The space of maps from a real projective space to a toric variety

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Abstract

The main purpose of this note is consider the homotopy type of the space of algebraic maps from a real projective space to a projective smooth toric variety as in [14]. The main result of this paper (Theorem 1.1) is also regarded as one of generalizations of the previous work of the second and third authors [19].

An irreducible normal algebraic variety $X$ (over $\mathbb{C}$) is called a toric variety if it has an algebraic action of algebraic torus $\mathbb{T}^r = (\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}^r \cdot *$ of some point $* \in X$ is dense in $X$ and isomorphic to $\mathbb{T}^r$. A finite correction $\Sigma$ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called a fan if every face of element of $\Sigma$ is belongs to $\Sigma$ and the intersection of any two elements of $\Sigma$ is a face of each. It is known that A toric variety $X$ is completely characterized up to isomorphism by its fan $\Sigma$, and we denote by $X_{\Sigma}$ the corresponding toric variety. For an $n$ dimensional lattice polytope $P$, we denote by $\Sigma_P$ the normal fan of $P$ in $\mathbb{R}^n$. It is known that the toric variety $X_{\Sigma}$ is projective if and only if $\Sigma = \Sigma_P$ for some $n$ dimensional lattice polytope $P$ in $\mathbb{R}^n$.

We shall use the symbols $\{z_k\}_{k=1}^r$ to denote variables of polynomials, and for $f_1, \cdots, f_s \in \mathbb{C}[z_1, \cdots, z_r]$, let $V(f_1, \cdots, f_s)$ denote the affine variety $V(f_1, \cdots, f_s) = \{x \in \mathbb{C}^r \mid f_k(x) = 0 \text{ for each } 1 \leq k \leq s\}$.

Let $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ denote the set of all one dimensional cones (or called a ray) in a fan $\Sigma$, and let $n_k \in \mathbb{Z}^n$ denote the generator of $\rho_k \cap \mathbb{Z}^n$ called the primitive element of $\rho_k$ for each $1 \leq k \leq r$. Define the affine variety $Z_{\Sigma} \subset \mathbb{C}^r$ by $Z_{\Sigma} = V(z^\sigma \mid \sigma \in \Sigma)$, where $z^\sigma$ denotes the monomial given by $z^\sigma = \prod_{1 \leq k \leq r, n_k \not\in \sigma} z_k \in \mathbb{Z}[z_1, \cdots, z_r]$ ($\sigma \in \Sigma$). Let $G_{\Sigma} \subset \mathbb{T}^r$ denote the subgroup consisting of all $r$-tuples $(\mu_1, \cdots, \mu_r) \in \mathbb{T}^r$ such that $\prod_{k=1}^r \mu_k^{(m,n_k)} = \cdots$
1 for any \( m \in \mathbb{Z}^n \), where we set \( \langle x, y \rangle = \sum_{k=1}^n x_k y_k \) for \( x = (x_1, \cdots, x_n) \), \( y = (y_1, \cdots, y_n) \in \mathbb{R}^n \). We say that a set of primitive elements \( \{n_1, \cdots, n_s\} \) is \textit{primitive} if they do not lie in any cone in \( \Sigma \) but every proper subset does. It is known that

\[
Z_{\Sigma} = \bigcup_{\{n_1, \cdots, n_s\} : \text{primitive}} V(z_1, \cdots, z_s).
\]

Note that \( Z_{\Sigma} \) is a closed variety of dimension \( 2(r - r_{\min}) \), where we set

\[
r_{\min} = \min\{ s \in \mathbb{Z}_{\geq 1} \mid \{n_1, \cdots, n_s\} \text{ is primitive} \}.
\]

It is also known that if the set \( \{n_1, \cdots, n_r\} \) spans \( \mathbb{R}^n \), there is an isomorphism \( X_{\Sigma} \cong (\mathbb{C}^r \setminus Z_{\Sigma})/G_{\Sigma} \), where the group \( G_{\Sigma} \) acts on the complement \( \mathbb{C}^r \setminus Z_{\Sigma} \) by the coordinate-wise multiplication.

