|\rho^\pm \mathbf{v}\|_{2, \Omega}^2 + \sigma (|\Gamma_{i}| - 4\pi R_0^2) \right\} + c_1 \|\mathbf{v}\|_{W_{2}^{1}(\Omega)}^2 \leq c \|f\|_{2, \Omega}^2. \quad (4.12)$$
In order to be able to use the Gronwall inequality for the generalized energy mentioned above, we should try to add in the left hand side the term of the type $|\Gamma_t| - 4\pi R_0^2$. To this end, we construct an auxiliary vector-valued function $W(x, t), \ x \in \Omega$.

Let the function $f_0(z)$ be defined on the sphere $S_{R_0}$ and $\int_{S_{R_0}} f_0 \, dS = 0$. We define $W_0(z)$ as a solenoidal vector field in the whole space $\mathbb{R}^3$, such that

$$W_0|_{S_{R_0}} = N f_0(z).$$

We also assume that supp$W_0$ is contained in the ball $B_{R_0+a} = \{|y| \leq R_0+a\}$, where $a > 0$ is not very large. In addition,

$$\|W_0\|_{W^2_2(\mathbb{R}^3)} \leq c \|f_0\|_{W^2_2(S_{R_0})}, \quad \|W_0\|_{2, \mathbb{R}^3} \leq c \|f_0\|_{2, S_{R_0}}, \quad (4.13)$$

and if $f_0 = f_0(z, t)$, then

$$\|D_t W_0\|_{2, \mathbb{R}^3} \leq c \|D_t f_0\|_{2, S_{R_0}}, \quad (4.14)$$

Such $W_0$ can be constructed (see, for instance, [9], Ch. I).

Further we set

$$f_0(y, t) = \bar{r}(y, t) \equiv r(y, t) - \overline{r}(t),$$

where $\overline{r}(t) = \frac{1}{4\pi R_0^2} \int_{S_{R_0}} r(y, t) \, dS_{R_0}$. For $a$ and $h$ sufficiently small the vector field $W_0$ vanishes near $S$ for all $t \leq T$.

Now we make the following coordinate transformation:

$$x = y + N^*(y) r^*(y, t) + h(t) = e_r(y, t), \quad y \in \Omega,$$

where $N^*$ is the extension of $N$ in $\Omega$, and $r^*(y, t)$ is the extension of $\bar{r}(y)$ with the support in the neighborhood of $S_{R_0}$ and $r^* = 0$ near $S$. We note that for small $r^*(y, t)$ and $h$ this transformation is invertible, and it maps $S_{R_0}$ on $\Gamma_t$. The vector field $W$ can be defined as

$$W(x, t) = \frac{L(y, t)}{L(y, t)} W_0(y, t)|_{y = e_r^{-1}(x)}.$$

Here $L$ is the Jacobi matrix of the transformation $e_r$:

$$L(y, t) = \left\{ \frac{\partial e_r(y, t)}{\partial y} \right\} = \left\{ \delta^i_j + \frac{\partial (N^*_i r^*)}{\partial y_j} \right\}_{i, j=1}^3.$$

Therefore

$$W(x, t) = \frac{L(y, t)}{L(y, t)} W_0(y, t)|_{y = e_r^{-1}(x)}.$$
$L=\det \mathcal{L}$. Let $\hat{\mathcal{L}}$ be a co-factors matrix of $\mathcal{L}$, i.e., $\hat{\mathcal{L}}(y, t) = L\mathcal{L}^{-1}(y, t)$.

The vector field $\mathbf{W}$ is divergence free as the function of $x$. Indeed,

$$\nabla_x \cdot \mathbf{W} = \mathcal{L}^{-1T} \nabla_y \cdot \mathbf{W} = L^{-1} \nabla_y \cdot (\hat{\mathcal{L}} \mathbf{W}) = L^{-1} \nabla_y \cdot \mathcal{L}_L \mathbf{W}_0 = 0$$

in view of the identity $\nabla_y \cdot \hat{\mathcal{L}} = 0$, that holds for the co-factors matrix of arbitrary coordinate transformation. In addition, $\mathbf{W}(x, t) = 0$ for $x \in S$ and

$$\mathbf{W} \cdot n|_{\Gamma_t} = \mathbf{W} \cdot \frac{\hat{\mathcal{L}}^T N}{|\hat{\mathcal{L}}^T N|} = \frac{\mathbf{W}_0 \cdot N}{|\hat{\mathcal{L}}^T N|} = \frac{\tilde{r}}{|\hat{\mathcal{L}}^T N|}|_{y = e_{r}^{-1}(x)}.$$  \hfill (4.15)

