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Leray's problem on $D$-solutions to the stationary Navier-Stokes equations past an obstacle (Mathematical Analysis of Incompressible Flow)

Author(s)
Heck, Horst; Kim, Hyunseok; KOZONO, Hideo

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Leray's problem on \( D \)-solutions to the stationary Navier-Stokes equations past an obstacle

Horst Heck
Department of Engineering and Information Technology
Bern University of Applied Sciences
CH-3400 Burgdorf, Switzerland
horst.heck@bfh.ch

Hyunseok Kim
Department of Mathematics
Sogang University
Seoul, 121-742, Korea
kimh@sogang.ac.kr

Hideo KOZONO
Department of Mathematics
Waseda University
Tokyo 169-8555, Japan
kozono@waseda.jp

Introduction.

Let \( \Omega \) be an exterior domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \in C^\infty \). We consider the stationary Navier-Stokes equations in \( \Omega \):

\[
\begin{aligned}
-\Delta u + u \cdot \nabla u + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega, \\
\quad u(x) \rightarrow u^\infty \quad \text{as } |x| \rightarrow \infty,
\end{aligned}
\]

(N-S)

where \( u = u(x) = (u_1(x), u_2(x), u_3(x)) \) and \( p = p(x) \) denote the unknown velocity vector and the unknown pressure at \( x = (x_1, x_2, x_3) \in \Omega \), while \( f = f(x) = (f_1(x), f_2(x), f_3(x)) \) is the given external force, and \( u^\infty = (u_1^\infty, u_2^\infty, u_3^\infty) \) is the prescribed constant vector in \( \mathbb{R}^3 \) at infinity. In the pioneer work of Leray [14], it was shown that for every \( f \in \dot{H}^{-1,2}(\Omega) \equiv \dot{H}_0^{1,2}(\Omega)^* \) and for every \( u^\infty \in \mathbb{R}^3 \), there exists at least one weak solution \( u \) of (N-S) with \( \int_\Omega |\nabla u(x)|^2 dx < \infty \) such that

\[
\int_\Omega |u(x) - u^\infty|^2 dx < \infty.
\]

Here and in what follows, \( \dot{H}_0^{1,q}(\Omega) \) denotes the closure of \( C_0^\infty(\Omega) \) with respect to the homogeneous norm \( \|\nabla u\|_{L^q} \) for \( 1 < q < \infty \). Leray named such a weak solution \( u \) a D-solution of (N-S) because it has a finite Dirichlet integral in \( \Omega \). The asymptotic behavior of D-solution \( u \) at infinity had been improved by Finn [3], Fujita [4] and Ladyzhenskaya [13] in such a way that

\[
u(x) \rightarrow u^\infty \quad \text{uniformly as } |x| \rightarrow \infty,
\]
provided \( f \) has a compact support in \( \Omega \). In his paper [14], Leray proposed the problem whether every \( D \)-solution \( u \) satisfies the energy identity

\[
(\text{EI}) \quad \int_{\Omega} \nabla u \cdot \nabla (u - a) \, dx + \int_{\Omega} u \cdot \nabla a \cdot (u - a) \, dx = \langle f, u - a \rangle
\]

for all \( a \in C^{1}(\overline{\Omega}) \) such that \( \text{div} \, a = 0 \) in \( \Omega \), \( a|_{\partial \Omega} = 0 \), \( a(x) \equiv u^{\infty} \) for all \( x \in \Omega \) satisfying \( |x| \geq R \) with some large \( R > 0 \). Here \( \langle \cdot, \cdot \rangle \) denotes the duality pairing of \( H^{-1,2}(\Omega) \) and \( H_{0}^{1,2}(\Omega) \). The second important question is a uniqueness problem of \( D \)-solutions. It is still an open question whether there exists a small constant \( \delta \) such that if \( ||f||_{H^{1,2}} + |u^{\infty}| \leq \delta \), then the \( D \)-solution \( u \) of (N-S) is unique. This is so-called a uniqueness theorem of \( D \)-solutions for arbitrary small given data \( f \in H^{-1,2}(\Omega) \) and \( u^{\infty} \in \mathbb{R}^{3} \).

