Leray’s problem on $D$-solutions to the stationary Navier-Stokes equations past an obstacle

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Introduction.

Let $\Omega$ be an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega \in C^\infty$. We consider the stationary Navier-Stokes equations in $\Omega$:

\[
\begin{cases}
-\Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in } \Omega, \\
\text{div } u = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
u(x) \to u^\infty \quad \text{as } |x| \to \infty,
\end{cases}
\]

(N-S)

where $u = u(x) = (u_1(x), u_2(x), u_3(x))$ and $p = p(x)$ denote the unknown velocity vector and the unknown pressure at $x = (x_1, x_2, x_3) \in \Omega$, while $f = f(x) = (f_1(x), f_2(x), f_3(x))$ is the given external force, and $u^\infty = (u_1^\infty, u_2^\infty, u_3^\infty)$ is the prescribed constant vector in $\mathbb{R}^3$ at infinity. In the pioneer work of Leray [14], it was shown that for every $f \in \dot{H}^{-1,2}(\Omega) \equiv \dot{H}_0^{1,2}(\Omega)^*$ and for every $u^\infty \in \mathbb{R}^3$, there exists at least one weak solution $u$ of (N-S) with $\int_\Omega |\nabla u(x)|^2 dx < \infty$ such that

\[
\int_\Omega |u(x) - u^\infty|^q dx < \infty.
\]

Here and in what follows, $\dot{H}_0^{1,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the homogeneous norm $\|\nabla u\|_{L^q}$ for $1 < q < \infty$. Leray named such a weak solution $u$ a $D$-solution of (N-S) because it has a finite Dirichlet integral in $\Omega$. The asymptotic behavior of $D$-solution $u$ at infinity had been improved by Finn [3], Fujita [4] and Ladyzhenskaya [13] in such a way that $u(x) \to u^\infty$ uniformly as $|x| \to \infty$. 
provided $f$ has a compact support in $\Omega$. In his paper [14], Leray proposed the problem whether every $D$-solution $u$ satisfies the energy identity

\[(EI) \quad \int_{\Omega} \nabla u \cdot \nabla (u - a) \, dx + \int_{\Omega} u \cdot \nabla a \cdot (u - a) \, dx = \langle f, u - a \rangle \]

for all $a \in C^1(\bar{\Omega})$ such that div $a = 0$ in $\Omega$, $a|_{\partial \Omega} = 0$, $a(x) \equiv u^\infty$ for all $x \in \Omega$ satisfying $|x| \geq R$ with some large $R > 0$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-1,2}(\Omega)$ and $H_0^{1,2}(\Omega)$. The second important question is a uniqueness problem of $D$-solutions. It is still an open question whether there exists a small constant $\delta$ such that if $\|f\|_{\dot{H}^{1,2}} + |u^\infty| \leq \delta$, then the $D$-solution $u$ of (N-S) is unique. This is so-called a uniqueness theorem of $D$-solutions for arbitrary small given data $f \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \in \mathbb{R}^3$.

In this article, we shall give final affirmative answers to these two questions provided $u^\infty \neq 0$. It should be noted that the corresponding results to those in the case $u^\infty = 0$ are still open questions. See e.g., Nakatsuka [15]. There is another notion of physically reasonable(PR) solutions introduced by Finn [2], [3]. We call the solution $u$ of (N-S) physically reasonable if it holds

\[(PR) \quad u(x) - u^\infty = O(|x|^{-\alpha}) \quad \text{as} \quad |x| \to \infty \]

for some $\alpha > 1/2$. If $u$ is a PR-solution of (N-S) with $f \in C_0^\infty(\Omega)$, then $u$ behaves like

\[(WR) \quad u(x) - u^\infty = O(|x|^{-1}(1 + s_x)^{-1}), \quad s_x \equiv |x| - \frac{x \cdot u^\infty}{|u^\infty|} \quad \text{as} \quad |x| \to \infty, \]

which exhibits a parabolic wake region behind the obstacle. It had been shown by Finn [3] that in the case when $f \in C_0^\infty(\Omega)$, every PR-solution $u$ becomes necessarily a $D$-solution. The converse assertion was treated by Babenko [1] who proved that if $f \equiv 0$, then every $D$-solution $u$ of (N-S) satisfies (PR) with $\alpha = 1$. As a result, it turns out that every $D$-solution with $f \equiv 0$ has a parabolic wake region such as (WR). Later on, Galdi [6], [7], [8], [9] and Farwig [5] succeeded to handle more general $f$ by introducing anisotropic weight functions, and obtained more precise asymptotic behavior of $u$ than (WR) in the class of PR-solutions. Furthermore, Kobayashi-Shibata [11] showed the stability of PR-solutions for small $f$ and $u^\infty$ in terms of the Oseen semi-group in $L^p$-spaces.

