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The space of maps from a real projective space to a toric variety

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Abstract

The main purpose of this note is consider the homotopy type of the space of algebraic maps from a real projective space to a projective smooth toric variety as in [14]. The main result of this paper (Theorem 1.1) is also regarded as one of generalizations of the previous work of the second and third authors [19].

An irreducible normal algebraic variety $X$ (over $\mathbb{C}$) is called a toric variety if it has an algebraic action of algebraic torus $\mathbb{T}^r = (\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}^r \cdot *$ of some point $* \in X$ is dense in $X$ and isomorphic to $\mathbb{T}^r$. A finite correction $\Sigma$ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called a fan if every face of element of $\Sigma$ is belongs to $\Sigma$ and the intersection of any two elements of $\Sigma$ is a face of each. It is known that A toric variety $X$ is completely characterized up to isomorphism by its fan $\Sigma$, and we denote by $X_\Sigma$ the corresponding toric variety. For an $n$ dimensional lattice polytope $P$, we denote by $\Sigma_P$ the normal fan of $P$ in $\mathbb{R}^n$. It is known that the toric variety $X_\Sigma$ is projective if and only if $\Sigma = \Sigma_P$ for some $n$ dimensional lattice polytope $P$ in $\mathbb{R}^n$.

We shall use the symbols $\{z_k\}_{k=1}^r$ to denote variables of polynomials, and for $f_1, \cdots, f_s \in \mathbb{C}[z_1, \cdots, z_r]$, let $V(f_1, \cdots, f_s)$ denote the affine variety $V(f_1, \cdots, f_s) = \{x \in \mathbb{C}^r | f_k(x) = 0 \text{ for each } 1 \leq k \leq s\}$.

Let $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ denote the set of all one dimensional cones (or called a ray) in a fan $\Sigma$, and let $n_k \in \mathbb{Z}^n$ denote the generator of $\rho_k \cap \mathbb{Z}^n$ called the primitive element of $\rho_k$ for each $1 \leq k \leq r$. Define the affine variety $Z_\Sigma \subset \mathbb{C}^r$ by $Z_\Sigma = V(z^\sigma | \sigma \in \Sigma)$, where $z^\sigma$ denotes the monomial given by $z^\sigma = \prod_{1 \leq k \leq r, n_k \not\in \sigma} z_k \in \mathbb{Z}[z_1, \cdots, z_r]$ ($\sigma \in \Sigma$). Let $G_\Sigma \subset \mathbb{T}^r$ denote the subgroup consisting of all $r$-tuples $(\mu_1, \cdots, \mu_r) \in \mathbb{T}^r$ such that $\prod_{k=1}^r \mu_k^{(m,n_k)} = \cdots$
1 for any $m \in \mathbb{Z}^n$, where we set $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$ for $x = (x_1, \cdots, x_n), \ y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. We say that a set of primitive elements $\{n_{i_1}, \cdots, n_{i_s}\}$ is primitive if they do not lie in any cone in $\Sigma$ but every proper subset does. It is known that

$$Z_\Sigma = \bigcup_{\{n_{i_1}, \cdots, n_{i_s}\} : \text{primitive}} V(z_{i_1}, \cdots, z_{i_s}).$$

Note that $Z_\Sigma$ is a closed variety of dimension $2(r - r_{\min})$, where we set

$$r_{\min} = \min \{s \in \mathbb{Z}_{\geq 1} \mid \{n_{i_1}, \cdots, n_{i_s}\} \text{ is primitive} \}.$$

It is also known that if the set $\{n_1, \cdots, n_r\}$ spans $\mathbb{R}^n$, there is an isomorphism $X_\Sigma \cong (\mathbb{C}^r \setminus Z_\Sigma)/G_{\Sigma}$, where the group $G_{\Sigma}$ acts on the complement $\mathbb{C}^r \setminus Z_\Sigma$ by the coordinate-wise multiplication.

