<table>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2014, 1875: 19-28</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195560">http://hdl.handle.net/2433/195560</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Leray's problem on $D$-solutions to the stationary Navier-Stokes equations past an obstacle

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Introduction.

Let $\Omega$ be an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial \Omega \in C^\infty$. We consider the stationary Navier-Stokes equations in $\Omega$:

\begin{equation}
\begin{cases}
-\Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in } \Omega, \\
div u = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega, \\
u(x) \to u^\infty \quad \text{as } |x| \to \infty,
\end{cases}
\end{equation}

where $u = u(x) = (u_1(x), u_2(x), u_3(x))$ and $p = p(x)$ denote the unknown velocity vector and the unknown pressure at $x = (x_1, x_2, x_3) \in \Omega$, while $f = f(x) = (f_1(x), f_2(x), f_3(x))$ is the given external force, and $u^\infty = (u_1^\infty, u_2^\infty, u_3^\infty)$ is the prescribed constant vector in $\mathbb{R}^3$ at infinity. In the pioneer work of Leray [14], it was shown that for every $f \in \dot{H}^{-1,2}(\Omega) \equiv \dot{H}_0^{1,2}(\Omega)^*$ and for every $u^\infty \in \mathbb{R}^3$, there exists at least one weak solution $u$ of (N-S) with $\int_{\Omega} |\nabla u(x)|^2 \, dx < \infty$ such that

$$
\int_{\Omega} |u(x) - u^\infty|^6 \, dx < \infty.
$$

Here and in what follows, $\dot{H}_0^{1,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the homogeneous norm $\|\nabla u\|_{L^q}$ for $1 < q < \infty$. Leray named such a weak solution $u$ a $D$-solution of (N-S) because it has a finite Dirichlet integral in $\Omega$. The asymptotic behavior of $D$-solution $u$ at infinity had been improved by Finn [3], Fujita [4] and Ladyzhenskaya [13] in such a way that

$$
u(x) \to u^\infty \quad \text{uniformly as } |x| \to \infty.$$


provided $f$ has a compact support in $\Omega$. In his paper [14], Leray proposed the problem whether every $D$-solution $u$ satisfies the energy identity

$$\text{(EI)} \quad \int_{\Omega} \nabla u \cdot \nabla (u - a) dx + \int_{\Omega} u \cdot \nabla a \cdot (u - a) dx = \langle f, u - a \rangle$$

for all $a \in C^1(\Omega)$ such that $\text{div} \ a = 0$ in $\Omega$, $a|_{\partial \Omega} = 0$, $a(x) \equiv u^\infty$ for all $x \in \Omega$ satisfying $|x| \geq R$ with some large $R > 0$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-1,2}(\Omega)$ and $\dot{H}_{0}^{1,2}(\Omega)$. The second important question is a uniqueness problem of $D$-solutions. It is still an open question whether there exists a small constant $\delta$ such that if $\|f\|_{H^{1,2}} + |u^\infty| \leq \delta$, then the $D$-solution $u$ of (N-S) is unique. This is so-called a uniqueness theorem of $D$-solutions for arbitrary small given data $f \in H^{-1,2}(\Omega)$ and $u^\infty \in \mathbb{R}^3$.

In this article, we shall give final affirmative answers to these two questions provided $u^\infty \neq 0$. It should be noted that the corresponding results to those in the case $u^\infty = 0$ are still open questions. See e.g., Nakatsuka [15]. There is another notion of physically reasonable(PR) solutions introduced by Finn [2], [3]. We call the solution $u$ of (N-S) physically reasonable if it holds

$$\text{(PR)} \quad u(x) - u^\infty = O(|x|^{-\alpha}) \quad \text{as} \quad |x| \to \infty$$

for some $\alpha > 1/2$. If $u$ is a PR-solution of (N-S) with $f \in C_0^\infty(\Omega)$, then $u$ behaves like

$$\text{(WR)} \quad u(x) - u^\infty = O(|x|^{-1}(1 + s_x)^{-1}), \quad s_x \equiv |x| - \frac{x \cdot u^\infty}{|u^\infty|} \quad \text{as} \quad |x| \to \infty,$$

