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1. Introduction

Let $G$ be a finite group. The Smith problem is as follows. Let $\Sigma$ be a homotopy sphere with smooth $G$-action such that $\Sigma$ has just two fixed points, say $a$ and $b$. Are tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic as real $G$-modules? Two real $G$-modules $U$ and $V$ are called Smith equivalent if there exists a smooth action of $G$ on a sphere $\Sigma$ such that $S^G = \{a, b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$ as real $G$-modules. We know infinitely many Oliver groups possessing non-isomorphic Smith equivalent real modules. We consider about the subset $\text{Sm}(G)$ of the real representation ring $\text{RO}(G)$ of $G$ consisting of all differences $U - V$ of Smith equivalent real $G$-modules. Recently we have several results corresponding to the Smith set. In this note, we study a sufficient condition for the Smith set to be an additive subgroup of the real representation ring $\text{RO}(G)$. This work is a continuous study from [24].

2. Smith Problem

The Smith problem asks whether the Smith set $\text{Sm}(G)$ is zero or not. There are many results corresponding to the Smith problem.

Atiyah and Bott [1] or Milnor [7] showed that for a homotopy sphere $\Sigma$ with semi-free smooth compact Lie group with just two fixed points, the tangential representations are isomorphic. Thus, any Smith equivalent real modules over an abelian simple group are isomorphic, that is, $\text{Sm}(C) = 0$ for a prime order cyclic group $C$. Sanchez [18] generalized the result as follows by computing $G$-signature and using Franz-Bass’s theorem. For a cyclic group $P$ of odd prime power order, Smith equivalent real $P$-modules are isomorphic. Therefore $\text{Sm}(P) = 0$ for any group $P$ of odd prime power order by combining the Smith theory.

On the other hand, Cappell and Shaneson [2] showed that there exists non-isomorphic, Smith equivalent real module over a cyclic group $C_{4n}$ of order $4n$ for $n \geq 2$, that is, $\text{Sm}(C_{4n}) \neq \{0\}$. Petrie [17] showed that the Smith set of an abelian group of odd order which has at least four non-cyclic subgroups is nontrivial, eg. $\text{Sm}(C_{pqrs} \times C_{pqrs}) \neq 0$.

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where $p, q, r, s$ are distinct odd primes. And in 1980's, Dovermann, Suh, Masuda, etc. studied the Smith equivalent real modules.

Oliver [13] showed that $G$ acts smoothly on a disk without fixed points if and only if there are no subgroups $P$ and $H$ such that $P$ is a $p$-group, $H/P$ is cyclic, $G/H$ is a $q$-group for some primes $p$ and $q$, possibly $p = q$. A group acting on a disk without fixed points is called an Oliver group. Laitinen and Morimoto [5] showed that $G$ is an Oliver group if and only if there exists a one fixed point $G$-action on sphere. Laitinen and Pawafowski [6] showed that there exists Smith equivalent, non-isomorphic real $G$-modules for a perfect group $G$ with $r_G \geq 2$ by connecting sum with a sphere with just one fixed point, where $r_G$ is the number of real conjugacy classes of elements of $G$ not of prime power order. After that, Pawafowski and Solomon [14] extended to that Sm$(G) \neq 0$ if $G$ is a gap Oliver group with $r_G \geq 2$ except Aut$(A_6)$ and $\text{P}S\text{L}(2,27)$. A group $G$ is a gap group if there exists a real $G$-module $V$ such that

- $\dim V^L = 0$ for any prime power index subgroup $L$ of $G$ and
- for any subgroups $P$ of prime power order and $H$ with $H > P$,

$$\dim V^P \geq 2 \dim V^H.$$ 

In particular, a perfect group $G$ with $r_G \geq 2$ is a gap Oliver group. A study for gap groups is seen in [12, 19, 20, 22, 23].