From (4.13), (4.14) the estimates

$$\|\mathbf{W}(\cdot, t)\|_{W^2_1(\Omega)} \leq c\|\mathbf{W}_0(\cdot, t)\|_{W^2_1(B_{R_0+a})} \leq c\|r(\cdot, t)\|_{W^{1/2}_{2}(S_{R_0})},$$

$$\|\mathbf{W}(\cdot, t)\|_{2, \Omega} \leq c\|r(\cdot, t)\|_{2, S_{R_0}},$$  \hfill (4.16)

follow.

In [5], due to (4.6), it was also proved the inequality

$$\|\mathcal{D}_t \mathbf{W}(\cdot, t)\|_{2, \Omega} \leq c(|\mathbf{v}|_{\cup \Omega_{t}^{\pm}}^{(1)})\{\|\mathbf{v}\|_{W^2_1(\Omega)} + \|r\|_{W^{1/2}_{2}(S_{1})}\}.$$

(4.17)

Now we multiply the first equation in (1.1) by $\rho^\pm \mathbf{W}$ and integrate by parts over $\Omega_t^- \cup \Omega_t^+$:

$$\frac{d}{dt} \int_{\Omega} \rho^\pm \mathbf{v} \cdot \mathbf{W} \, dx + \int_{\Omega} \rho^\pm \mathbf{v} \cdot (\mathcal{D}_t \mathbf{W} + (\mathbf{v} \cdot \nabla) \mathbf{W}) \, dx \quad \hfill (4.18)$$

$$+ \int_{\Omega} \frac{\mu^\pm}{2} \mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{W}) \, dx - \sigma \int_{\Gamma_t} (H + \frac{2}{R_0}) n \cdot \mathbf{W} \, d\Gamma = \int_{\Omega} f \cdot \mathbf{W} \, dx,$$

where $\mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{W}) = S_{ij}(\mathbf{v}) S_{ij}(\mathbf{W})$. We add equation (4.18) multiplied by a small $\gamma$ to (4.12). Using (4.15) and the Korn inequality (3.3), we obtain

$$\frac{d}{dt} \left\{ \frac{1}{2} \|\rho^\pm \mathbf{v}\|_{2, \Omega}^2 + \gamma \int_{\Omega} \rho^\pm \mathbf{v} \cdot \mathbf{W} \, dx + \sigma (|\Gamma_t| - 4\pi R_0^2) \right\} + c_1 \|\mathbf{v}\|_{W^2_1(\Omega)}^2$$

$$- \gamma \int_{\Omega} \rho^\pm \mathbf{v} \cdot (\mathcal{D}_t \mathbf{W} + (\mathbf{v} \cdot \nabla) \mathbf{W}) \, dx + \gamma \int_{\Omega} \frac{\mu^\pm}{2} \mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{W}) \, dx$$

$$- \gamma \sigma \int_{S_{R_0}} (H + \frac{2}{R_0}) \tilde{r} \, dS_{R_0} \leq c(\epsilon_1) \|f\|_{2, \Omega}^2 + \epsilon_1 \gamma \|\mathbf{W}\|_{2, \Omega}^2.$$  \hfill (4.19)
We have used here the equality $d\Gamma(x) = |\hat{\mathcal{L}}^T N| dS_{R_0}(y)$, [8, p. 227].) For arbitrary small $\gamma$ the expression under the sign of derivative, that can be called a generalized energy $\mathcal{E}(t)$, is positive. Indeed, by theorem 3 in [6], under conditions (4.3), (4.8), (4.9) the inequality

$$E_1(R, R_0) \equiv |\Gamma_t| - 4\pi R_0^2 \geq c_2 \|r\|_{W^2_2(S_1)}^2,$$  \hspace{1cm} (4.20)

holds with the constant $c_2$ independent of $\delta_1$ and $R_0$.