which exhibits a parabolic wake region behind the obstacle. It had been shown by Finn [3] that in the case when $f \in C_0^\infty(\Omega)$, every PR-solution $u$ becomes necessarily a $D$-solution. The converse assertion was treated by Babenko [1] who proved that if $f \equiv 0$, then every $D$-solution $u$ of (N-S) satisfies (PR) with $\alpha = 1$. As a result, it turns out that every $D$-solution with $f \equiv 0$ has a parabolic wake region such as (WR). Later on, Galdi [6], [7], [8], [9] and Farwig [5] succeeded to handle more general $f$ by introducing anisotropic weight functions, and obtained more precise asymptotic behavior of $u$ than (WR) in the class of PR-solutions. Furthermore, Kobayashi-Shibata [11] showed the stability of PR-solutions for small $f$ and $u^\infty$ in terms of the Oseen semi-group in $L^p$-spaces.

1 Results.

Before stating our results, let us introduce some notation and then give our definition of $D$-solutions of (N-S). $C_{0,0}^\infty(\Omega)$ is the set of all $C^\infty$-vector functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with compact support in $\Omega$, such that $\text{div} \ \varphi = 0$. For $1 < q < \infty$, $L^q(\Omega)$ stands for all $L^q$-summable vector functions on $\Omega$ with the norm $\| \cdot \|_{L^q}$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $L^q(\Omega)$ and $L^{q'}(\Omega)$, where $1/q + 1/q' = 1$. $\dot{H}_0^{1,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the homogeneous norm $\| \nabla \varphi \|_{L^q}$, where $\nabla \varphi = \left( \frac{\partial \varphi_i}{\partial x_j} \right), \quad i, j = 1, 2, 3$. $\dot{H}^{-1,q}(\Omega)$ is the dual space of $\dot{H}_0^{1,q}(\Omega)$, and $\langle f, \phi \rangle$ denotes the duality pairing between $f \in \dot{H}^{-1,q}(\Omega)$ and $\phi \in \dot{H}_0^{1,q}(\Omega)$. Finally, for $u^\infty \in \mathbb{R}^3$, we define the space $A(u^\infty)$ by

$$A(u^\infty) \equiv \{ a \in C^1(\Omega); \text{div} \ a = 0, a|_{\partial \Omega} = 0, a(x) \equiv u^\infty \quad \text{for all} \ x \in \mathbb{R}^3 \text{ satisfying } |x| > R \}$$
with some $R > 0$.

Our definition of $D$-solutions to (N-S) reads as follows.

**Definition.** Let $f \in \dot{H}^{-1,2}({\Omega})$ and $u^\infty \in {\mathbb R}^3$. A measurable function $u$ on $\Omega$ is called a $D$-solution of (N-S) if the following conditions (i), (ii) and (iii) are satisfied.

(i) $\nabla u \in L^2(\Omega)$ with $\text{div} u = 0$ in $\Omega$ and $u = 0$ on $\partial \Omega$;
(ii) $u(\cdot) - u^\infty \in L^6(\Omega)$;
(iii) it holds that

\[
\begin{align*}
(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) &= \langle f, \varphi \rangle 
\quad \text{for all } \varphi \in C^\infty_0(\Omega).
\end{align*}
\]

**Remark.** For every $D$-solution $u$ of (N-S), there exists a unique scalar function $p \in L^2_{\text{loc}}(\Omega)$ up to an additive constant such that

\[
\begin{align*}
(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) + (p, \text{div } \phi) &= \langle f, \phi \rangle 
\quad \text{for all } \phi \in C^\infty(\Omega).
\end{align*}
\]

Our first result on the energy identity (EI) now reads:

**Theorem 1.1** Assume that $f \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \in {\mathbb R}^3$ with $u^\infty \neq 0$. Then every $D$-solution $u$ of (N-S) satisfies

\[
\begin{align*}
(\nabla u, \nabla u) - (\nabla u, \nabla a) + (u \cdot \nabla a, u - a) &= \langle f, u - a \rangle 
\quad \text{for all } a \in A(u^\infty).
\end{align*}
\]

Moreover, if in addition $f \in H^{-1,2}(\Omega) \cap L^q(\Omega)$ for some $1 < q < 2$, then it holds that

\[
\begin{align*}
\int_{\Omega} |\nabla u|^2 dx + u^\infty \cdot \int_{\partial \Omega} T(u, p) \cdot \nu dS &= \langle f, u - u^\infty \rangle,
\end{align*}
\]

where $T(u, p) \equiv \left( \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} - \delta_{ij} p \right)_{1 \leq i, j \leq 3}$ denotes the stress tensor and where $\nu$ is the unit outer normal to $\partial \Omega$.