1 Results.

Before stating our results, let us introduce some notation and then give our definition of $D$-solutions of (N-S). $C_0^\infty(\Omega)$ is the set of all $C^\infty$-vector functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with compact support in $\Omega$, such that div $\varphi = 0$. For $1 < q < \infty$, $L^q(\Omega)$ stands for all $L^q$-summable vector functions on $\Omega$ with the norm $\| \cdot \|_{L^q}$. We denote by $\langle \cdot, \cdot \rangle$ the duality paring between $L^q(\Omega)$ and $L^{q'}(\Omega)$, where $1/q + 1/q' = 1$. $\dot{H}_0^{1,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the homogeneous norm $\| \nabla \varphi \|_{L^q}$, where $\nabla \varphi = (\frac{\partial \varphi_i}{\partial x_j})$, $i, j = 1, 2, 3$. $\dot{H}^{1,q}(\Omega)$ is the dual space of $\dot{H}_0^{1,q}(\Omega)$, and $\langle f, \phi \rangle$ denotes the duality pairing between $f \in \dot{H}^{-1,q}(\Omega)$ and $\phi \in \dot{H}_0^{1,q}(\Omega)$. Finally, for $u^\infty \in \mathbb{R}^3$, we define the space $A(u^\infty)$ by

$$A(u^\infty) \equiv \{ a \in C^1(\bar{\Omega}); \text{div} \ a = 0, a|_{\partial \Omega} = 0, a(x) \equiv u^\infty \quad \text{for all} \quad x \in \mathbb{R}^3 \quad \text{satisfying} \quad |x| > R \}$$
Our definition of $D$-solutions to (N-S) reads as follows.

**Definition.** Let $f \in \dot{H}^{-1,2} (\Omega)$ and $u^\infty \in \mathbb{R}^3$. A measurable function $u$ on $\Omega$ is called a $D$-solution of (N-S) if the following conditions (i), (ii) and (iii) are satisfied.

(i) $\nabla u \in L^2 (\Omega)$ with div $u = 0$ in $\Omega$ and $u = 0$ on $\partial \Omega$; 
(ii) $u(\cdot) - u^\infty \in L^6 (\Omega)$; 
(iii) it holds that

$$ (E) \quad (\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) = \langle f, \varphi \rangle \quad \text{for all } \varphi \in C^\infty_{0, \sigma} (\Omega). $$

**Remark.** For every $D$-solution $u$ of (N-S), there exists a unique scalar function $p \in L_{loc}^2 (\Omega)$ up to an additive constant such that

$$ (E') \quad (\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) + (p, \text{div } \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in C^\infty (\Omega). $$

Our first result on the energy identity (EI) now reads:

**Theorem 1.1** Assume that $f \in \dot{H}^{-1,2} (\Omega)$ and $u^\infty \in \mathbb{R}^3$ with $u^\infty \neq 0$. Then every $D$-solution $u$ of (N-S) satisfies

$$ (1.1) \quad (\nabla u, \nabla u) - (\nabla u, \nabla a) + (u \cdot \nabla a, u - a) = \langle f, u - a \rangle \quad \text{for all } a \in A(u^\infty). $$

Moreover, if in addition $f \in H^{-1,2} (\Omega) \cap L^q (\Omega)$ for some $1 < q < 2$, then it holds that

$$ (1.2) \quad \int_\Omega |\nabla u|^2 \, dx + u^\infty \cdot \int_{\partial \Omega} T(u, p) \cdot \nu \, dS = \langle f, u - u^\infty \rangle, $$

where $T(u, p) \equiv \left( \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} - \delta_{ij} p \right)_{1 \leq i, j \leq 3}$ denotes the stress tensor and where $\nu$ is the unit outer normal to $\partial \Omega$.