For connected spaces $X$ and $Y$, let $\text{Map}(X, Y)$ be the space of all continuous maps $f : X \to Y$, and let $\text{Map}^*(X, Y)$ denote the corresponding subspace of all based continuous maps. If $m \geq 2$ and $g \in \text{Map}^*(\mathbb{R}P^{m-1}, X)$, let $F(\mathbb{R}P^m, X; g)$ denote the subspace of $\text{Map}^*(\mathbb{R}P^m, X)$ given by

$$F(\mathbb{R}P^m, X; g) = \{f \in \text{Map}^*(\mathbb{R}P^m, X) : f|\mathbb{R}P^{m-1} = g\},$$

where we identify $\mathbb{R}P^{m-1} \subset \mathbb{R}P^m$ by putting $x_m = 0$. It is known that there is a homotopy equivalence $F(\mathbb{R}P^m, X; g) \simeq \Omega^m X$.

From now on, we assume that the following two conditions are satisfied:

1. Let $\Sigma$ be a fan in $\mathbb{R}^n$, $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ be the set of all one-dimension cones in $\Sigma$, and all primitive elements $\{n_1, \cdots, n_r\}$ of the fan $\Sigma$ spans $\mathbb{R}^n$, where $n_k \in \mathbb{Z}^n$ denotes the primitive element of $\rho_k$ for $1 \leq k \leq r$.

2. Let $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an $r$-tuple of integers such that $\sum_{k=1}^{r} d_k n_k = 0$.

Then, we can identify $X_\Sigma = (\mathbb{C}^r \setminus Z_\Sigma)/G_{\Sigma}$ as above. For each $(a_1, \cdots, a_r) \in \mathbb{C}^r \setminus Z_\Sigma$, we denote by $[a_1, \cdots, a_r]$ the corresponding element of $X_\Sigma$. Let $\mathcal{H}_{d,m} \subset \mathbb{C}[z_0, \cdots, z_m]$ denote the subspace consisting of all homogeneous polynomials of degree $d$. Let $A_D(m)$ denote the space

$$A_D(m) = \mathcal{H}_{d_1,m} \times \mathcal{H}_{d_2,m} \times \cdots \times \mathcal{H}_{d_r,m}$$

and let $A_{D,\Sigma}(m) \subset A_D(m)$ denote the subspace consisting of all $r$-tuples $(f_1, \cdots, f_r) \in A_D(m)$ such that $(f_1(x), \cdots, f_r(x)) \notin Z_\Sigma$ for any $x \in \mathbb{R}^{m+1} \setminus \{0\}$. Let $x_0 \in X_\Sigma$ be the base point such that $x_0 = [x_{1,0}, \cdots, x_{r,0}]$ for some fixed $(x_{1,0}, \cdots, x_{r,0}) \in \mathbb{C}^r \setminus Z_\Sigma$. Then let $A_D(m, X_\Sigma) \subset A_{D,\Sigma}(m)$ denote
the subspace consisting of all $r$-tuples $(f_1, \cdots, f_r) \in A_{D, \Sigma}(m)$ satisfying the condition $(f_1(e_1), \cdots, f_r(e_1)) = (x_{10}, \cdots, x_{r0})$, where $e_1 = (1,0,\cdots,0) \in \mathbb{R}^{m+1}$, and let us choose $[e_1] = [1 : 0 : \cdots : 0]$ as the base-point of $\mathbb{R}P^m$.

Define the natural map $j''_D : A_{D, \Sigma}(m) \to \text{Map}(\mathbb{R}P^m, X_\Sigma)$ by

$$j''_D(f_1, \cdots, f_r)([x_0 : \cdots : x_m]) = [f_1(x), \cdots, f_r(x)]$$

for $x = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}$. Since the space $A_{D, \Sigma}(m)$ is connected, the image of $j''_D$ lies in a connected component of $\text{Map}(\mathbb{R}P^m, X_\Sigma)$, which is denoted by $\text{Map}_D(\mathbb{R}P^m, X_\Sigma)$.