In this article, we shall give final affirmative answers to these two questions provided \( u^{\infty} \neq 0 \). It should be noted that the corresponding results to those in the case \( u^{\infty} = 0 \) are still open questions. See e.g., Nakatsuka [15]. There is another notion of physically reasonable(PR) solutions introduced by Finn [2], [3]. We call the solution \( u \) of (N-S) physically reasonable if it holds

\[
(\text{PR}) \quad u(x) - u^{\infty} = O(|x|^{-\alpha}) \quad \text{as} \quad |x| \to \infty
\]

for some \( \alpha > 1/2 \). If \( u \) is a PR-solution of (N-S) with \( f \in C_{0}^{\infty}(\Omega) \), then \( u \) behaves like

\[
(\text{WR}) \quad u(x) - u^{\infty} = O(|x|^{-1}(1 + s_{x})^{-1}), \quad s_{x} \equiv |x| - \frac{x \cdot u^{\infty}}{|u^{\infty}|} \quad \text{as} \quad |x| \to \infty,
\]

which exhibits a parabolic wake region behind the obstacle. It had been shown by Finn [3] that in the case when \( f \in C_{0}^{\infty}(\Omega) \), every PR-solution \( u \) becomes necessarily a \( D \)-solution. The converse assertion was treated by Babenko [1] who proved that if \( f \equiv 0 \), then every \( D \)-solution \( u \) of (N-S) satisfies (PR) with \( \alpha = 1 \). As a result, it turns out that every \( D \)-solution with \( f \equiv 0 \) has a parabolic wake region such as (WR). Later on, Galdi [6], [7], [8], [9] and Farwig [5] succeeded to handle more general \( f \) by introducing anisotropic weight functions, and obtained more precise asymptotic behavior of \( u \) than (WR) in the class of PR-solutions. Furthermore, Kobayashi-Shibata [11] showed the stability of PR-solutions for small \( f \) and \( u^{\infty} \) in terms of the Oseen semi-group in \( L^{p} \)-spaces.

1 Results.

Before stating our results, let us introduce some notation and then give our definition of \( D \)-solutions of (N-S). \( C_{0}^{\infty}(\Omega) \) is the set of all \( C^{\infty} \)-vector functions \( \varphi = (\varphi_{1}, \varphi_{2}, \varphi_{3}) \) with compact support in \( \Omega \), such that \( \text{div} \, \varphi = 0 \). For \( 1 < q < \infty \), \( L^{q}(\Omega) \) stands for all \( L^{q} \)-summable vector functions on \( \Omega \) with the norm \( \| \cdot \|_{L^{q}} \). We denote by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( L^{q}(\Omega) \) and \( L^{q'}(\Omega) \), where \( 1/q + 1/q' = 1 \). \( H_{0}^{1,q}(\Omega) \) denotes the closure of \( C_{0}^{\infty}(\Omega) \) with respect to the homogeneous norm \( \| \nabla \varphi \|_{L^{q}} \), where \( \nabla \varphi = (\frac{\partial \varphi_{i}}{\partial x_{j}}) \), \( i, j = 1, 2, 3 \). \( \dot{H}^{-1,q}(\Omega) \) is the dual space of \( H_{0}^{1,q}(\Omega) \), and \( \langle f, \phi \rangle \) denotes the duality pairing between \( f \in \dot{H}^{-1,q}(\Omega) \) and \( \phi \in H_{0}^{1,q}(\Omega) \). Finally, for \( u^{\infty} \in \mathbb{R}^{3} \), we define the space \( A(u^{\infty}) \) by

\[
A(u^{\infty}) \equiv \{ a \in C^{1}(\Omega); \text{div} \, a = 0, a|_{\partial \Omega} = 0, a(x) \equiv u^{\infty} \quad \text{for all} \quad x \in \mathbb{R}^{3} \quad \text{satisfying} \quad |x| > R \}
\]
Our definition of $D$-solutions to (N-S) reads as follows.

**Definition.** Let $f \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \in \mathbb{R}^3$. A measurable function $u$ on $\Omega$ is called a $D$-solution of (N-S) if the following conditions (i), (ii) and (iii) are satisfied.