**Remarks.** (i) Galdi [8] and Farwig [5] showed a similar result to that of Theorem 1.1 under the assumption that $f \in H^{-1,2} (\Omega) \cap L^3 (\Omega) \cap L^3 (\Omega)$. On the other hand, for the validity of the energy identity (1.1), we do not need any condition on $f$ except for $f \in H^{-1,2} (\Omega)$.

(ii) The corresponding problem for $u^\infty = 0$ is still open. Indeed, up to the present, the energy identity (1.1) is shown under the hypothesis that $u \in \dot{H}^{1,2} (\Omega) \cap L^{3,\infty} (\Omega)$, where $L^{q,r} (\Omega)$ denotes the Lorentz space on $\Omega$. For instance, see Kozono-Yamazaki [12].

Next, we consider the uniqueness of $D$-solutions under the smallness assumption on the given data.

**Theorem 1.2** There is a constant $\delta_1 = \delta_1 (\Omega) > 0$ such that if $u^\infty \neq 0$ and $f \in L^{-1,2} (\Omega)$ satisfy

$$ (1.3) \quad \|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_1 |u^\infty|^\frac{1}{2}, $$

with some $R > 0$. 


then there exists a unique D-solution \( u \) of (N-S). Moreover, such a solution \( u \) is necessarily subject to the estimate

\[
|u^\infty|^\frac{1}{4} \|u - u^\infty\|_{L^4} + \|\nabla u\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|),
\]

where \( C = C(\Omega) \).

Remarks. (i) Galdi [8] showed that if \( u^\infty \neq 0 \) and \( f \in L^\frac{6}{5}(\Omega) \cap L^\frac{3}{2}(\Omega) \) satisfy

\[
\|f\|_{L^\frac{6}{5}} + |u^\infty| \leq \delta_1,
\]

then there exists a unique D-solution. Since \( L^\frac{6}{5}(\Omega) \subset \dot{H}^{-1,2}(\Omega) \), our result covers that of Galdi [8]. Furthermore, we do not need any redundant assumption such as \( f \in L^\frac{3}{2}(\Omega) \). Hence, Theorem 1.2 seems to be a final answer to Leray's question on uniqueness of D-solutions for small data.

(ii) The case when \( u^\infty = 0 \), such a uniqueness result as in Theorem 1.2 is known in more restrictive situations. For instance, Nakatsuka [15] treated the case \( u^\infty = 0 \), and proved that for every \( 3 < r < \infty \) there is a constant \( \delta = \delta(r) > 0 \) such that if \( \{u, p\} \) and \( \{v, q\} \) with \( \nabla u, \nabla v, p, q \in L^\frac{3}{2,\infty}(\Omega) \) satisfy \((E')\) and if

\[
\|u\|_{L^3,\infty} \leq \delta, \quad v \in L^3(\Omega) + L^r(\Omega),
\]

then it holds that

\[
\{u, p\} = \{v, q\}.
\]

In his result, it is necessary to assume the smallness of one solution \( u \) and some redundant regularity on another solution \( v \). It is still an open question whether any norm of solutions \( u \) of (N-S) with \( u^\infty = 0 \) can be controlled by \( f \). For details, we refer to Kim-Kozono [10].

# 2 Oseen equations.

In this section, we investigate the following Oseen equations.

\[
(Os) \quad \begin{cases}
-\Delta v + u^\infty \cdot \nabla v + \nabla \pi = f & \text{in } \Omega, \\
\text{div } v = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega, \\
v(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]

Let us introduce the two function spaces \( \tilde{H}^{1,q}(\Omega) \) and \( \tilde{H}^{2,q}(\Omega) \) defined by

\[
\tilde{H}^{1,q}(\Omega) \equiv \{v \in L^\frac{4q}{4q-4}(\Omega); \nabla v \in L^q(\Omega)\}, \quad 1 < q < 4,
\]

\[
\tilde{H}^{2,q}(\Omega) \equiv \{v \in \tilde{H}^{1,\frac{4q}{4q-4}}(\Omega); \nabla^2 v \in L^q(\Omega)\}, \quad 1 < q < 2.
\]

Then we have the following results on unique solvability of \((Os)\).