Now we make the formula

$$E_1(R, R_0) = \int_{S_1} \left( R\sqrt{R^2 + |\nabla_{S_1}R|^2} - R_0^2 \right) d\omega$$

our starting point. By decomposition of $E_1(R, R_0)$ in the Taylor series at the point $R_0$, it was shown in [5] that $E_1(R, R_0)$ depends only on $r$ and $\nabla r$, therefore we write $E_1(r) \equiv E_1(R, R_0)$. Under condition (4.9), it holds

$$E_1(r) \leq c_3 \|r\|_{W^2_2(S_1)}^2$$  \hspace{1cm} (4.21)

Next, it was proved that the surface integral

$$E_2(r) \equiv - \int_{S_{R_0}} \left( H + \frac{2}{R_0} \right) \tilde{r} dS_{R_0}$$

was also positive definite, the well known formula for twice the mean curvature

$$H[R] = \frac{1}{R} \nabla_{S_1} \cdot \frac{\nabla_{S_1} R}{\sqrt{g}} - \frac{2}{\sqrt{g}},$$

being used in the spherical coordinates, $g = R^2 + |\nabla_{S_1} R|^2$,

$$\nabla_{S_1} R = R'_\theta e_\theta + \frac{1}{\sin \theta} R'_\varphi e_\varphi, \hspace{1cm} \nabla_{S_1} \cdot u = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta u_\theta) + \frac{1}{\sin \theta} \frac{\partial u_\varphi}{\partial \varphi}.$$  

Thus, we have

$$E_2(r) \geq c \|r\|_{W^2_2(S_1)}^2.$$  \hspace{1cm} (4.22)

And we set

$$\mathcal{E}(t) = \frac{1}{2} \|\rho^\pm v\|_{2,\Omega}^2 + \gamma \int_{\Omega} \rho^\pm v \cdot W dx + \sigma E_1(r).$$

On the basis of (4.16), (4.20), (4.21) we conclude that for sufficiently small $\gamma$

$$c_4 (\|v\|_{2,\Omega}^2 + \|r\|_{W^2_2(S_1)}^2) \leq \mathcal{E}(t) \leq c_5 (\|v\|_{2,\Omega}^2 + \|r\|_{W^2_2(S_1)}^2).$$  \hspace{1cm} (4.23)
Let us denote by $E_1(t)$ the terms outside the derivative $D_t E(t)$ in (4.19). It is easily seen, in view of (4.16), (4.17), (4.22), (4.23), that for small $\gamma$ there exists such a constant $b_1 > 0$ that

$$E_1(t) \geq 2b_1 E(t),$$

which implies the estimate

$$D_t E(t) + 2b_1 E(t) \leq c\|f\|_{2, \Omega}^2.$$

Then from the Gronwall lemma it follows that

$$E(t) \leq e^{-2b_1 t} E(0) + \int_0^t e^{-2b_1 (t-\tau)} \|f(\cdot, \tau)\|_{2, \Omega}^2 d\tau$$

$$\leq c e^{-2b_1 t} (\|v_0\|_{2, \Omega}^2 + \|r_0\|_{W^1_2(S_1)}^2 + \|e^{b_1 \tau} f\|_{2, Q_t}^2)$$

and (4.10), due to (4.23). By integrating from 0 to $T$, we obtain inequality (4.11). \(\square\)

**Corollary 4.1.** The coordinates of the barycenter of $\Omega^+_t$ satisfy the inequality

$$|h(t)| \leq c \left\{ \|e^{b_1 \tau} f\|_{2, Q_\infty} + \|v_0\|_{2, \Omega} + \|r_0\|_{W^1_2(S_1)} \right\}, \quad \forall t \in [0, T]. \quad (4.24)$$

**Proof.** From formula (4.7) it follows the estimate

$$|h(t)| \leq \frac{1}{|\Omega^+_t|^{1/2}} \int_0^t \|v(\cdot, \tau)\|_{2, \Omega^+_t} d\tau,$$

which, together with (4.11) implies inequality (4.24). \(\square\)