**Remarks.** (i) Galdi [8] and Farwig [5] showed a similar result to that of Theorem 1.1 under the assumption that $f \in H^{-1,2}(\Omega) \cap L^3(\Omega) \cap L^3(\Omega)$. On the other hand, for the validity of the energy identity (1.1), we do not need any condition on $f$ except for $f \in H^{-1,2}(\Omega)$.

(ii) The corresponding problem for $u^\infty = 0$ is still open. Indeed, up to the present, the energy identity (1.1) is shown under the hypothesis that $u \in \dot{H}^{1,2}(\Omega) \cap L^{3,\infty}(\Omega)$, where $L^{p,r}(\Omega)$ denotes the Lorentz space on $\Omega$. For instance, see Kozono-Yamazaki [12].

Next, we consider the uniqueness of $D$-solutions under the smallness assumption on the given data.

**Theorem 1.2** There is a constant $\delta_1 = \delta_1(\Omega) > 0$ such that if $u^\infty \neq 0$ and $f \in \dot{H}^{-1,2}(\Omega)$ satisfy

\[
\begin{align*}
\|f\|_{\dot{H}^{-1,2}} + |u^\infty| &\leq \delta_1 |u^\infty|^\frac{1}{2},
\end{align*}
\]

...
then there exists a unique $D$-solution $u$ of (N-S). Moreover, such a solution $u$ is necessarily subject to the estimate

\[(1.4) \quad |u^\infty|^\frac{1}{4} \|u - u^\infty\|_{L^4} + \|\nabla u\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|),\]

where $C = C(\Omega)$.

**Remarks.** (i) Galdi [8] showed that if $u^\infty \neq 0$ and $f \in L^\frac{8}{5}(\Omega) \cap L^\frac{3}{2}(\Omega)$ satisfy

\[\|f\|_{L^\frac{8}{5}} + |u^\infty| \leq \delta_1\]

then there exists a unique $D$-solution. Since $L^\frac{8}{5}(\Omega) \subset \dot{H}^{-1,2}(\Omega)$, our result covers that of Galdi [8]. Furthermore, we do not need any redundant assumption such as $f \in L^\frac{3}{2}(\Omega)$.

Hence, Theorem 1.2 seems to be a final answer to Leray’s question on uniqueness of $D$-solutions for small data.

(ii) The case when $u^\infty = 0$, such a uniqueness result as in Theorem 1.2 is known in more restrictive situations. For instance, Nakataoka [15] treated the case $u^\infty = 0$, and proved that for every $3 < r < \infty$ there is a constant $\delta = \delta(r) > 0$ such that if $\{u, p\}$ and $\{v, q\}$ with $\nabla u, \nabla v, p, q \in L^\frac{3}{2,\infty}(\Omega)$ satisfy (E') and if

\[\|u\|_{L^3,\infty} \leq \delta, \quad v \in L^3(\Omega) + L^r(\Omega),\]

then it holds that

\[\{u, p\} = \{v, q\}.\]

In his result, it is necessary to assume the smallness of one solution $u$ and some redundant regularity on another solution $v$. It is still an open question whether any norm of solutions $u$ of (N-S) with $u^\infty = 0$ can be controlled by $f$. For details, we refer to Kim-Kozono [10].

## 2 Oseen equations.

In this section, we investigate the following Oseen equations.

\[(Os) \quad \begin{cases} -\Delta v + u^\infty \cdot \nabla v + \nabla \pi = f \quad \text{in } \Omega, \\
 \text{div } v = 0 \quad \text{in } \Omega, \\
 v = 0 \quad \text{on } \partial \Omega, \\
 v(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}\]

Let us introduce the two function spaces $\tilde{H}^{1,q}(\Omega)$ and $\tilde{H}^{2,q}(\Omega)$ defined by

\[\tilde{H}^{1,q}(\Omega) \equiv \{v \in L^\frac{4q}{4q-4}(\Omega); \nabla v \in L^q(\Omega)\}, \quad 1 < q < 4,\]

\[\tilde{H}^{2,q}(\Omega) \equiv \{v \in \tilde{H}^{1,\frac{4q}{q-4}}(\Omega); \nabla^2 v \in L^q(\Omega)\}, \quad 1 < q < 2.\]

Then we have the following results on unique solvability of (Os).