(i) $\nabla u \in L^2(\Omega)$ with $\text{div} u = 0$ in $\Omega$ and $u = 0$ on $\partial \Omega$;
(ii) $u(\cdot) - u^\infty \in L^6(\Omega)$;
(iii) it holds that

\[ (\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) = \langle f, \varphi \rangle \quad \text{for all } \varphi \in C_{0,\sigma}^\infty(\Omega). \]

**Remark.** For every $D$-solution $u$ of (N-S), there exists a unique scalar function $p \in L_{\text{loc}}^2(\Omega)$ up to an additive constant such that

\[ (\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) + (p, \text{div} \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in C^\infty(\Omega). \]

Our first result on the energy identity (EI) now reads:

**Theorem 1.1** Assume that $f \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \in \mathbb{R}^3$ with $u^\infty \neq 0$. Then every $D$-solution $u$ of (N-S) satisfies

\[ (\nabla u, \nabla u) - (\nabla u, \nabla a) + (u \cdot \nabla a, u - a) = \langle f, u - a \rangle \quad \text{for all } a \in A(u^\infty). \]

Moreover, if in addition $f \in H^{-1,2}(\Omega) \cap L^q(\Omega)$ for some $1 < q < 2$, then it holds that

\[ \int_\Omega |\nabla u|^2 dx + u^\infty \cdot \int_{\partial \Omega} T(u, p) \cdot \nu dS = \langle f, u - u^\infty \rangle, \]

where $T(u, p) \equiv \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \delta_{ij} p \right)_{1 \leq i,j \leq 3}$ denotes the stress tensor and where $\nu$ is the unit outer normal to $\partial \Omega$.

**Remarks.** (i) Galdi [8] and Farwig [5] showed a similar result to that of Theorem 1.1 under the assumption that $f \in H^{-1,2}(\Omega) \cap L^3(\Omega) \cap L^3(\Omega)$. On the other hand, for the validity of the energy identity (1.1), we do not need any condition on $f$ except for $f \in \dot{H}^{-1,2}(\Omega)$.

(ii) The corresponding problem for $u^\infty = 0$ is still open. Indeed, up to the present, the energy identity (1.1) is shown under the hypothesis that $u \in \dot{H}^{1,2}(\Omega) \cap L^{3,\infty}(\Omega)$, where $L^{q,r}(\Omega)$ denotes the Lorentz space on $\Omega$. For instance, see Kozono-Yamazaki [12].

Next, we consider the uniqueness of $D$-solutions under the smallness assumption on the given data.

**Theorem 1.2** There is a constant $\delta_1 = \delta_1(\Omega) > 0$ such that if $u^\infty \neq 0$ and $f \in \dot{H}^{-1,2}(\Omega)$ satisfy

\[ \|f\|_{\dot{H}^{-1,2}} + |u^\infty| \leq \delta_1 |u^\infty|^\frac{1}{2}, \]

with some $R > 0$. 

**Proof.**
then there exists a unique $D$-solution $u$ of (N-S). Moreover, such a solution $u$ is necessarily subject to the estimate

$$|u^\infty|^\frac{1}{4} \|u - u^\infty\|_{L^4} + \|\nabla u\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|),$$

where $C = C(\Omega)$.

**Remarks.** (i) Galdi [8] showed that if $u^\infty \neq 0$ and $f \in L^\frac{8}{5}(\Omega) \cap L^\frac{3}{2}(\Omega)$ satisfy

$$\|f\|_{L^\frac{8}{5}} + |u^\infty| \leq \delta_1$$

then there exists a unique $D$-solution. Since $L^\frac{8}{5}(\Omega) \subset \dot{H}^{-1,2}(\Omega)$, our result covers that of Galdi [8]. Furthermore, we do not need any redundant assumption such as $f \in L^\frac{3}{2}(\Omega)$. Hence, Theorem 1.2 seems to be a final answer to Leray’s question on uniqueness of $D$-solutions for small data.

(ii) The case when $u^\infty = 0$, such a uniqueness result as in Theorem 1.2 is known in more restrictive situations. For instance, Nakatsuka [15] treated the case $u^\infty = 0$, and proved that for every $3 < r < \infty$ there is a constant $\delta = \delta(r) > 0$ such that if $\{u, p\}$ and $\{v, q\}$ with $\nabla u, \nabla v, p, q \in L^\frac{3}{2,\infty}(\Omega)$ satisfy $(E')$ and if

$$\|u\|_{L^3,\infty} \leq \delta, \ v \in L^3(\Omega) + L^r(\Omega),$$

then it holds that

$$\{u, p\} = \{v, q\}.$$

In his result, it is necessary to assume the smallness of one solution $u$ and some redundant regularity on another solution $v$. It is still an open question whether any norm of solutions $u$ of (N-S) with $u^\infty = 0$ can be controlled by $f$. For details, we refer to Kim-Kozono [10].