**Lemma 2.1** Let \( u^\infty \neq 0 \). Assume that \( 1 < q_1, q_2 < 4 \). The solution \( \{v, \pi\} \in \tilde{H}^{1,q_1}(\Omega) + \tilde{H}^{1,q_2}(\Omega) \times L^1_{loc}(\Omega) \) of \((Os)\) is unique.
Lemma 2.2 (i) For $f \in H^{-1,q}(\Omega)$ with $\frac{3}{2} < q < 4$, there exists a unique solution $\{v, \pi\} \in H^{1,q}(\Omega) \times L^q(\Omega)$ of $(Os)$. Moreover, for every $\frac{3}{2} < q < 3$ and every $M > 0$ there is a constant $C = C(q, M, \Omega)$ such that if $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega)$ is a solution of $(Os)$ with $|u^\infty| \leq M$, then it holds that

$$k_1||v||_{L^{\frac{4q}{4-q}}} + ||\nabla v||_{L^q} + ||\pi||_{L^q} \leq C||f||_{H^{-1,q}},$$

where $k_1 \equiv \min\{1, |u^\infty|^\frac{1}{4}\}$.

(ii) For every $f \in L^q(\Omega)$ with $1 < q < 2$, there exists a unique solution $\{v, \pi\} \in \tilde{H}^{2,q}(\Omega) \times L^{q^*}(\Omega)$ of $(Os)$ with $\nabla \pi \in L^q(\Omega)$, where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{3}$. Moreover, for every $1 < q < \frac{3}{2}$ and every $M > 0$ there is a constant $C = C(q, M, \Omega)$ such that if $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega)$ is a solution of $(Os)$ with $|u^\infty| \leq M$, then it holds that

$$k_2||v||_{L^{2q}} + k_1||\nabla v||_{L^{\frac{4q}{4-q}}} + ||\nabla^2 v||_{L^q} + ||\pi||_{L^{q^*}} + ||\pi||_{L^q} \leq C||f||_{L^q},$$

where $k_2 = k_1^2 \equiv \min\{1, |u^\infty|^\frac{1}{2}\}$.

3 Proof of Theorems.

The following lemma is based on Lemma 2.2 and plays a key role for the proof of Theorem 1.1.

Lemma 3.1 Let $u^\infty \neq 0$ and $f \in \dot{H}^{-1,2}(\Omega)$. Let $u$ be a $D$-solution of (N-S).

(i) If in addition $f \in \dot{H}^{-1,2}(\Omega) \cap \dot{H}^{-1,q}(\Omega)$ for $\frac{4}{3} < q < 4$, then it holds that

$$u - u^\infty \in L^{\frac{4q}{4-q}}(\Omega), \quad u^\infty \cdot \nabla u \in \dot{H}^{-1,q}(\Omega),$$

$$\nabla u \in L^q(\Omega), \quad p - p^\infty \in L^q(\Omega) \quad \text{for some constant } p^\infty.$$

(ii) If in addition $f \in \dot{H}^{-1,2}(\Omega) \cap L^q(\Omega)$ for $1 < q < 2$, then it holds that

$$u - u^\infty \in L^{\frac{4q}{4-q}}(\Omega), \quad \nabla u \in L^{\frac{4q}{4-q}}(\Omega) \cap L^{\frac{4q}{3q}}(\Omega),$$

$$p - p^\infty \in L^{\frac{3q}{3q-2}}(\Omega) \quad \text{for some constant } p^\infty,$$

$$\nabla^2 u, \nabla p, u^\infty \cdot \nabla u \in L^q(\Omega).$$

By taking $q = 2$ in this lemma, we have

Corollary 3.1 Every $D$-solution $u$ of (N-S) with $u^\infty \neq 0$ and $f \in \dot{H}^{-1,2}(\Omega)$ satisfies

$$u - u^\infty \in L^4(\Omega), \quad u^\infty \cdot \nabla u \in \dot{H}^{-1,2}(\Omega), \quad p - p^\infty \in L^2(\Omega)$$

for some constant $p^\infty$.