Now we need some notations. A real conjugacy class $(x)^*$ of an element $x$ of $G$ is the union of the conjugacy class

$$(x) = \{g^{-1}xg \mid g \in G\}$$

of $x$ and one of its inverse $x^{-1}$. We denote by NPP$(G)$ the set of elements of $G$ not of prime power order, by NNNP$(G)$ the set of elements of the real conjugacy classes of elements of NPP$(G)$. Then $r_G$ is the cardinality of the set NNNP$(G)$. For a prime $p$, let $N_p(G)$ be the set of normal subgroups $N$ of $G$ with $[G : N] \leq p$. We denote by RO$(G)$ the real representation ring, by $P(G)$ the set of all subgroups of $G$ of prime power, possibly 1, order, by $O^p(G)$ the Dress subgroup of type $p$ for a prime $p$ defined as

$$O^p(G) = \bigcap_{L \leq G, [G:L]=p^a} L,$$

and by $L(G)$ the set of all prime power, possible 1, index subgroups of $G$. Then for $L \in L(G), L$ contains $O^p(G)$ for some prime $p$. We put

$$\cap_p(G) = \cap_{N \leq N_p(G)} N$$

which quotient is an elementary abelian $p$-group and denote by $G^{\text{nil}}$ the smallest normal subgroup of $G$ by which quotient is nilpotent:

$$G^{\text{nil}} = \bigcap_p O^p(G).$$
Note that
\[ G \supseteq \cap p(G) \supseteq O^p(G) \supseteq G^{nil}. \]

For families \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of subgroups of \( G \) and a subset \( A \) of \( \text{RO}(G) \), we put
\[ A_{\mathcal{F}_1} = \bigcap_{P \in \mathcal{F}_1} \ker(\text{Res}_P^G : \text{RO}(G) \to \text{RO}(P)) \cap A, \]
\[ A_{\mathcal{F}_2} = \bigcap_{L \in \mathcal{F}_2} \ker(\text{Fix}^L : \text{RO}(G) \to \text{RO}(N_G(L)/L)) \cap A, \]
and
\[ A_{\mathcal{F}_1}^{\mathcal{F}_2} = A_{\mathcal{F}_1} \cap A_{\mathcal{F}_2} = \bigcap_{P \in \mathcal{F}_1} \ker(\text{Res}_P^G) \cap \bigcap_{L \in \mathcal{F}_2} \ker(\text{Fix}^L) \cap A. \]

The automorphism group \( \text{Aut}(A_6) \) of the alternating group \( A_6 \) is not a gap group, \( r_{\text{Aut}(A_6)} = 2 \), and \( \text{Sm}(G) = 0 \) [8]. Morimoto [8] gave a condition
\[ \text{Sm}(G) \subset \text{RO}(G)^{N_2(G)} = \text{RO}(G)^{O^2(G)} \]
for Smith equivalent real modules. The rank of \( \text{RO}(G)^{N_2(G)} \) is equal to
\[ r_G - r_{G,\cap 2(G)}, \]
where \( r_{G,\cap 2(G)} \) is the cardinality of the set \( \pi(\overline{\text{bpp}}(G)) \) for a canonical projection \( \pi : G \to G/\cap 2(G) \) (cf. [14]). This condition implies that there are Oliver solvable groups \( G \) such that \( r_G \geq 2 \) and \( \text{Sm}(G) = 0 \) [15]. The group \( \text{PSL}(2, 27) \) is an extension of \( \text{PSL}(2, 27) \) by a field automorphism group of order 3 which is a gap non-solvable group, \( r_{\text{PSL}(2, 27)} = 2 \) and \( \text{Sm}(\text{PSL}(2, 27)) \neq 0 \) [9]. Moreover, putting together with [16], for an Oliver non-solvable group \( G \) with \( r_G \geq 2 \), \( \text{Sm}(G) = 0 \) if and only if \( G \) is isomorphic to \( \text{Aut}(A_6) \).

3. SUBSETS OF THE SMITH SET

Sanchez’s criterion and Petrie’s observation says that
\[ \text{Sm}(G) \subset \text{RO}(G)^{[G]}_{\mathcal{P}_o(G)}, \]
where \( \mathcal{P}_o(G) \) is the set of subgroups of \( G \) of order 1, 2, 4, or odd prime power. Thus we have
\[ \text{Sm}(G) \subset \text{RO}(G)^{N_2(G)}_{\mathcal{P}_o(G)}. \]