## 2 Oseen equations.

In this section, we investigate the following Oseen equations.

\[(Os)\]

\[
\begin{aligned}
-\Delta v + u^\infty \cdot \nabla v + \nabla \pi &= f \quad \text{in } \Omega, \\
\text{div } v &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega, \\
v(x) \to 0 &\quad \text{as } |x| \to \infty.
\end{aligned}
\]

Let us introduce the two function spaces $\tilde{H}^{1,q}(\Omega)$ and $\tilde{H}^{2,q}(\Omega)$ defined by

$$\tilde{H}^{1,q}(\Omega) \equiv \{v \in L^{\frac{4q}{4-q}}(\Omega); \nabla v \in L^q(\Omega)\}, \quad 1 < q < 4,$$

$$\tilde{H}^{2,q}(\Omega) \equiv \{v \in \tilde{H}^{1,\frac{4q}{4-q}}(\Omega); \nabla^2 v \in L^q(\Omega)\}, \quad 1 < q < 2.$$

Then we have the following results on unique solvability of (Os).

**Lemma 2.1** Let $u^\infty \neq 0$. Assume that $1 < q_1, q_2 < 4$. The solution $\{v, \pi\} \in \tilde{H}^{1,q_1}(\Omega) + \tilde{H}^{1,q_2}(\Omega) \times L^1_{loc}(\Omega)$ of (Os) is unique.
Lemma 2.2 (i) For \( f \in \dot{H}^{-1,q}(\Omega) \) with \( \frac{3}{2} < q < 4 \), there exists a unique solution \( \{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega) \) of \((O\tilde{s})\). Moreover, for every \( \frac{3}{2} < q < 3 \) and every \( M > 0 \) there is a constant \( C = C(q, M, \Omega) \) such that if \( \{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega) \) is a solution of \((O\tilde{s})\) with \( |u^\infty| \leq M \), then it holds that

\[
k_1 \|v\|_{L^{\frac{4q}{4-q}}} + \|\nabla v\|_{L^q} + \|\pi\|_{L^q} \leq C \|f\|_{\dot{H}^{-1,q}},\]

where \( k_1 \equiv \min\{1, |u^\infty|^\frac{1}{4}\} \).

(ii) For every \( f \in L^q(\Omega) \) with \( 1 < q < 2 \), there exists a unique solution \( \{v, \pi\} \in \tilde{H}^{2,q}(\Omega) \times L^{q^*}(\Omega) \) of \((O\tilde{s})\) with \( \nabla \pi \in L^q(\Omega) \), where \( \frac{1}{q^*} = \frac{1}{q} - \frac{1}{3} \). Moreover, for every \( 1 < q < \frac{3}{2} \) and every \( M > 0 \) there is a constant \( C = C(q, M, \Omega) \) such that if \( \{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega) \) is a solution of \((O\tilde{s})\) with \( |u^\infty| \leq M \), then it holds that

\[
k_2 \|v\|_{L^\frac{4q}{4-q}} + k_1 \|\nabla v\|_{L^\frac{4q}{4-q}} + \|\nabla^2 v\|_{L^q} + \|\pi\|_{L^{q^*}} + \|\pi\|_{L^q} \leq C \|f\|_{L^q},\]

where \( k_2 = k_1^2 \equiv \min\{1, |u^\infty|^\frac{1}{2}\} \).

3 Proof of Theorems.

The following lemma is based on Lemma 2.2 and plays a key role for the proof of Theorem 1.1.

Lemma 3.1 Let \( u^\infty \neq 0 \) and \( f \in \dot{H}^{-1,2}(\Omega) \). Let \( u \) be a \( D \)-solution of \((N-S)\).