## 5 Proof of the solvability of problem (1.1), (1.3), (1.4) with $\sigma > 0$ in global

**Proposition 5.1.** Let the solution of problem (1.1), (1.3), (1.4) be defined in the interval $(0, T]$ and let the estimate

$$N_{(0,T)}[v, p_1] \equiv |u|_{D_T}^{2+\alpha,1+\alpha/2} + |\nabla q|_{D_T}^{(\alpha,\alpha/2)} + |q|_{D_T}^{(\gamma, 1+\alpha)} \leq \mu.$$

hold, where $(u, q)$ is a solution of problem (1.1), (1.3), (1.4) written as a function of the Lagrangian coordinates.
Then
\[
N_{(t_0-\tau_0,t_0)}[v,p_1,r] \equiv N_{(t_0-\tau_0,t_0)}[v,p_1] + \sup_{t_0-\tau_0<\tau<t_0} |r(\cdot, \tau)|_{S_1}^{(3+\alpha)}
\]
\[
\leq c_1(\delta, \tau_0) \left\{ |f|_{\cup Q_0}^{(\alpha, 1+\alpha-\gamma/2)} + |\nabla f|_{\cup Q_0}^{(\alpha, 1+\alpha-\gamma/2)}
\right.
\]
\[
\left. + \int_{t_0-2\tau_0}^{t_0} (\|v(\cdot, \tau)\|_{2,\Omega} + \|r(\cdot, \tau)\|_{W_2^1(S_1)}) d\tau \right\},
\]
where \( t_0 \in (0, T], \tau_0 \in (0, t_0/2), \) \( \tau_0 \) depends on \( \mu \) and on the constant \( \delta \) in (3.13), \( \cup Q_0 = \cup Q_{(t_0-2\tau_0,t_0)}' \).

This proposition was demonstrated in [5] for \( f = 0 \) in a similar way as Proposition 3.2.

**Lemma 5.1.** Let \( r_0 \in C^{1+\alpha}(S_1) \) and \( u \in C^{1+\alpha,0}(D_{T_0}), \alpha \in (0, 1) \).

Then \( r(\cdot, t) \in C^{1+\alpha}(S_1) \) for arbitrary \( t \in (0, T_0) \) and the inequality
\[
|r(\cdot, t)|_{S_1}^{(1+\alpha)} \leq c_2 (|r_0|_{S_1}^{(1+\alpha)} + t |u|_{\xi,D_t}^{(1+\alpha)}),
\]
holds, if the norms \( r_0 \) and \( u \) are small.

This lemma is proved by the passage to the Lagrangian coordinates [5].

**Proof of Theorem 4.1.** By Theorem 2.1, there exists a local solution \((v, p_1)\) in \((0, T_0], T_0 > 1\) when \( \epsilon \) in (4.1) is small enough. For \((v, p_1)\), estimate (2.4) holds therefore
\[
N_{(0,T_0)}[v,p_1] \leq c \left( |f|_{Q_{T_0}}^{(\alpha, 1+\alpha-\gamma/2)} + |D_x f|_{Q_{T_0}}^{(\alpha, 1+\alpha-\gamma/2)} + |v_0|_{\cup Q_0}^{(2+\alpha)} + |H_0 + \frac{2}{R_0} r|_{S_1}^{(1+\alpha)} \right)
\]
\[
\leq c \left( |f|_{Q_{T_0}}^{(\alpha, 1+\alpha-\gamma/2)} + |D_x f|_{Q_{T_0}}^{(\alpha, 1+\alpha-\gamma/2)} + |v_0|_{\cup Q_0}^{(2+\alpha)} + |r_0|_{S_1}^{(3+\alpha)} \right) \leq c_3 \epsilon \equiv \mu.
\]