This also gives the natural map $i''_D : A_{D, \Sigma}(m) \to \text{Map}_D^{*}(\mathbb{R}P^m, X_\Sigma)$. Note that $j''_D(f_1, \cdots, f_r) \in \text{Map}_D^*(\mathbb{R}P^m, X_\Sigma)$ if $(f_1, \cdots, f_r) \in A_D(m, X_\Sigma)$. Hence, if we set $\text{Map}_D^*(\mathbb{R}P^m, X_\Sigma) = \text{Map}_D^*(\mathbb{R}P^m, X_\Sigma) \cap \text{Map}_D(\mathbb{R}P^m, X_\Sigma)$, we have the natural map $i''_D = j''_D|A_D(m, X_\Sigma) : A_D(m, X_\Sigma) \to \text{Map}_D^*(\mathbb{R}P^m, X_\Sigma)$.

Suppose that $m \geq 2$ and let us choose a fixed element $(g_1, \cdots, g_r) \in A_D(m-1, X_\Sigma)$. For each $1 \leq k \leq r$, let $B_k = \{g_k + zm : h \in \mathcal{H}_{d_k-1,m}\}$. Then define the subspace $A_D(m, X_\Sigma; g) \subset A_D(m, X_\Sigma)$ by

$$A_D(m, X_\Sigma; g) = A_D(m, X_\Sigma) \cap (B_1 \times B_2 \times \cdots \times B_r).$$

It is easy to see that $i''_D(f_1, \cdots, f_r)|_{\mathbb{R}P^{m-1}} = g$ if $(f_1, \cdots, f_r) \in A_D(m, X_\Sigma; g)$, where $g$ denotes the map in $\text{Map}_D^*(\mathbb{R}P^{m-1}, X_\Sigma)$ given by

$$g([x_0 : \cdots : x_{m-1}]) = [g_1(x), \cdots, g_r(x)] \quad \text{for} \quad x = (x_0, \cdots, x_{m-1}) \in \mathbb{R}^m \setminus \{0\}.$$

Then, define the map $i''_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}P^m, X_\Sigma)$ by the restriction $i''_D = i''_D|A_D(m, X_\Sigma; g)$. Now define the equivalence relation "≈" on $A_{D, \Sigma}(m)$ by $(f_1, \cdots, f_r) \sim (g_1, \cdots, g_r)$ if there exists some element $\lambda \in \mathbb{R}^*$ such that $f_k = \lambda^{d_k}g_k$ for any $1 \leq k \leq r$. We denote by $A_D(m, X_\Sigma)$ the quotient space $A_D(m, X_\Sigma) = A_{D, \Sigma}(m)/\sim$. Then define the map $j_D : A_D(m, X_\Sigma) \to \text{Map}_D(\mathbb{R}P^m, X_\Sigma)$ by $j_D([f_1, \cdots, f_r])([x_0, \cdots, x_r]) = [f_1(x), \cdots, f_r(x)]$ for $x = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}$.

A map $f : \mathbb{R}P^m \to X_\Sigma$ is called an algebraic map of degree $D$ if it can be represented as a rational map (or regular map) of the form

$$f = j''_D(f_1, \cdots, f_r) = [f_1, \cdots, f_r] \quad \text{for some} \quad (f_1, \cdots, f_r) \in A_{D, \Sigma}(m).$$

We denote by $\text{Alg}_D(\mathbb{R}P^m, X_\Sigma)$ the space of all algebraic maps $f : \mathbb{R}P^m \to X_\Sigma$ of degree $D$. Consider the natural projection $\Gamma'_D : A_{D, \Sigma}(m) \to \text{Alg}_D(\mathbb{R}P^m, X_\Sigma)$ given by $\Gamma'_D(f_1, \cdots, f_r) = j''_D(f_1, \cdots, f_r) = [f_1, \cdots, f_r]$. Then it clearly induces a natural projection $\Gamma_D : A_D(m, X_\Sigma) \to \text{Alg}_D(\mathbb{R}P^m, X_\Sigma)$. 
For \( g \in \text{Alg}^*_{D}(\mathbb{R}P^{m-1}, X_{\Sigma}) \), let \( \text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \) and \( \text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}; g) \) denote the subspaces of \( \text{Alg}_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \) given by

\[
\left\{ \begin{align*}
\text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}) & = \text{Alg}_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \cap \text{Map}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \\
\text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}; g) & = \text{Alg}_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \cap F(\mathbb{R}P^{m}, X_{\Sigma}; g)
\end{align*} \right\
\]