Note that if \( G \) has no element of order 8 then \( \mathcal{P}_o(G) = \mathcal{P}(G) \). Recall that two real \( G \)-modules \( U \) and \( V \) are Smith equivalent if there exists a smooth action of \( G \) on a sphere \( \Sigma \) such that \( S^G = \{a, b\}, T_a(\Sigma) \cong U \) and \( T_b(\Sigma) \cong V \) as real \( G \)-modules and put
\[ \text{Sm}(G) = \{[U] - [V] \mid U \text{ and } V \text{ are Smith equivalent}\}. \]

Similarly we consider the sets \( \text{PSm}(G) \) (resp. \( \text{LSm}(G) \)) of all differences \([U] - [V]\) such that \( U \) and \( V \) are Smith equivalent and in addition the homotopy sphere \( \Sigma \) satisfies that \( \Sigma^p \) is connected for any prime power order subgroups \( P \) of \( G \) (resp. for any 2-groups of \( G \)).
The set $\text{PSm}^c(G)$ (resp. $\text{LSm}(G)$) is empty if and only if $G$ is of order prime power (resp. 2-power). It holds the inclusions
$$\text{PSm}^c(G) \subset \text{LSm}(G) \subset \text{Sm}(G)$$
and
$$\text{LSm}(G) \subset \text{RO}(G)_{\mathcal{P}(G)}.$$
not have an example for an Oliver group \( G \) such that \( PSm^c(G) \neq Sm(G)_{\mathcal{P}(G)} \). It remains the problem whether \( PSm^c(G) = Sm(G)_{\mathcal{P}(G)} \) for an Oliver group.

4. CRITERION FOR THE SMITH SET TO BE A GROUP

We discuss for Oliver groups \( G \) such that \( PSm^c(G) \) is a subgroup of \( RO(G) \). We introduce two condition. One is a part of a sufficient condition to show \( Sm(G)_{\mathcal{P}(G)} \setminus Sm(G)_{\mathcal{P}(G)}^2 \neq \emptyset \) and the other is a sufficient condition so that \( Sm(G)_{\mathcal{P}(G)} \) is a group.

Let \( Q = \cap_{p^2 \leq 2} O^{p2}(G) \) be a normal subgroup of \( G \) with odd index and let \( N \) be a normal subgroup of \( G \) with \( G^{nil} \leq N \leq \cap 2(G) \cap Q \). Then

\[ Q \geq \cap 2(G) \cap Q \geq N \geq G^{nil} \geq O^2(Q). \]

**Definition 4.1.** We say that \( G \) satisfies the quasi-\( N\)-\( \mathcal{P} \)-condition if there are real \( Q \)-modules \( U \) and \( V \) such that
- \( \dim U^{\cap 2(G) \cap Q} = \dim V^{N} = 0 \) and
- \( [\mathbb{R} \oplus U] - [V] \in RO(Q)_{\mathcal{P}(Q)}. \)

In particular, the quasi-\( G^{nil} \)-\( \mathcal{P} \)-condition is simply called quasi-nil-\( \mathcal{P} \)-condition.

**Definition 4.2.** We say that \( G \) satisfies the weak-nil-\( \mathcal{P} \)-condition if there are real \( G \)-modules \( U \) and \( V \) such that
- \( \dim U^{\cap 2(G)} = \dim V^{G^{nil}} = 0 \) and
- \( [\mathbb{R} \oplus U] - [V] \in RO(G)_{\mathcal{P}(G)}. \)

**Lemma 4.3.** If \( G \) satisfies the quasi-nil-\( \mathcal{P} \)-condition, then \( G \) satisfies the weak nil-\( \mathcal{P} \)-condition.

**Proposition 4.4** (cf. [10, Lemma 15]). Let \( G \) be a finite group with \( O^2(G) = G \). The following statements are equivalent.

1. \( G^{nil} \) has a sub-quotient isomorphic to \( D_{2pq} \) for distinct primes \( p, q \).
2. \( G \) satisfies the quasi-nil-\( \mathcal{P} \)-condition.