**Lemma 2.1** Let $u^\infty \neq 0$. Assume that $1 < q_1, q_2 < 4$. The solution $\{v, \pi\} \in \tilde{H}^{1,q_1}(\Omega) + \tilde{H}^{1,q_2}(\Omega) \times L^1_{loc}(\Omega)$ of (Os) is unique.
Lemma 2.2 (i) For $f \in \dot{H}^{-1,q}(\Omega)$ with $\frac{3}{2} < q < 4$, there exists a unique solution $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^{q}(\Omega)$ of $(Os)$. Moreover, for every $\frac{3}{2} < q < 3$ and every $M > 0$ there is a constant $C = C(q, M, \Omega)$ such that if $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^{q}(\Omega)$ is a solution of $(Os)$ with $|u^\infty| \leq M$, then it holds that

$$k_1 \|v\|_{L^{\frac{4q}{4-q}}} + \|\nabla v\|_{L^q} + \|\pi\|_{L^q} \leq C \|f\|_{\dot{H}^{-1,q}},$$

where $k_1 \equiv \min \{1, |u^\infty|^\frac{1}{4}\}$.

(ii) For every $f \in L^q(\Omega)$ with $1 < q < 2$, there exists a unique solution $\{v, \pi\} \in \tilde{H}^{2,q}(\Omega) \times L^{q^*}(\Omega)$ of $(Os)$ with $\nabla \pi \in L^q(\Omega)$, where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{3}$. Moreover, for every $1 < q < \frac{3}{2}$ and every $M > 0$ there is a constant $C = C(q, M, \Omega)$ such that if $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^{q}(\Omega)$ is a solution of $(Os)$ with $|u^\infty| \leq M$, then it holds that

$$k_2 \|v\|_{L^{\frac{4q}{4-q}}} + k_1 \|\nabla v\|_{L^q} + \|\nabla^2 v\|_{L^q} + \|\pi\|_{L^{q^*}} + \|\pi\|_{L^q} \leq C \|f\|_{L^q},$$

where $k_2 = k_1^2 \equiv \min \{1, |u^\infty|^\frac{1}{2}\}$.

3 Proof of Theorems.

The following lemma is based on Lemma 2.2 and plays a key role for the proof of Theorem 1.1.

Lemma 3.1 Let $u^\infty \neq 0$ and $f \in \dot{H}^{-1,2}(\Omega)$. Let $u$ be a $D$-solution of $(N-S)$.

(i) If in addition $f \in \dot{H}^{-1,2}(\Omega) \cap \dot{H}^{-1,q}(\Omega)$ for $\frac{4}{3} < q < 4$, then it holds that

$$u - u^\infty \in L^{\frac{4q}{4-q}}(\Omega), \quad u^\infty \cdot \nabla u \in \dot{H}^{-1,q}(\Omega), \quad \nabla u \in L^q(\Omega), \quad p - p^\infty \in L^q(\Omega) \quad \text{for some constant } p^\infty.$$

(ii) If in addition $f \in \dot{H}^{-1,2}(\Omega) \cap L^q(\Omega)$ for $1 < q < 2$, then it holds that

$$u - u^\infty \in L^{\frac{4q}{4-q}}(\Omega), \quad \nabla u \in L^{4^*}(\Omega) \cap L^{\frac{3q}{3q-2}}(\Omega), \quad p - p^\infty \in L^{\frac{3q}{3q-2}}(\Omega) \quad \text{for some constant } p^\infty,$$

$$\nabla^2 u, \nabla p, u^\infty \cdot \nabla u \in L^q(\Omega).$$

By taking $q = 2$ in this lemma, we have

Corollary 3.1 Every $D$-solution $u$ of $(N-S)$ with $u^\infty \neq 0$ and $f \in \dot{H}^{-1,2}(\Omega)$ satisfies

$$u - u^\infty \in L^4(\Omega), \quad u^\infty \cdot \nabla u \in \dot{H}^{-1,2}(\Omega), \quad p - p^\infty \in L^2(\Omega)$$

for some constant $p^\infty$.