Then the projection \( \Gamma'_D \) induces the projection maps by the restrictions

\[
\left\{ \begin{align*}
\Psi_D : A_D(m, X_{\Sigma}) & \to \text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \\
\Psi_D' : A_D(m, X_{\Sigma}; g) & \to \text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}; g)
\end{align*} \right\
\]

Let

\[
\left\{ \begin{align*}
j_{D,C} : \text{Alg}_{D}(\mathbb{R}P^{m}, X_{\Sigma}) & \to \text{Map}_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \\
i_{D,C} : \text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}) & \to \text{Map}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}) \\
i_D' : \text{Alg}^*_{D}(\mathbb{R}P^{m}, X_{\Sigma}; g) & \to F(\mathbb{R}P^{m}, X_{\Sigma}; g) \simeq \Omega^m X_{\Sigma}
\end{align*} \right\
\]

denote the inclusions. It is easy to see that the following equalities hold:

\[
\left\{ \begin{align*}
j_D = j_{D,C} \circ \Gamma_D : \tilde{A}_D(m, X_{\Sigma}) & \to \text{Map}_D(\mathbb{R}P^{m}, X_{\Sigma}) \\
i_D = i_{D,C} \circ \Psi_D : A_D(m, X_{\Sigma}) & \to \text{Map}^*_D(\mathbb{R}P^{m}, X_{\Sigma}) \\
i_D' = i_D' \circ \Psi_D' : A_D(m, X_{\Sigma}; g) & \to F(\mathbb{R}P^{m}, X_{\Sigma}; g) \simeq \Omega^m X_{\Sigma}
\end{align*} \right\
\]

Let \( g \in \text{Alg}^*_{D}(\mathbb{R}P^{m-1}, X_{\Sigma}) \) be any fixed algebraic map of degree \( D \) and we choose an element \( (g_1, \cdots, g_r) \in A_D(m-1, X_{\Sigma}) \) such that \( g = [g_1, \cdots, g_r] \).

Now we can state the our main result as follows.

**Theorem 1.1** ([14]). Let \( D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r \) and let \( \Sigma \) be a complete smooth fan in \( \mathbb{R}^n \) satisfying the above conditions (1.1) and (1.2). Then if \( 2 \leq m \leq 2(r_{\min} - 1) \) and \( X_{\Sigma} \) is a smooth compact toric variety, the maps

\[
\left\{ \begin{align*}
j_D : \tilde{A}_D(m, X_{\Sigma}) & \to \text{Map}_D(\mathbb{R}P^{m}, X_{\Sigma}) \\
i_D : A_D(m, X_{\Sigma}) & \to \text{Map}^*_D(\mathbb{R}P^{m}, X_{\Sigma}) \\
i_D' : A_D(m, X_{\Sigma}; g) & \to F(\mathbb{R}P^{m}, X_{\Sigma}; g) \simeq \Omega^m X_{\Sigma}
\end{align*} \right\
\]

are homology equivalences through dimension \( D(d_1, \cdots, d_r; m) \), where the number \( D(d_1, \cdots, d_r; m) \) is given by

\[
D(d_1, \cdots, d_r; m) = (2r_{\min} - m - 1) \min\{d_1, d_2, \cdots, d_r\} - 2.
\]

**Remark.** A map \( f : X \to Y \) is called a homology equivalence through dimension \( N \) if the induced homomorphism \( f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z}) \) is an isomorphism for any \( k \leq N \).
References


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