Morimoto and Qi [11] obtained a sufficient condition for an Oliver group \( G \) to hold that \( Sm(G)_{\mathcal{P}(G)} \) is not equal to \( Sm(G)_{\mathcal{P}(G)}^{G(G)} \). This result supplies that \( Sm(G) = Sm(G)_{\mathcal{P}(G)} \cong \mathbb{Z} \) for \( G = SG(864, 2666) \) or \( SG(864, 4666) \). For \( G = SG(864, 2666) \) or \( SG(864, 4666) \), \( G/G^{nil} \) is a cyclic group of order 3 and \( RO(G)_{\mathcal{P}(G)} \) is generated by two element \( \mathbb{R}[G/G^{nil}] + X_1 \) and \( 3(\mathbb{R}[G/G^{nil}] - \mathbb{R}) + X_2 \) for some elements \( X_1, X_2 \in RO(G)^{G^{nil}} \) and thus, \( G \) satisfies the weak-nil-\( \mathcal{P} \)-condition since \( G/G^{nil} \) is a cyclic group of order 3. We see it in the next section. Indeed, \( G \) has a sub-quotient isomorphic to \( D_{12} \) and \( G \) satisfies the quasi-nil-\( \mathcal{P} \)-condition.

**Definition 4.5.** For a normal subgroup \( N \) of \( G \), we say that \( G \) satisfies the \( N\)-\( \mathcal{P} \)-condition if there are real \( G \)-modules \( U \) and \( V \) such that \( U^N = V^N = 0 \) and \( [\mathbb{R} \oplus U] - [V] \in RO(G)_{\mathcal{P}(G)}. \) If \( N = G^{nil} \) we say that \( G \) satisfies the nil-\( \mathcal{P} \)-condition.
Lemma 4.6 or Theorem 4.8 in [9] essentially yields us the following two theorems.

**Theorem 4.6.** If a gap Oliver group $G$ satisfies the weak-Nil-$\mathcal{P}$-condition with $\text{NPP}(G) \cap G^{\text{nil}} \neq \emptyset$ and has an element of $\text{NPP}(G)$ outside $O^p(G)$ for some prime $p$, then

$$\text{PSm}^c(G) \setminus \text{RO}(G)^{L(G)}_{\mathcal{P}(G)} \neq 0.$$  

Note that under the assumption that $\text{NPP}(G) \cap G^{\text{nil}} \neq \emptyset$ the inequality $\text{RO}(G)^{N_2(G)}_{\mathcal{P}(G)} \neq \text{RO}(G)^{L(G)}_{\mathcal{P}(G)}$ if and only if $\text{NPP}(G) \setminus O^p(G)$ is not empty for some prime $p$. By using the multiplication of RO($G$), we get the following theorem.

**Theorem 4.7.** Let $G$ be a gap Oliver group satisfying the Nil-$\mathcal{P}$-condition. Then

$$\text{PSm}^c(G) = \text{RO}(G)^{N_2(G)}_{\mathcal{P}(G)} = \text{Sm}(G)_{\mathcal{P}(G)}$$

and in particular $\text{Sm}(G)_{\mathcal{P}(G)}$ is an additive group.

If a Sylow 2-subgroup of $G$ is normal, $G$ does not satisfy the Nil-$\mathcal{P}$-condition. Although the Nil-$\mathcal{P}$-condition is a sufficient one for an Oliver group $G$ such that $\text{Sm}(G)_{\mathcal{P}(G)}$ is an additive group, it is not a necessary condition. For example, $A_5 \times C_4$ does not satisfy the Nil-$\mathcal{P}$-condition but the following result holds.

**Proposition 4.8.** $\text{PSm}^c(A_5 \times C_4) = \text{Sm}(A_5 \times C_4) = \text{RO}(A_5 \times C_4)^{|A_5|}$.

**Problem.** $\text{PSm}^c(A_5 \times (C_4)^n) = \text{Sm}(A_5 \times (C_4)^n)$ holds. Is it true that $\text{PSm}^c(A_5 \times (C_4)^n) = \text{RO}(A_5 \times (C_4)^n)^{|A_5 \times (C_4)|}$ for $n \geq 2$?

### 5. Quasi-Nil-$\mathcal{P}$-Condition

In this section we study properties for the weak-Nil-$\mathcal{P}$-condition. Remark that there is an Oliver group which satisfies the weak-Nil-$\mathcal{P}$-condition but does not satisfy the Nil-$\mathcal{P}$-condition (eg. $\text{SG}(864, 2666)$, $\text{SG}(864, 4666)$).