(i) If in addition \( f \in \dot{H}^{-1,2}(\Omega) \cap \dot{H}^{-1,q}(\Omega) \) for \( \frac{4}{3} < q < 4 \), then it holds that

\[
\begin{align*}
\nabla u &\in L^q(\Omega), \quad p - p^\infty \in L^q(\Omega) \quad \text{for some constant } p^\infty.
\end{align*}
\]

(ii) If in addition \( f \in \dot{H}^{-1,2}(\Omega) \cap L^q(\Omega) \) for \( 1 < q < 2 \), then it holds that

\[
\begin{align*}
\nabla^2 u, \nabla p, u^\infty \cdot \nabla u &\in L^q(\Omega).
\end{align*}
\]

By taking \( q = 2 \) in this lemma, we have

Corollary 3.1 Every \( D \)-solution \( u \) of \((N-S)\) with \( u^\infty \neq 0 \) and \( f \in \dot{H}^{-1,2}(\Omega) \) satisfies

\[
u - u^\infty \in L^4(\Omega), \quad u^\infty \cdot \nabla u \in \dot{H}^{-1,2}(\Omega), \quad p - p^\infty \in L^2(\Omega)
\]

for some constant \( p^\infty \).

To deal with the nonlinear term, we need

Proposition 3.1 Let \( u, w \in \dot{H}^{1,2}_0(\Omega) \cap L^4(\Omega) \).

(i) If \( u \in L^4(\Omega) \) with \( \text{div} u = 0 \) in \( \Omega \), then it holds that

\[
(u \cdot \nabla v, w) = -(u \cdot \nabla w, v).
\]

(ii) If \( u^\infty \cdot \nabla v \in \dot{H}^{-1,2}(\Omega) \) and \( u^\infty \cdot \nabla w \in \dot{H}^{-1,2}(\Omega) \), then it holds that

\[
\langle u^\infty \cdot \nabla v, w \rangle = -(u^\infty \cdot \nabla w, v), \quad \langle a \cdot \nabla v, w \rangle = -(a \cdot \nabla w, v) \quad \text{for all } a \in A(u^\infty).
\]
3.1 Proof of Theorem 1.1.

By Definition of $D$-solutions, we have

\[(f, \phi) = (\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) - (p, \text{div} \phi)\]

(3.1)

\[= (\nabla u, \nabla \phi) + ((u - a) \cdot \nabla u, \phi) + (a \cdot \nabla u, \phi) - (p - p_{\infty}, \text{div} \phi)\]

for all $\phi \in C^\infty_0(\Omega)$. Since $C^\infty_0(\Omega)$ is dense in $H^{1,2}_0(\Omega) \cap L^4(\Omega)$, we have

\[(f, \phi) = (\nabla u, \nabla \phi) + ((u - a) \cdot \nabla u, \phi) + (a \cdot \nabla u, \phi) - (p - p_{\infty}, \text{div} \phi)\]

(3.2)

for all $\phi \in H^{1,2}_0(\Omega) \cap L^4(\Omega)$. By Corollary 3.1 it holds that $u - a = u - u^\infty + u^\infty - a \in H^{1,2}_0(\Omega) \cap L^4(\Omega)$. Hence, taking $\phi = u - a$ in (3.2), we have

\[(f, u - a) = (\nabla u, \nabla (u - a)) + ((u - a) \cdot \nabla u, u - a) + (a \cdot \nabla u, u - a).\]

Furthermore by Proposition 3.1, it holds that

\[((u - a) \cdot \nabla u, u - a) + (a \cdot \nabla u, u - a)\]

\[= ((u - a) \cdot \nabla (u - a), u - a) + (a \cdot \nabla (u - a), u - a)\]

\[+ ((u - a) \cdot \nabla a, u - a) + (a \cdot \nabla a, u - a)\]

\[= (u \cdot \nabla a, u - a),\]

from which and (3.3) we obtain

\[\|\nabla u\|^2_{L^2} - (\nabla u, \nabla a) + (u \cdot \nabla a, u - a) = (f, u - a).\]

This proves (1.1).

Assume in addition that $f \in \dot{H}^{-1,2}(\Omega) \cap L^q(\Omega)$ for some $1 < q < 2$. By Lemma 3.1 (ii), we have

\[-\Delta u + u \cdot \nabla u + \nabla p = f \quad \text{a.e. in} \ \Omega.\]

Note that

\[a - u^\infty \in C^\infty_{0,\sigma}(\mathbb{R}^3), \quad a - u^\infty = 0 \quad \text{on} \ \partial \Omega.\]

By integration by parts, we have

\[(f, a - u^\infty) = (-\Delta u + u \cdot \nabla u + \nabla p, a - u^\infty)\]

\[= (-\text{div} \ (T(u,p), a - u^\infty) + (u \cdot \nabla a, a - u^\infty))\]