To deal with the nonlinear term, we need

Proposition 3.1 Let $u, w \in \dot{H}_0^{1,2}(\Omega) \cap L^4(\Omega)$.

(i) If $u \in L^4(\Omega)$ with $\text{div} u = 0$ in $\Omega$, then it holds that

$$(u \cdot \nabla v, w) = -(u \cdot \nabla w, v).$$

(ii) If $u^\infty \cdot \nabla v \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \cdot \nabla w \in \dot{H}^{-1,2}(\Omega)$, then it holds that

$$(u^\infty \cdot \nabla v, w) = -(u^\infty \cdot \nabla w, v),$$

$$(a \cdot \nabla v, w) = -(a \cdot \nabla w, v) \quad \text{for all } a \in A(u^\infty).$$
3.1 Proof of Theorem 1.1.

By Definition of $D$-solutions, we have

$$\langle f, \phi \rangle = (\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) - (p, \text{div} \phi)$$

(3.1)

$$= (\nabla u, \nabla \phi) + ((u-a) \cdot \nabla u, \phi) + (a \cdot \nabla u, \phi) - (p - p_\infty, \text{div} \phi)$$

for all $\phi \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $\dot{H}^{1,2}_0(\Omega) \cap L^4(\Omega)$, we have

$$\langle f, \phi \rangle = (\nabla u, \nabla \phi) + ((u-a) \cdot \nabla u, \phi) + (a \cdot \nabla u, \phi) - (p - p_\infty, \text{div} \phi)$$

(3.2)

for all $\phi \in \dot{H}^{1,2}_0(\Omega) \cap L^4(\Omega)$. By Corollary 3.1 it holds that $u - a = u - u^\infty + u^\infty - a \in \dot{H}^{1,2}_0(\Omega) \cap L^4(\Omega)$. Hence, taking $\phi = u - a$ in (3.2), we have

$$\langle f, u - a \rangle = (\nabla u, \nabla (u - a)) + ((u-a) \cdot \nabla u, u - a) + (a \cdot \nabla u, u - a).$$

Furthermore by Proposition 3.1, it holds that

$$((u-a) \cdot \nabla u, u - a) + (a \cdot \nabla u, u - a)$$

$$= ((u-a) \cdot \nabla(u - a), u - a) + (a \cdot \nabla(u - a), u - a)$$

$$+ ((u-a) \cdot \nabla a, u - a) + (a \cdot \nabla a, u - a)$$

$$= (u \cdot \nabla a, u - a),$$

from which and (3.3) we obtain

$$\|\nabla u\|_{L^2}^2 - (\nabla u, \nabla a) + (u \cdot \nabla a, u - a) = \langle f, u - a \rangle.$$  

This proves (1.1).

Assume in addition that $f \in \dot{H}^{-1,2}(\Omega) \cap L^q(\Omega)$ for some $1 < q < 2$. By Lemma 3.1 (ii), we have

$$-\Delta u + u \cdot \nabla u + \nabla p = f \quad \text{a.e. in } \Omega.$$

Note that

$$a - u^\infty \in C_0^\infty(\mathbb{R}^3), \quad a - u^\infty = 0 \quad \text{on } \partial \Omega.$$

By integration by parts, we have

$$\langle f, a - u^\infty \rangle = (-\Delta u + u \cdot \nabla u + \nabla p, a - u^\infty)$$

$$= (-\text{div} \left(T(u, p), a - u^\infty\right)) + (u \cdot \nabla u, a - u^\infty)$$

(3.4)

$$= (\nabla u, \nabla a) + u^\infty \cdot \int_{\partial \Omega} T(u, p) \cdot \nu dS - (u \cdot \nabla a, u).$$

Addition of (3.4) and (1.1) yields that

$$\|\nabla u\|_{L^2}^2 + u^\infty \cdot \int_{\partial \Omega} T(u, p) \cdot \nu dS - (u \cdot \nabla a, a) = \langle f, u - u^\infty \rangle.$$

(3.5)

Since supp $\nabla a$ is compact, we see easily

$$u \cdot \nabla a, a = 0,$$

from which and (3.5) we obtain the desired identity (1.2). This proves Theorem 1.1.
3.2 Proof of Theorem 1.2.