For connected spaces \( X \) and \( Y \), let \( \text{Map}(X, Y) \) be the space of all continuous maps \( f : X \to Y \), and let \( \text{Map}^*(X, Y) \) denote the corresponding subspace of all based continuous maps. If \( m \geq 2 \) and \( g \in \text{Map}^*(\mathbb{R}P^{m-1}, X) \), let \( F(\mathbb{R}P^m, X; g) \) denote the subspace of \( \text{Map}^*(\mathbb{R}P^m, X) \) given by

\[
F(\mathbb{R}P^m, X; g) = \{ f \in \text{Map}^*(\mathbb{R}P^m, X) : f|\mathbb{R}P^{m-1} = g \},
\]

where we identify \( \mathbb{R}P^{m-1} \subset \mathbb{R}P^m \) by putting \( x_m = 0 \). It is known that there is a homotopy equivalence \( F(\mathbb{R}P^m, X; g) \simeq \Omega^m X \).

From now on, we assume that the following two conditions are satisfied:

(1.1) Let \( \Sigma \) be a fan in \( \mathbb{R}^n \), \( \Sigma(1) = \{ \rho_1, \cdots, \rho_r \} \) be the set of all one-dimension cones in \( \Sigma \), and all primitive elements \( \{n_1, \cdots, n_r\} \) of the fan \( \Sigma \) spans \( \mathbb{R}^n \), where \( n_k \in \mathbb{Z}^n \) denotes the primitive element of \( \rho_k \) for \( 1 \leq k \leq r \).

(1.2) Let \( D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r \) be an \( r \)-tuple of integers such that \( \sum_{k=1}^r d_k n_k = 0 \).

Then, we can identify \( X_{\Sigma} = (\mathbb{C}^r \setminus Z_{\Sigma})/G_{\Sigma} \) as above. For each \( (a_1, \cdots, a_r) \in \mathbb{C}^r \setminus Z_{\Sigma} \), we denote by \( [a_1, \cdots, a_r] \) the corresponding element of \( X_{\Sigma} \). Let \( \mathcal{H}_{d,m} \subset \mathbb{C}[z_0, \cdots, z_m] \) denote the subspace consisting of all homogeneous polynomials of degree \( d \). Let \( A_D(m) \) denote the space

\[
A_D(m) = \mathcal{H}_{d_1,m} \times \mathcal{H}_{d_2,m} \times \cdots \times \mathcal{H}_{d_r,m}
\]

and let \( A_{D,\Sigma}(m) \) denote the subspace consisting of all \( r \)-tuples \( (f_1, \cdots, f_r) \in A_D(m) \) such that \( (f_1(x), \cdots, f_r(x)) \notin Z_{\Sigma} \) for any \( x \in \mathbb{R}^{m+1} \setminus \{0\} \). Let \( x_0 \in X_{\Sigma} \) be the base point such that \( x_0 = [x_{1,0}, \cdots, x_{r,0}] \) for some fixed \( (x_{1,0}, \cdots, x_{r,0}) \in \mathbb{C}^r \setminus Z_{\Sigma} \). Then let \( A_D(m, X_{\Sigma}) \subset A_{D,\Sigma}(m) \) denote
the subspace consisting of all $r$-tuples $(f_1, \cdots, f_r) \in A_{D, \Sigma}(m)$ satisfying the condition $(f_1(e_1), \cdots, f_r(e_1)) = (x_{10}, \cdots, x_{r0})$, where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^{m+1}$, and let us choose $[e_1] = [1 : 0 : \cdots : 0]$ as the base-point of $\mathbb{R}P^m$.

Define the natural map $j'_D : A_{D, \Sigma}(m) \to \text{Map}(\mathbb{R}P^m, X_\Sigma)$ by

$$j'_D(f_1, \cdots, f_r)([x_0 : \cdots : x_m]) = [f_1(x), \cdots, f_r(x)]$$

for $x = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}$. Since the space $A_{D, \Sigma}(m)$ is connected, the image of $j'_D$ lies in a connected component of $\text{Map}(\mathbb{R}P^m, X_\Sigma)$, which is denoted by $\text{Map}_D(\mathbb{R}P^m, X_\Sigma)$.