By Prop. 5.1, there exists \( \tau_0 < T_0/2 \) such that (3.13) is satisfied and for \((v, p_1), T_0 \) estimate (5.1) holds. Lemma 5.1 guarantees the inequality
\[
|r|_{S_1 \times (0,T_0)}^{(1+\alpha,0)} \leq c_2 (|r_0|_{S_1}^{(1+\alpha)} + c_3 \epsilon T_0) \leq \delta_1 R_0
\]
\((|r|_{S_1 \times (0,T_0)}^{(1+\alpha,0)} \equiv \sup_{0<\tau<T_0} |r(\cdot, \tau)|_{S_1}^{(1+\alpha)}), \) when \( \epsilon \) is sufficiently small. This permits us to apply Prop. 4.1, and from (4.10), (5.1) we obtain
\[
N_{(t_0-\tau_0,t_0)}[v,p_1,r] \leq c_4 e^{-b_1(t_0-2\tau_0)} \left\{ |e^{b_1 t f}|_{\cup Q_0}^{(\alpha, 1+\alpha\gamma/2)} + |e^{b_1 t \nabla f}|_{\cup Q_0}^{(\alpha, 1+\alpha\gamma/2)}
\right.
\]
\[
\left. + |e^{b_1 t r}|_{\cup Q_0}^{2} + \|v_0\|_{2,\Omega} + \|r_0\|_{W_2^1(S_1)} \right\}
\]
\[
\leq c_5 (\tau_0) e^{-b_1 t_0} (|\Omega|^\frac{1}{2} + 2\pi^{\frac{1}{2}}) \epsilon,
\]
where $|\Omega|$ is the measure of $\Omega$, and $t_0 \in (2\tau_0, T_0]$.

For $t_0 = T_0$, we have the estimate
\[
|v(\cdot, T_0)|^{(2+\alpha)}_{\Omega_{T_0}^{\pm}} + |r(\cdot, T_0)|^{(3+\alpha)}_{S_1} \leq \mu.
\]

Next, we use Theorem 2.1 again to obtain solution in $(T_0, T_0 + T_1]$ for the initial data $v(\cdot, T_0)$, $|r(\cdot, T_0)$. The norm of the solution
\[
N_{(T_0, T_0+T_1)}[v, p_1] \leq \mu_1
\]

Due to Prop. 5.1, we can find $0 < \tau_1 < T_1/2$ such that satisfies (3.13) and
\[
N_{(T_0+T_1-\tau_1, T_0+T_1)}[v, p_1, r] \leq c(\delta, \tau_1)\left\{\left|f\right|^{(\alpha, \frac{3+\alpha}{2})}_{Q(T_0,T_1-\tau_1,T_0+T_1)} + \left|\nabla f\right|^{(\alpha, \frac{3+\alpha}{2})}_{Q(T_0+T_1-\tau_1,T_0+T_1)}
\right.
+ \int_{T_0+T_1-2\tau_1}^{T_0+T_1} (|v(\cdot, \tau)|_{2, \Omega} + |r(\cdot, 0)|_{W_2^{1}(S_1)}) d\tau\right\}. \quad (5.3)
\]

Since in view of (5.2)
\[
|r|^{(1+\alpha, 0)}_{S_1 \times (T_0, T_0+T_1)} \leq c_2 \left(\left|f(\cdot, 0)\right|^{(1+\alpha)}_{S_1} + T_1 |u|^{(1+\alpha)}_{D(T_0, T_0+T_1)}\right) \leq c_2 (c_1 \varepsilon + T_1 \mu_1) \leq \delta_1 R_0,
\]
similarly to (3.18) we continue (5.3) by Prop. 4.1 as follows
\[
N_{(T_0+T_1-\tau_1, T_0+T_1)}[v, p_1, r] \leq c_0(\delta, \tau_1)e^{-b(T_0+T_1)}(1 + |\Omega|^{1/2} + 2\pi^{1/2})\varepsilon.
\]

Choose $\varepsilon$ so small that $c_0(\delta, \tau_1)(1 + |\Omega|^{1/2} + 2\pi^{1/2})\varepsilon \leq \mu$.

Hence,
\[
|v(\cdot, T_0 + T_1)|^{(2+\alpha)}_{\cup \Omega_{T_0+T_1}^{\pm}} + |r(\cdot, T_0 + T_1)|^{(3+\alpha)}_{S_1} \leq \mu e^{-b(T_0+T_1)}.
\]

Thus, the norms of the initial data do not increase. Therefore we can extend the solution in the interval $(T_0 + T_1, T_0 + 2T_1]$. This procedure may be repeated again and again as long as we like.

