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<td>著者</td>
<td>Morimoto, Masaharu</td>
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<tr>
<td>引用</td>
<td>数理解析研究所講究録 1876: 112-119</td>
</tr>
<tr>
<td>発行日</td>
<td>2014-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195566">http://hdl.handle.net/2433/195566</a></td>
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<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>publisher</td>
</tr>
<tr>
<td>管理機関</td>
<td>Kyoto University</td>
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AN EQUIVARIANT TRANSVERSALITY THEOREM
AND ITS APPLICATIONS

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To the memory of the late Professor Minoru Nakaoka

Abstract. Let $G$ be a finite group. In this article, we recall an equivariant transversality theorem and discuss its applications to semifree $G$-actions on closed manifolds and to Smith-equivalent real $G$-modules.

1. INTRODUCTION

Unless otherwise stated, let $G$ be a finite group. We mean by a manifold a paracompact smooth manifold. A submanifold, $M$ say, of a manifold, $N$ say, should be read as a regular smooth submanifold such that $M$ is a closed subset of $N$. We mean by a $G$-manifold a smooth manifold with a smooth $G$-action. In particular, each connected component of a manifold in the present article is $\sigma$-compact, and an arbitrary $G$-manifold can be equipped with a $G$-invariant Riemannian metric.

Let $M$ and $N$ be manifolds, $B$ a subset of $M$, $Y$ a submanifold of $N$, and $f : M \to N$ a continuous map. We say that $f$ is transversal on $B$ to $Y$ in $N$ if $f$ is smooth on a neighborhood of $f^{-1}(Y) \cap B$ in $M$ and the linear map

$$ T_x(M) \xrightarrow{df_x} T_y(N) \to T_y(N)/T_y(Y) $$

is surjective for every $y \in Y$ and $x \in f^{-1}(y) \cap B$, where $T_x(M)$ stands for the tangent space of $M$ at $x$. There have been obtained several versions of equivariant transversality theorems, e.g. A. Wasserman [19, Lemma 3.3], T. Petrie [16, §1,

2010 Mathematics Subject Classification. Primary 57S17; Secondary 20C15.

Key words and phrases. Transversality theorem, fixed point, tangential representation, Smith equivalent.

This article partially contains results by joint work with Takashi Matsunaga.
In this paper we will discuss applications of the next version.

**Theorem 1.1.** Let $M$ be a $G$-manifold, $N$ a $G$-manifold with a $G$-invariant Riemannian metric, $A$ a $G$-invariant closed subset of $M$, and $Y$ a $G$-submanifold of $N$. Let $f : M \to N$ be a smooth $G$-map transversal on $A$ to $Y$ in $N$. Suppose the $G$-action on $M \setminus A$ is free. Then for an arbitrary $G$-invariant positive continuous function $\delta : M \to \mathbb{R}$, there exists a smooth $G$-map $g : M \to N$ satisfying the following conditions.

1. $g$ is transversal on $M$ to $Y$ in $N$.
2. $g|_A = f|_A$.
3. $d_N(f(x), g(x)) < \delta(x)$ for all $x \in M$, where $d_N$ stands for the distance function on $N$ induced from the Riemannian metric of $N$.

We mean by a real (resp. complex) $G$-module a real (resp. complex) $G$-representation space of finite dimension. For a real $G$-module $V$ (of finite dimension), let $S(V)$ denote the unit sphere of $V$ with respect to some $G$-invariant inner product on $V$. The following two propositions have been known.

**Proposition ([14, Lemma 2.1]).** If $G$ is a group of order 2 and $M$ is a connected closed $G$-manifold of positive dimension then $|M^G| \neq 1$.

**Proposition ([7, Lemma 2.2]).** If $M$ is a connected closed orientable $G$-manifold of positive dimension such that the $G$-action on $M \setminus M^G$ is free, then $|M^G| \neq 1$.

The latter proposition is generalized to the next result.