To deal with the nonlinear term, we need

Proposition 3.1 Let $v, w \in \dot{H}_0^{1,2}(\Omega) \cap L^4(\Omega)$.

(i) If $u \in L^4(\Omega)$ with $\text{div} u = 0$ in $\Omega$, then it holds that

$$(u \cdot \nabla v, w) = -(u \cdot \nabla w, v).$$

(ii) If $u^\infty \cdot \nabla v \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \cdot \nabla w \in \dot{H}^{-1,2}(\Omega)$, then it holds that

$$(u^\infty \cdot \nabla v, w) = -(u^\infty \cdot \nabla w, v),$$

$$(a \cdot \nabla v, w) = -(a \cdot \nabla w, v) \quad \text{for all } a \in A(u^\infty).$$
3.1 Proof of Theorem 1.1.

By Definition of $D$-solutions, we have

\[
\langle f, \phi \rangle = (\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) - (p, \text{div} \phi)
\]

(3.1)

\[
= (\nabla u, \nabla \phi) + ((u-a) \cdot \nabla u, \phi) + (a \cdot \nabla u, \phi) - (p - p_{\infty}, \text{div} \phi)
\]

for all $\phi \in C_{0}^{\infty}(\Omega)$. Since $C_{0}^{\infty}(\Omega)$ is dense in $\dot{H}_{0}^{1,2}(\Omega) \cap L^{4}(\Omega)$, we have

\[
\langle f, \phi \rangle = (\nabla u, \nabla \phi) + ((u-a) \cdot \nabla u, \phi) + (a \cdot \nabla u, \phi) - (p - p_{\infty}, \text{div} \phi)
\]

(3.2)

for all $\phi \in \dot{H}_{0}^{1,2}(\Omega) \cap L^{4}(\Omega)$. By Corollary 3.1 it holds that $u - a = u - u^{\infty} + u^{\infty} - a \in \dot{H}_{0}^{1,2}(\Omega) \cap L^{4}(\Omega)$. Hence, taking $\phi = u - a$ in (3.2), we have

\[
\langle f, u - a \rangle = (\nabla u, \nabla (u - a)) + ((u - a) \cdot \nabla u, u - a) + (a \cdot \nabla u, u - a).
\]

(3.3)

Furthermore by Proposition 3.1, it holds that

\[
((u - a) \cdot \nabla u, u - a) + (a \cdot \nabla u, u - a)
\]

\[
= ((u - a) \cdot \nabla (u - a), u - a) + (a \cdot \nabla (u - a), u - a)
\]

\[
+ ((u - a) \cdot \nabla a, u - a) + (a \cdot \nabla a, u - a)
\]

\[
= (u \cdot \nabla a, u - a),
\]

from which and (3.3) we obtain

\[
\|\nabla u\|_{L^{2}}^{2} - (\nabla u, \nabla a) + (u \cdot \nabla a, u - a) = \langle f, u - a \rangle.
\]

This proves (1.1).

Assume in addition that $f \in \dot{H}^{-1,2}(\Omega) \cap L^{q}(\Omega)$ for some $1 < q < 2$. By Lemma 3.1 (ii), we have

\[-\Delta u + u \cdot \nabla u + \nabla p = f \quad \text{a.e. in} \quad \Omega.
\]

Note that

\[a - u^{\infty} \in C_{0,\sigma}^{\infty}(\mathbb{R}^{3}), \quad a - u^{\infty} = 0 \quad \text{on} \quad \partial \Omega.
\]

By integration by parts, we have

\[
\langle f, a - u^{\infty} \rangle = (-\Delta u + u \cdot \nabla u + \nabla p, a - u^{\infty})
\]

\[
= (-\text{div} (T(u,p), a - u^{\infty}) + (u \cdot \nabla u, a - u^{\infty})
\]

(3.4)

\[
= (\nabla u, \nabla a) + u^{\infty} \cdot \int_{\partial \Omega} T(u,p) \cdot \nu dS - (u \cdot \nabla a, u).
\]

Addition of (3.4) and (1.1) yields that

\[
\|\nabla u\|_{L^{2}}^{2} + u^{\infty} \cdot \int_{\partial \Omega} T(u,p) \cdot \nu dS - (u \cdot \nabla a, a) = \langle f, u - u^{\infty} \rangle.
\]

(3.5)

Since $\text{supp} \nabla a$ is compact, we see easily

\[
(u \cdot \nabla a, a) = 0,
\]

from which and (3.5) we obtain the desired identity (1.2). This proves Theorem 1.1.
3.2 Proof of Theorem 1.2.