**Proposition 5.1.** Let $K$ be a subgroup of $G$ such that $\cap 2(G) \cdot K = G$. If $K$ satisfies the weak-$(G^{\text{nil}} \cap K)$-$\mathcal{P}$-condition, then $G$ satisfies the weak-Nil-$\mathcal{P}$-condition.

**Theorem 5.2.** Let $G$ be a gap Oliver group. Suppose that $\text{NPP}(G) \cap G^{\text{nil}}$ is not empty and that there is an element $\text{NPP}(G)$ outside of $O^p(G)$ for some prime $p$. If an odd index subgroup $K$ of $G$ satisfies the weak-$(G^{\text{nil}} \cap K)$-$\mathcal{P}$-condition, then

$$\text{PSm}^c(G) \setminus \text{RO}(G)^{L(G)}_{\mathcal{P}(G)} \neq 0.$$  

Morimoto and Qi [10, Lemma 21 and Theorem 22] showed that $\text{Sm}(G)_{\mathcal{P}(G)} \neq \text{Sm}(G)^{L(G)}_{\mathcal{P}(G)}$ for an odd integer $n > 1$, an odd prime $p$, and $G = D_{2n} \int C_p$, the wreath product of the
dihedral group $D_{2n}$ of order $2n$ by a cyclic group $C_p$ of order $p$. The group $G$ satisfies the assumption of Proposition 5.1 as follows. The group $G$ has a presentation

$$a_i^2 = b_i^2 = (ab_i)^2 = 1, \ (\forall i),$$

$$\langle a_1, b_1, \ldots, a_p, b_p, c \mid a_i a_j = a_j a_i, a_i b_j = b_j a_i, b_i b_j = b_j b_i, \ (i \neq j), \rangle,$$

$$c^2 = 1, \ c^{-1}a_i c = a_{i+1}, \ c^{-1}b_i c = b_{i+1}, \ (\forall i)$$

where $a_{p+1} = a_1$ and $b_{p+1} = b_1$. The group $G^{\text{nil}}$ is a subgroup of $G$ generated by elements $a_1, \ldots, a_p$ and $b_1, b_j$ ($i < j$), and then $G/G^{\text{nil}} \cong C_{2p}$. Thus $G$ is a gap Oliver group. Put $K = O^p(G)$. Let $f: D_{2n}^2 \to D_{2n}$ be the first projection and let $\hat{U}$ and $\hat{V}$ be $\mathcal{P}(D_{2n})$-matched real $D_{2n}$-modules such that $\hat{U}^{D_{2n}} = \mathbb{R}$ and $\hat{V}^{D_{2n}} = 0$. The real $K$-modules $f^* \hat{U}$ and $f^* \hat{V}$ implies that $K$ satisfies the assumption of Proposition 5.1 since $f(G^{\text{nil}}) = D_{2n}$. (Or directly, two real $G$-modules $\text{Ind}_K^G f^* \hat{U}$ and $\text{Ind}_K^G f^* \hat{V}$ implies that $G$ satisfies the weak-nil-$\mathcal{P}$-condition.)

Before closing this section, we should say the strongness of the weak-nil-$\mathcal{P}$-condition. Let $G$ be a finite group such that $G/G^{\text{nil}}$ is a nilpotent group of odd order and there are an element of $G^{\text{nil}}$ not of prime power order and an element of $G$ outside $G^{\text{nil}}$ not of prime power order. Then

$$\text{RO}(G)^{G^{\text{nil}}}_{\mathcal{P}(G)} \neq \text{RO}(G)^{G}_{\mathcal{P}(G)}.$$

Note that if a Sylow 2-subgroup of $G$ is normal then $\text{Sm}(G) \subset \text{RO}(G)^{N_{r}(G)|s}_{\mathcal{P}}$ (cf. [4]) and $G$ does not satisfy the weak-nil-$\mathcal{P}$-condition. Otherwise, if $G$ has a sub-quotient isomorphic to $D_{2qr}$ for some distinct primes $q$ and $r$, there are real $G$-modules $U$ and $V$ such that the equalities $U^{G^{\text{nil}}} = 0 = V^{G^{\text{nil}}}$ hold and that $\mathbb{R}[G/G^{\text{nil}}] \oplus U$ and $V$ are $\mathcal{P}(G)$-matched:

$$\mathbb{R} + [(\mathbb{R}[G/G^{\text{nil}}] - \mathbb{R}) \oplus U] - [V] = \mathbb{R}[G/G^{\text{nil}}] + [U] - [V] \in \text{RO}(G)_{\mathcal{P}(G)}.$$