By repeating our argument, we should pass to the Lagrangian coordinates according to the formula
\[
X = \xi^{(1)} + \int_{T_0}^{t} \tilde{u}(\xi^{(1)}, \tau) d\tau, \quad \xi^{(1)} \in \cup \Omega_{T_0}^{\pm}, \quad t \in (T_0, T_0 + T_1).
\]

In fact, due to the additivity of the integral, (5.4) coincides with (4.2):
\[
X(\xi, t) = \xi + \int_{0}^{T_0} u(\xi, \tau) d\tau + \int_{T_0}^{t} u(\xi, \tau) d\tau, \quad \xi \in \cup \Omega_{0}^{\pm}, \quad t \in (T_0, T_0 + T_1),
\]
because \( \tilde{u}(\xi^{(1)}, \tau) = u(\xi, \tau) \).

The same remark is valid for the coordinates of inner fluid barycenter, since the volume of the fluid is conserved:

\[
\begin{align*}
\mathbf{h}(t) &= \mathbf{h}(T_0) + \int_{T_0}^{t} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x, \tau) \, dx \, d\tau = \int_{0}^{T_0} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x, \tau) \, dx \, d\tau \\
&+ \int_{T_0}^{t} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x, \tau) \, dx \, d\tau = \frac{3}{4\pi R_0^3} \int_{0}^{T_0} \int_{\Omega_t^+} \mathbf{v}(x, \tau) \, dx \, d\tau + \int_{T_0}^{t} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x, \tau) \, dx \, d\tau, \quad t > T_0.
\end{align*}
\]

Hence, one can conclude that

\[
|h(t)| \leq a, \quad t \leq T_0 + T_1,
\]

where

\[
a = \frac{c_8}{|\Omega_0^+|^{1/2} b_1} \left\{ \|e^{b_1 t} \mathbf{f}\|_{2,Q_{\infty}} + \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{W_2^1(S_1)} \right\}.
\]

The solution of the system (1.1), (1.3), (1.4) can be extended in this way with respect to \( t \) as far as necessary and it will satisfy the inequality (4.2).

The limiting position of the barycenter is estimated from Corollary 4.1:

\[
|h_{\infty}| \leq a \leq c_7 \left( \|e^{b_1 t} \mathbf{f}\|_{2,Q_{\infty}} + \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{S_1}^{(3+\alpha)} \right) \leq c_7 \varepsilon. \tag{5.5}
\]

Inequality (5.5) implies that the initial distance between the surfaces \( \Gamma \) and \( S \) should be strictly larger than \( c_7 \left( \|e^{b_1 t} \mathbf{f}\|_{2,Q_{\infty}} + \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{S_1}^{(3+\alpha)} \right) + \delta_1 R_0 \)

with \( \delta_1, R_0 \) from (4.9), to exclude their intersection in the future.

The uniqueness of a solution follows from the local existence and uniqueness theorem.

**Remark 5.1.** If \( \sigma = 0 \), we need evaluate the integral \( I \equiv \int_0^\infty |u(\cdot, t)|_{\Omega^+} \, dt \) in order to estimate the distance between the solid boundary and the interface. In virtue of (3.2),

\[
|u(\cdot, t_0)|_{\Omega^+} \leq c(\delta, \tau_0) e^{-bt_0} \varepsilon. \tag{5.6}
\]

By using (2.4) and integrating (5.6) with respect to \( t_0 > T_0/2 \), we arrive at

\[
I \leq \frac{T_0}{2} |u|_{Q_{T_0/2}^{++}} + \int_{T_0/2}^\infty |u(\cdot, t)|_{\Omega^+} \, dt \leq \frac{T_0}{2} \varepsilon + c(\delta, \tau_0) \frac{1}{b} \varepsilon \leq c_8 \varepsilon.
\]

Thus, if the initial distance between the surfaces is greater than \( c_8 \varepsilon \), \( \Gamma_t \) will never intersect \( S \).

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References


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