This also gives the natural map $j'_D : A_{D, \Sigma}(m) \to \text{Map}_D^{*}(\mathbb{R}P^m, X_\Sigma)$. Note that $j'_D(f_1, \cdots, f_r) \in \text{Map}^{*}(\mathbb{R}P^m, X_\Sigma)$ if $(f_1, \cdots, f_r) \in A_D(m, X_\Sigma)$. Hence, if we set $\text{Map}^{*}_D(\mathbb{R}P^m, X_\Sigma) = \text{Map}^{*}(\mathbb{R}P^m, X_\Sigma) \cap \text{Map}_D(\mathbb{R}P^m, X_\Sigma)$, we have the natural map $i_D = j'_D|A_D(m, X_\Sigma) : A_D(m, X_\Sigma) \to \text{Map}^{*}_D(\mathbb{R}P^m, X_\Sigma)$.

Suppose that $m \geq 2$ and let us choose a fixed element $(g_1, \cdots, g_r) \in A_D(m-1, X_\Sigma)$. For each $1 \leq k \leq r$, let $B_k = \{g_k + z_m h : h \in \mathcal{H}_{d_k-1,m}\}$. Then define the subspace $A_D(m, X_\Sigma; g) \subset A_D(m, X_\Sigma)$ by

$$A_D(m, X_\Sigma; g) = A_D(m, X_\Sigma) \cap (B_1 \times B_2 \times \cdots \times B_r).$$

It is easy to see that $i_D(f_1, \cdots, f_r)|\mathbb{R}P^{m-1} = g$ if $(f_1, \cdots, f_r) \in A_D(m, X_\Sigma; g)$, where $g$ denotes the map in $\text{Map}^{*}_D(\mathbb{R}P^{m-1}, X_\Sigma)$ given by

$$g([x_0 : \cdots : x_{m-1}]) = [g_1(x), \cdots, g_r(x)] \quad \text{for } x = (x_0, \cdots, x_{m-1}) \in \mathbb{R}^m \setminus \{0\}.$$

Then, define the map $i'_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}P^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma$ by the restriction $i'_D = i_D|A_D(m, X_\Sigma; g)$. Now define the equivalence relation "~" on $A_{D_{\Sigma}}(m)$ by $(f_1, \cdots, f_r) \sim (g_1, \cdots, g_r)$ if there exists some element $\lambda \in \mathbb{R}^*$ such that $f_k = \lambda^k g_k$ for any $1 \leq k \leq r$. We denote by $\overline{A_D}(m, X_\Sigma)$ the quotient space $A_D(m, X_\Sigma) = A_{D, \Sigma}(m)/\sim$. Then define the map $j_D : \overline{A_D}(m, X_\Sigma) \to \text{Map}_D(\mathbb{R}P^m, X_\Sigma)$ by $j_D([f_1, \cdots, f_r])([x_0, \cdots, x_r]) = [f_1(x), \cdots, f_r(x)]$ for $x = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}$.

A map $f : \mathbb{R}P^m \to X_\Sigma$ is called an algebraic map of degree $D$ if it can be represented as a rational map (or regular map) of the form

$$f = j'_D(f_1, \cdots, f_r) = [f_1, \cdots, f_r] \quad \text{for some } (f_1, \cdots, f_r) \in A_{D, \Sigma}(m).$$

We denote by $\text{Alg}_D(\mathbb{R}P^m, X_\Sigma)$ the space of all algebraic maps $f : \mathbb{R}P^m \to X_\Sigma$ of degree $D$. Consider the natural projection $\Gamma'_D : A_{D, \Sigma}(m) \to \text{Alg}_D(\mathbb{R}P^m, X_\Sigma)$ given by $\Gamma'_D(f_1, \cdots, f_r) = j'_D(f_1, \cdots, f_r) = [f_1, \cdots, f_r]$. Then it clearly induces a natural projection $\Gamma_D : A_D(m, X_\Sigma) \to \text{Alg}_D(\mathbb{R}P^m, X_\Sigma)$. 
For $g \in \text{Alg}^*_D(\mathbb{R}P^{m-1}, X_\Sigma)$, let $\text{Alg}^*_D(\mathbb{R}P^m, X_\Sigma)$ and $\text{Alg}(\mathbb{R}P^m, X_\Sigma; g)$ denote the subspaces of $\text{Alg}_D(\mathbb{R}P^m, X_\Sigma)$ given by