Step 1. We first show that there are constants $\delta_* = \delta_*(\Omega)$ and $C_* (\Omega) > 0$ such that if

$$\|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^{\frac{1}{2}},$$

then every $D$-solution $u$ of (N-S) satisfies

$$|u^\infty|^{\frac{1}{4}} \|u-a\|_{L^4} + \|\nabla u\|_{L^2} \leq C_* (\|f\|_{H^{-1,2}} + |u^\infty|)$$

for some $a \in A(u^\infty)$. Indeed, taking $0 < R_0 < R_1 < \infty$ and $a \in A(u^\infty)$ in such a way that

$$\Omega^c = \mathbb{R}^3 \setminus \Omega \subset B_{R_0}(0), \quad \text{supp} \, \nabla a \subset \{R_0 < |x| < R_1\},$$

we have

$$\|a\|_{L^\infty} + \|\nabla a\|_{L^1 \cap L^\infty} \leq C |u^\infty|$$

with $C = C(\Omega)$.

By (1.1), we see that

$$\|\nabla u\|_{L^2}^2 = \langle f, u-a \rangle + (\nabla u, \nabla a) + (u \cdot \nabla a, u-a),$$

from which and (3.8) with the aid of the Young inequality it follows that

$$\|\nabla u\|_{L^2}^2 \leq \left(\frac{1}{2} + C |u^\infty|\right) \|\nabla u\|_{L^2}^2 + C \|f\|_{H^{-1,2}}^2 + C (|u^\infty|^2 + |u^\infty|^4).$$

Hence, under the assumption

$$|u^\infty| \leq \delta_*^{(1)} \equiv \min \{1, \frac{1}{4C} \},$$

we have

$$\frac{1}{4} \|\nabla u\|_{L^2}^2 \leq C \|f\|_{H^{-1,2}}^2 + C (|u^\infty|^2 + |u^\infty|^4) \leq C \|f\|_{H^{-1,2}}^2 + |u^\infty|^2, $$

which yields that

$$\|\nabla u\|_{L^2} \leq C (\|f\|_{H^{-1,2}} + |u^\infty|).$$

Next, we show the bound of $\|u-a\|_{L^4}$. Define $v = u-a$, and we have by (3.8) and (3.9) that

$$v \in H^1_{0,2}(\Omega), \quad \|\nabla v\|_{L^2} \leq C (\|f\|_{H^{-1,2}} + |u^\infty|),$$

and that

$$\begin{cases} -\Delta v + u^\infty \cdot \nabla v + \nabla \pi = f - Q(v) \quad \text{in} \quad \Omega, \\ \text{div} \, v = 0 \quad \text{in} \quad \Omega, \\ v = 0 \quad \text{on} \quad \partial \Omega, \\ v(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases}$$