(3.4)

\[= (\nabla u, \nabla a) + u^\infty \cdot \int_{\partial\Omega} T(u,p) \cdot \nu dS - (u \cdot \nabla a, u).\]

Addition of (3.4) and (1.1) yields that

\[\|\nabla u\|^2_{L^2} + u^\infty \cdot \int_{\partial\Omega} T(u,p) \cdot \nu dS - (u \cdot \nabla a, a) = (f, u - u^\infty).\]

(3.5)

Since supp $\nabla a$ is compact, we see easily

\[u \cdot \nabla a, a = 0,\]

from which and (3.5) we obtain the desired identity (1.2). This proves Theorem 1.1.
3.2 Proof of Theorem 1.2.

Step 1. We first show that there are constants $\delta_* = \delta_*(\Omega)$ and $C_*(\Omega) > 0$ such that if

$$\|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^\frac{1}{2},$$

then every $D$-solution $u$ of (N-S) satisfies

$$|u^\infty|^\frac{1}{4} \|u-a\|_{L^4} + \|\nabla u\|_{L^2} \leq C_*(\|f\|_{H^{-1,2}} + |u^\infty|)$$

for some $a \in A(u^\infty)$. Indeed, taking $0 < R_0 < R_1 < \infty$ and $a \in A(u^\infty)$ in such a way that

$$\Omega^c = \mathbb{R}^3 \setminus \Omega \subset B_{R_0}(0), \text{ supp } \nabla a \subset \{R_0 < |x| < R_1\},$$

we have

$$\|a\|_{L^\infty} + \|\nabla a\|_{L^1 \cap L^\infty} \leq C |u^\infty| \text{ with } C = C(\Omega).$$

By (1.1), we see that

$$\|\nabla u\|_{L^2}^2 = \langle f, u-a \rangle + (\nabla u, \nabla a) + (u \cdot \nabla a, u-a),$$

from which and (3.8) with the aid of the Young inequality it follows that

$$\|\nabla u\|_{L^2}^2 \leq \left(\frac{1}{2} + C |u^\infty|\right) \|\nabla u\|_{L^2}^2 + C \|f\|_{H^{-1,2}}^2 + C(|u^\infty|^2 + |u^\infty|^4).$$

Hence, under the assumption

$$|u^\infty| \leq \delta_*^{(1)} \equiv \min\{1, \frac{1}{4C}\},$$

we have

$$\frac{1}{4} \|\nabla u\|_{L^2}^2 \leq C \|f\|_{H^{-1,2}}^2 + C(|u^\infty|^2 + |u^\infty|^4) \leq C(\|f\|_{H^{-1,2}}^2 + |u^\infty|^2),$$

which yields that

$$\|\nabla u\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|).$$

Next, we show the bound of $\|u-a\|_{L^4}$. Define $v = u-a$, and we have by (3.8) and (3.9) that

$$v \in H_0^{1,2}(\Omega), \quad \|\nabla v\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|),$$

and that

$$\begin{cases}
-\Delta v + u^\infty \cdot \nabla v + \nabla \pi = f - Q(v) \text{ in } \Omega,
\text{div } v = 0 \text{ in } \Omega,
v = 0 \text{ on } \partial \Omega,
v(x) \to 0 \text{ as } |x| \to \infty,
\end{cases}$$