Step 1. We first show that there are constants \( \delta_* = \delta_*(\Omega) \) and \( C_*(\Omega) > 0 \) such that if

\[
\|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^\frac{1}{2},
\]

then every \( D \)-solution \( u \) of (N-S) satisfies

\[
|u^\infty|^\frac{1}{4} \|u - a\|_{L^4} + \|\nabla u\|_{L^2} \leq C_*(\|f\|_{H^{-1,2}} + |u^\infty|)
\]

for some \( a \in A(u^\infty) \). Indeed, taking \( 0 < R_0 < R_1 < \infty \) and \( a \in A(u^\infty) \) in such a way that

\[
\Omega^c = \mathbb{R}^3 \setminus \Omega \subset B_{R_0}(0), \quad \text{supp } \nabla a \subset \{R_0 < |x| < R_1\}.
\]

we have

\[
\|a\|_{L^\infty} + \|\nabla a\|_{L^1 \cap L^\infty} \leq C|u^\infty|
\]

with \( C = C(\Omega) \). By (1.1), we see that

\[
\|\nabla u\|_{L^2}^2 = \langle f, u - a \rangle + (\nabla u, \nabla a) + (u \cdot \nabla a, u - a),
\]

from which and (3.8) with the aid of the Young inequality it follows that

\[
\|\nabla u\|_{L^2}^2 \leq \left(\frac{1}{2} + C|u^\infty|\right) \|\nabla u\|_{L^2}^2 + C\|f\|_{H^{-1,2}}^2 + C(|u^\infty|^2 + |u^\infty|^4).
\]

Hence, under the assumption

\[
|u^\infty| \leq \delta_*(1) = \min\{1, \frac{1}{4C}\},
\]

we have

\[
\frac{1}{4}\|\nabla u\|_{L^2}^2 \leq C\|f\|_{H^{-1,2}}^2 + C(|u^\infty|^2 + |u^\infty|^4)
\]

\[
\leq C(\|f\|_{H^{-1,2}}^2 + |u^\infty|^2),
\]

which yields that

\[
\|\nabla u\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|).
\]

Next, we show the bound of \( \|u - a\|_{L^4} \). Define \( v = u - a \), and we have by (3.8) and (3.9) that

\[
v \in H^1_0(\Omega), \quad \|\nabla v\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|),
\]

and that

\[
\begin{cases}
-\Delta v + u^\infty \cdot \nabla v + \nabla \pi = f - Q(v) \quad \text{in } \Omega, \\
\text{div } v = 0 \quad \text{in } \Omega, \\
v = 0 \quad \text{on } \partial \Omega, \\
v(x) \to 0 \quad \text{as } |x| \to \infty,
\end{cases}
\]

where

\[
Q(v) \equiv v \cdot \nabla v + (a - u^\infty) \cdot \nabla v + v \cdot \nabla a - \Delta a + a \cdot \nabla a.
\]
By (3.8) and (3.11), it holds that
\[
\|v \cdot \nabla v\|_{L^4} \leq \|v\|_{L^4} \|\nabla v\|_{L^2} \leq C(\|f\|_{H^{-1,2}} + |u^\infty|)\|v\|_{L^4}
\]
\[
\|Q(v) - v \cdot \nabla v\|_{H^{-1,2}}
= \|(a - u^\infty) \cdot \nabla v + v \cdot \nabla a - \Delta a \cdot \nabla v\|_{H^{-1,2}}
\leq C(\|\nabla v\|_{L^2} + |u^\infty|)
\leq C(\|f\|_{H^{-1,2}} + |u^\infty|).
\]

Hence, it follows from Lemma 2.1 and Lemma 2.2 with \(q = 2\) in (i) and with \(q = \frac{4}{3}\) in (ii) that
\[
\|v\|_{L^4} \leq C \left( \frac{1}{k_1} \|f - Q(v) - v \cdot \nabla v\|_{H^{-1,2}} + \frac{1}{k_2} \|v \cdot \nabla v\|_{L^4} \right)
\]
\[
\leq C \left( \frac{1}{k_1} (\|f\|_{H^{-1,2}} + |u^\infty|) + \frac{1}{k_2} (\|f\|_{H^{-1,2}} + |u^\infty|)\|v\|_{L^4} \right).
\]