Thus, $G$ satisfies the weak-nil-$\mathcal{P}$-condition and in addition if $G$ is a gap Oliver group then

$$\text{PSm}^c(G)^{G^{\text{nil}}}_{\mathcal{P}(G)} \neq \text{PSm}^c(G).$$

6. Nil-$\mathcal{P}$-condition

In this section we study properties for the Nil-$\mathcal{P}$-condition.

**Proposition 6.1.** If $G$ satisfies the Nil-$\mathcal{P}$-condition, then $G$ satisfies the weak-Nil-$\mathcal{P}$-condition.

**Proposition 6.2.** If a quotient group of $G$ satisfies the Nil-$\mathcal{P}$-condition, then $G$ also satisfies the Nil-$\mathcal{P}$-condition.

**Proposition 6.3.** Let $N$ be a normal subgroup of $G$. If there are a subgroup $K$ of $G$ and an epimorphism $f: K \to H$ such that $f(K \cap N) = H$, $KN = G$ and $H$ has sub-quotient isomorphic to $D_{2pq}$, where $p$ and $q$ are distinct primes, then $G$ satisfies the $N$-$\mathcal{P}$-condition.
For a perfect group $G$, the weak-Nil-$\mathcal{P}$-condition and Nil-$\mathcal{P}$-condition are equivalent and moreover equivalent to that $G$ has a sub-quotient isomorphic to a dihedral group $D_{2pq}$ for distinct primes $p$ and $q$.

**Proposition 6.4** (cf. [21]). *Simple groups except the following groups satisfy the Nil-$\mathcal{P}$-condition.*

1. Cyclic group
2. Projective special linear groups: $\text{PSL}(2,4) = \text{PSL}(2,5) = A_5$, $\text{PSL}(2,7) = \text{PSL}(3,2)$, $\text{PSL}(2,8)$, $\text{PSL}(2,9) = A_6$, $\text{PSL}(2,17)$, $\text{PSL}(3,4)$, $\text{PSL}(3,8)$
3. Suzuki groups $\text{Sz}(8), \text{Sz}(32)$
4. Projective unitary groups: $\text{PSU}(3,3), \text{PSU}(3,4), \text{PSU}(3,8)$

**Theorem 6.5.** Let $q > 1$ be a prime power. *The following groups are gap groups satisfying the Nil-$\mathcal{P}$-condition.*

1. Symmetric groups $S_n$, $n \geq 7$
2. Projective general linear groups $\text{PGL}(2,q)$, $q \neq 2,3,4,5,7,8,9,17$
3. Projective general linear groups $\text{PGL}(3,q)$, $q \neq 2,4,8$
4. Projective general linear groups $\text{PGL}(n,q)$, $n \geq 4$
5. General linear groups $\text{GL}(2,q)$, $q \neq 2,3,4,5,7,8,9,17$
6. General linear groups $\text{GL}(3,q)$, $q \neq 2,4,8$
7. General linear groups $\text{GL}(n,q)$, $n \geq 4$
8. The automorphism group of sporadic groups

The Smith sets of $\text{PGL}(2,q)$ and $\text{PGL}(3,q)$ have been already obtained in [24]. This can be proved by finding subgroups as in Proposition 6.3. The groups listed up in Theorem 6.5 are non-solvable gap group. Then we have the following theorem.

**Theorem 6.6.** Let $G$ be a group which has quotient isomorphic to a group in Theorem 6.5. Then
\[
\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)} = \text{RO}(G)_{\mathcal{P}(G)}^{N_{\mathcal{P}}(G)}.
\]

**Corollary 6.7.** Let $K$ be a group in Theorem 6.5 and $F$ any finite group. Then for $G = K \times F$,
\[
\text{PSm}^c(G) = \text{Sm}(G)_{\mathcal{P}(G)} = \text{RO}(G)_{\mathcal{P}(G)}^{N_{\mathcal{P}}(G)}.
\]

**References**


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