where

$$Q(v) \equiv v \cdot \nabla v + (a - u^\infty) \cdot \nabla v + v \cdot \nabla a - \Delta a + a \cdot \nabla a.$$
By (3.8) and (3.11), it holds that
\[
\|v \cdot \nabla v\|_{L^4} \leq \|v\|_{L^4}\|\nabla v\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|)\|v\|_{L^4}
\]
\[
\|Q(v) - v \cdot \nabla v\|_{H^{-1,2}} = \|(a - u^\infty) \cdot \nabla v + v \cdot \nabla a - \Delta a + a \cdot \nabla a\|_{H^{-1,2}} \leq C(\|\nabla v\|_{L^2} + |u^\infty|)
\]
\[
\leq C(\|f\|_{H^{-1,2}} + |u^\infty|).
\]
Hence, it follows from Lemma 2.1 and Lemma 2.2 with $q = 2$ in (i) and with $q = \frac{4}{3}$ in (ii) that
\[
\|v\|_{L^4} \leq C \left( \frac{1}{k_1} \|f\|_{H^{-1,2}} + \frac{1}{k_2} \|v \cdot \nabla v\|_{L^4} \right)
\]
(3.12)
\[
\leq C \left( \frac{1}{k_1} (\|f\|_{H^{-1,2}} + |u^\infty|) + \frac{1}{k_2} (\|f\|_{H^{-1,2}} + |u^\infty|)\|v\|_{L^4} \right).
\]
Hence, under the assumption
\[
\frac{1}{k_2} (\|f\|_{H^{-1,2}} + |u^\infty|) \leq \delta_* \equiv \min \{\delta_*^{(1)}, \frac{1}{2C}\},
\]
we have
\[
\|u - a\|_{L^4} = \|v\|_{L^4} \leq \frac{C}{k_1} (\|f\|_{H^{-1,2}} + |u^\infty|).
\]
Since the assumption (3.13) necessarily implies the assumption (3.9), we see by (3.10) and (3.14) that if
\[
\|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^\frac{1}{2},
\]
then it holds that
\[
|u^\infty|^\frac{1}{4} \|u - a\|_{L^4} + \|\nabla u\|_{L^4} \leq (\|f\|_{H^{-1,2}} + |u^\infty|),
\]
which implies (3.7)

**Step 2.** We next show uniqueness. Let $u_1$ and $u_2$ be two $D$-solutions of (N-S). Define $v_1 = u_1 - a$ and $v_2 = u_2 - a$ with $a \in A(u^\infty)$ as in Step 1. Then $v \equiv v_1 - v_2 = u_1 - u_2$ fulfills
\[
\begin{cases}
-\Delta v + u^\infty \cdot \nabla v + \nabla \pi = -v_1 \cdot \nabla v - v \cdot \nabla u_2 \quad \text{in } \Omega, \\
\text{div } v = 0 \quad \text{in } \Omega, \\
v = 0 \quad \text{on } \partial \Omega, \\
v(x) \to 0 \quad \text{as } |x| \to \infty,
\end{cases}
\]
Hence it follows from Lemmata 2.1 and 2.1 with
\[
f = -v_1 \cdot \nabla v = \text{div} (v_1 \otimes v) \quad \text{for } q = 2 \text{ in (i)},
\]
\[
f = -v \cdot \nabla u_2 \quad \text{for } q = \frac{4}{3} \text{ in (ii)}
\]
that
\[ \|v\|_{L^4} \leq C \left( \frac{1}{k_1} \| \text{div} \, (v_1 \otimes v) \|_{H^{-1,2}} + \frac{1}{k_2} \| v \cdot \nabla u_2 \|_{L^2} \right) \]
\[ \leq C \left( \frac{1}{k_1} \| v_1 \otimes v \|_{L^2} + \frac{1}{k_2} \| v \|_{L^4} \| \nabla u_2 \|_{L^2} \right) \]
\[ \leq C \left( \frac{1}{k_1} \| v_1 \|_{L^4} + \frac{1}{k_2} \| v \|_{L^4} \| \nabla u_2 \|_{L^2} \right) \| v \|_{L^4}. \]
(3.15)

By Step1, under the assumption
\[ \|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^\frac{1}{2}, \]
we have
\[ \|v_1\|_{L^4} \leq \frac{C}{k_1} (\|f\|_{H^{-1,2}} + |u^\infty|), \quad \|\nabla u_2\|_{L^2} \leq C (\|f\|_{H^{-1,2}} + |u^\infty|), \]
from which and (3.15) with \( k_1^2 = k_2 \) it follows that
\[ \|v\|_{L^4} \leq \frac{C}{k_2} (\|f\|_{H^{-1,2}} + |u^\infty|) \|v\|_{L^4}. \]
(3.16)

Now, define \( \delta_1 = \delta_1(\Omega) \) so that
\[ \delta_1 = \min \{ \delta_*, \frac{1}{2C} \}. \]
Then under the assumption
\[ \|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_1 |u^\infty|^\frac{1}{2}, \]
it follows from (3.16) with the aid of the relation \( k_2 = \min \{ 1, |u^\infty|^\frac{1}{2} \} \) that
\[ \|v\|_{L^4} \leq 0, \]
which yields the desired uniqueness result. This completes the proof of Theorem 1.2.

References