where

$$Q(v) \equiv v \cdot \nabla v + (a-u^\infty) \cdot \nabla v + v \cdot \nabla a - \Delta a + a \cdot \nabla a.$$
By (3.8) and (3.11), it holds that
\[
\| v\cdot \nabla v \|_{L^4} \leq \| v \|_{L^4} \| \nabla v \|_{L^2} \leq C(\| f \|_{H^{-1,2}} + |u^\infty|) \| v \|_{L^4}
\]
\[
\| Q(v) - v\cdot \nabla v \|_{H^{-1,2}} = \| (a - u^\infty) \cdot \nabla v + v \cdot \nabla a - \Delta a + a \cdot \nabla a \|_{H^{-1,2}} 
\leq C(\| \nabla v \|_{L^2} + |u^\infty|)
\leq C(\| f \|_{H^{-1,2}} + |u^\infty|).
\]
Hence, it follows from Lemma 2.1 and Lemma 2.2 with \( q = 2 \) in (i) and with \( q = \frac{4}{3} \) in (ii) that
\[
\begin{align*}
\| v \|_{L^4} & \leq \frac{1}{k_1}(\| f - Q(v) - v\cdot \nabla v \|_{H^{-1,2}} + \| v \cdot \nabla v \|_{L^4}) \\
& \leq C \left( \frac{1}{k_1} (\| f \|_{H^{-1,2}} + |u^\infty|) + \frac{1}{k_2} (\| f \|_{H^{-1,2}} + |u^\infty|) \| v \|_{L^4} \right).
\end{align*}
\]
(3.12)
Hence, under the assumption
\[
\frac{1}{k_2}(\| f \|_{H^{-1,2}} + |u^\infty|) \leq \delta_* \equiv \min\{\delta_*^{(1)}, \frac{1}{2C}\},
\]
we have
\[
\| u - a \|_{L^4} = \| v \|_{L^4} \leq \frac{C}{k_1}(\| f \|_{H^{-1,2}} + |u^\infty|).
\]
(3.13)
Since the assumption (3.13) necessarily implies the assumption (3.9), we see by (3.10) and (3.14) that if
\[
\| f \|_{H^{-1,2}} + |u^\infty| \leq \delta_*|u^\infty|^\frac{1}{2},
\]
then it holds that
\[
|u^\infty|^\frac{1}{4} \| u - a \|_{L^4} + \| \nabla u \|_{L^4} \leq (\| f \|_{H^{-1,2}} + |u^\infty|),
\]
which implies (3.7)

**Step 2.** We next show uniqueness. Let \( u_1 \) and \( u_2 \) be two \( D \)-solutions of (N-S). Define \( v_1 = u_1 - a \) and \( v_2 = u_2 - a \) with \( a \in A(u^\infty) \) as in Step 1. Then \( v \equiv v_1 - v_2 = u_1 - u_2 \) fulfills
\[
\begin{align*}
-\Delta v + u^\infty \cdot \nabla v + \nabla \pi &= -v_1 \cdot \nabla v - v \cdot \nabla u_2 \quad \text{in } \Omega, \\
\text{div } v &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega, \\
v(x) &\to 0 \quad \text{as } |x| \to \infty,
\end{align*}
\]
Hence it follows from Lemmata 2.1 and 2.1 with
\[
\begin{align*}
f &= -v_1 \cdot \nabla v = \text{div } (v_1 \otimes v) \quad \text{for } q = 2 \text{ in (i)}, \\
f &= -v \cdot \nabla u_2 \quad \text{for } q = \frac{4}{3} \text{ in (ii)}
\end{align*}
\]
that
\[
\|v\|_{L^4} \leq C \left( \frac{1}{k_1} \|\text{div} (v_1 \otimes v)\|_{H^{-1,2}} + \frac{1}{k_2} \|v \cdot \nabla u_2\|_{L^\frac{4}{3}} \right)
\]
\[
\leq C \left( \frac{1}{k_1} \|v_1 \otimes v\|_{L^2} + \frac{1}{k_2} \|v\|_{L^4} \|\nabla u_2\|_{L^2} \right)
\]
\[
\leq C \left( \frac{1}{k_1} \|v_1\|_{L^4} + \frac{1}{k_2} \|\nabla u_2\|_{L^2} \right) \|v\|_{L^4}.
\]
(3.15)

By Step1, under the assumption
\[
\|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^\frac{1}{2},
\]
we have
\[
\|v_1\|_{L^4} \leq \frac{C}{k_1} (\|f\|_{H^{-1,2}} + |u^\infty|), \quad \|\nabla u_2\|_{L^2} \leq C (\|f\|_{H^{-1,2}} + |u^\infty|),
\]
from which and (3.15) with \(k_1^2 = k_2\) it follows that
\[
\|v\|_{L^4} \leq \frac{C}{k_2} (\|f\|_{H^{-1,2}} + |u^\infty|) \|v\|_{L^4}.
\]
(3.16)

Now, define \(\delta_1 = \delta_1(\Omega)\) so that
\[
\delta_1 = \min \{\delta_*, \frac{1}{2C} \}.
\]
Then under the assumption
\[
\|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_1 |u^\infty|^\frac{1}{2},
\]
it follows from (3.16) with the aid of the relation \(k_2 = \min \{1, |u^\infty|^\frac{1}{2} \}\) that
\[
\|v\|_{L^4} \leq 0,
\]
which yields the desired uniqueness result. This completes the proof of Theorem 1.2.

References