Hence, under the assumption
\[
\frac{1}{k_2} (\|f\|_{H^{-1,2}} + |u^\infty|) \leq \delta_* \equiv \min\{\delta^{(1)}_*, \frac{1}{2C}\},
\]
we have
\[
\|u - a\|_{L^4} = \|v\|_{L^4} \leq C \left( \frac{1}{k_1} (\|f\|_{H^{-1,2}} + |u^\infty|) \right).
\]

Since the assumption (3.13) necessarily implies the assumption (3.9), we see by (3.10) and (3.14) that if
\[
\|f\|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^\frac{1}{2},
\]
then it holds that
\[
|u^\infty|^\frac{1}{4} \|u - a\|_{L^4} + \|\nabla u\|_{L^4} \leq (\|f\|_{H^{-1,2}} + |u^\infty|),
\]
which implies (3.7)

**Step 2.** We next show uniqueness. Let \(u_1\) and \(u_2\) be two \(D\)-solutions of (N-S). Define \(v_1 = u_1 - a\) and \(v_2 = u_2 - a\) with \(a \in A(u^\infty)\) as in Step 1. Then \(v \equiv v_1 - v_2 = u_1 - u_2\) fulfills
\[
\begin{aligned}
-\Delta v + u^\infty \cdot \nabla v + \nabla \pi = & -v_1 \cdot \nabla v - v \cdot \nabla u_2 \quad \text{in } \Omega, \\
\text{div } v = & 0 \quad \text{in } \Omega, \\
v = & 0 \quad \text{on } \partial \Omega, \\
v(x) \to & 0 \quad \text{as } |x| \to \infty,
\end{aligned}
\]

Hence it follows from Lemmata 2.1 and 2.1 with
\[
f = -v_1 \cdot \nabla v = \text{div} (v_1 \otimes v) \quad \text{for } q = 2 \text{ in (i)},
\]
\[
f = -v \cdot \nabla u_2 \quad \text{for } q = \frac{4}{3} \text{ in (ii)}
\]
that
\[ \|v\|_{L^4} \leq C \left( \frac{1}{k_1} \| \text{div} (v_1 \otimes v) \|_{H^{-1,2}} + \frac{1}{k_2} \| v \cdot \nabla u_2 \|_{L^4} \right) \]
\[ \leq C \left( \frac{1}{k_1} \| v_1 \otimes v \|_{L^2} + \frac{1}{k_2} \| v \|_{L^4} \| \nabla u_2 \|_{L^2} \right) \]
\[ \leq C \left( \frac{1}{k_1} \| v_1 \|_{L^4} + \frac{1}{k_2} \| \nabla u_2 \|_{L^2} \right) \| v \|_{L^4}. \]  
(3.15)

By Step 1, under the assumption
\[ \| f \|_{H^{-1,2}} + |u^\infty| \leq \delta_* |u^\infty|^\frac{1}{2}, \]
we have
\[ \| v_1 \|_{L^4} \leq \frac{C}{k_1} (\| f \|_{H^{-1,2}} + |u^\infty|), \quad \| \nabla u_2 \|_{L^2} \leq C (\| f \|_{H^{-1,2}} + |u^\infty|), \]
from which and (3.15) with \( k_1^2 = k_2 \) it follows that
\[ \| v \|_{L^4} \leq \frac{C}{k_2} (\| f \|_{H^{-1,2}} + |u^\infty|) \| v \|_{L^4}. \]  
(3.16)

Now, define \( \delta_1 = \delta_1(\Omega) \) so that
\[ \delta_1 = \min \{ \delta_*, \frac{1}{2C} \} . \]

Then under the assumption
\[ \| f \|_{H^{-1,2}} + |u^\infty| \leq \delta_1 |u^\infty|^\frac{1}{2}, \]
it follows from (3.16) with the aid of the relation \( k_2 = \min \{ 1, |u^\infty|^\frac{1}{2} \} \) that
\[ \| v \|_{L^4} \leq 0, \]
which yields the desired uniqueness result. This completes the proof of Theorem 1.2.

References