\[
\begin{align*}
\text{Alg}^*_D(\mathbb{R}P^m, X_\Sigma) &= \text{Alg}_D(\mathbb{R}P^m, X_\Sigma) \cap \text{Map}^*(\mathbb{R}P^m, X_\Sigma) \\
\text{Alg}_D(\mathbb{R}P^m, X_\Sigma; g) &= \text{Alg}_D(\mathbb{R}P^m, X_\Sigma) \cap F(\mathbb{R}P^m, X_\Sigma; g)
\end{align*}
\]

Then the projection $\Gamma_D'$ induces the projection maps by the restrictions

\[
\begin{align*}
\Psi_D : A_D(m, X_\Sigma) &\to \text{Alg}^*_D(\mathbb{R}P^m, X_\Sigma) \\
\Psi_D : A_D(m, X_\Sigma; g) &\to \text{Alg}^*_D(\mathbb{R}P^m, X_\Sigma; g)
\end{align*}
\]

Let $j_{D,C} : \text{Alg}_D(\mathbb{R}P^m, X_\Sigma) \to \text{Map}_D(\mathbb{R}P^m, X_\Sigma)$

\[
\begin{align*}
j_D &= j_{D,C} \circ \Gamma_D : \tilde{A}_D(m, X_\Sigma) \to \text{Map}_D(\mathbb{R}P^m, X_\Sigma) \\
i_D &= i_{D,C} \circ \Psi_D : A_D(m, X_\Sigma) \to \text{Map}_D^*(\mathbb{R}P^m, X_\Sigma) \\
i'_D &= i'_{D,C} \circ \Psi_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}P^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma
\end{align*}
\]

denote the inclusions. It is easy to see that the following equalities hold:

\[
\begin{align*}
j_D &= j_{D,C} \circ \Gamma_D : \tilde{A}_D(m, X_\Sigma) \to \text{Map}_D(\mathbb{R}P^m, X_\Sigma) \\
i_D &= i_{D,C} \circ \Psi_D : A_D(m, X_\Sigma) \to \text{Map}_D^*(\mathbb{R}P^m, X_\Sigma) \\
i'_D &= i'_{D,C} \circ \Psi_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}P^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma
\end{align*}
\]

Let $g \in \text{Alg}^*_D(\mathbb{R}P^{m-1}, X_\Sigma)$ be any fixed algebraic map of degree $D$ and we choose an element $(g_1, \cdots, g_r) \in A_D(m - 1, X_\Sigma)$ such that $g = [g_1, \cdots, g_r]$.

Now we can state the our main result as follows.

**Theorem 1.1** ([14]). Let $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ and let $\Sigma$ be a complete smooth fan in $\mathbb{R}^n$ satisfying the above conditions (1.1) and (1.2). Then if $2 \leq m \leq 2(r_{\min} - 1)$ and $X_\Sigma$ is a smooth compact toric variety, the maps

\[
\begin{align*}
j_D : \tilde{A}_D(m, X_\Sigma) &\to \text{Map}_D(\mathbb{R}P^m, X_\Sigma) \\
i_D : A_D(m, X_\Sigma) &\to \text{Map}_D^*(\mathbb{R}P^m, X_\Sigma) \\
i'_D : A_D(m, X_\Sigma; g) &\to F(\mathbb{R}P^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma
\end{align*}
\]

are homology equivalences through dimension $D(d_1, \cdots, d_r; m)$, where the number $D(d_1, \cdots, d_r; m)$ is given by

\[
D(d_1, \cdots, d_r; m) = (2r_{\min} - m - 1) \min\{d_1, d_2, \cdots, d_r\} - 2. \quad \square
\]

**Remark.** A map $f : X \to Y$ is called a homology equivalence through dimension $N$ if the induced homomorphism $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$ is an isomorphism for any $k \leq N$. 


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