**Theorem 1.2.** Let $M$ be a connected closed oriented $G$-manifold of dimension $n+1$, and $\Sigma$ an oriented homotopy sphere of dimension $n$. Suppose the $G$-action on $M$ is semifree and preserves the orientation of $M$. If $M^G$ is a finite set then the congruence

$$\sum_{x \in M^G} \deg(f_x) \equiv 0 \mod |G|$$

holds, where $T_x(M)$ is the tangent space of $M$ at $x$ and $f_x$ is an arbitrary (continuous) $G$-map $S(T_x(M)) \to \Sigma$ for each $x \in M^G$. 
Theorem 1.3. Let $M$ be a connected closed oriented $G$-manifold of positive dimension such that the $G$-action on $M$ is semifree and $M^G = \{a, b\}$. Then the spheres $S(T_a(M))$ and $S(T_b(M))$ are $G$-homotopy equivalent to each other.

Corollary 1.4. Let $V$ and $W$ be real $G$-representations satisfying $\dim V = \dim W$. If $S(V) \amalg S(W)$ is the boundary of a compact orientable $G$-manifold $M$ such that the $G$-action on $M$ is free then $S(V)$ and $S(W)$ are $G$-homotopy equivalent to each other.

A homotopy sphere $\Sigma$ with a $G$-action is called a Smith sphere if $\Sigma^G$ consists of exactly 2 points. Two real $G$-modules $V$ and $W$ are said to be Smith equivalent if there exists a Smith sphere $\Sigma$ such that $\Sigma^G = \{a, b\}$ with $V \cong T_a(\Sigma)$ and $W \cong T_b(\Sigma)$ as real $G$-modules.

Theorem 1.5. Let $G$ be a finite group and let $V$ and $W$ be Smith-equivalent real $G$-modules. For any normal subgroup $H$ of $G$ such that $|G/H|$ is a prime and a Sylow 2-subgroup $H_2$ of $H$ is normal in $H$, $S(V^H)$ and $S(W^H)$ are $G/H$-homotopy equivalent to each other.

2. Transversality of maps

In this section, let us recall classical transversality theorems. First, recall a result by A. Wasserman.

Lemma (A. Wasserman [19, Lemma 3.3]). Let $G$ be a compact Lie group, $M$ and $N$ $G$-manifolds, $f : M \to N$ a smooth $G$-map, $W \subset N$ a closed invariant submanifold, and $C$ a closed subset of $M^G$. Suppose $f|_{M^G}$ is transversal on $C$ to $W^G$ in $N^G$. Then there exists a homotopy $f_t$ such that $f_0 = f$, $f_t|_C = f|_C$, and $f_1|_{M^G}$ is transversal on $M^G$ to $W^G$ in $N^G$.

T. Petrie gave several versions and the next one may be most basic.

Proposition (Petrie [16, §1, p.188]). Let $G$ be a compact Lie group. Let $M$, $N$ and $Y \subset N$ be $G$-manifolds and $f : M \to N$ a proper $G$-map. Suppose $f : M \to N$ is transversal to $Y$ on a $G$-neighborhood of a closed subset $Z$ of $M$. Let $K$ be a maximal closed subgroup such that $(M \setminus Z)^K \neq \emptyset$. Then $f$ is $G$-homotopic rel $Z$ to a proper $G$-map $h : M \to N$ such that $h^K$ is transversal on $M^K$ to $Y^K$ in $Z^K$. 

As its proof, T. Petrie wrote as follows. (Note that $N(K)/K$ acts freely on $M^K \setminus Z$.) “This uses the Thom Transversality Lemma [11] for the case of trivial group action and the $G$-homotopy extension lemma [19, Lemma 3.2].” Here the reference [11] should be replaced by an appropriate one.

Another version is obtained by E. Bierstone’s theory, namely from the following three results.

**Theorem** (Bierstone [1, Theorem 1.3]). Let $G$ be a compact Lie group. If $P$ is a closed $G$-submanifold of $N$, then the set of smooth equivariant maps $F : M \to N$ which are in general position with respect to $P$ is open in Whitney topology.

**Theorem** ([1, Theorem 1.4]). Let $G$ be a compact Lie group. If $P$ is an invariant submanifold of $N$, then the set of smooth equivariant maps $F : M \to N$ which are in general position with respect to $P$ is a countable intersection of open dense sets (in the Whitney of $C^\infty$ topology).

**Proposition** ([1, Proposition 6.3]). If a smooth equivariant map $F : M \to N$ is in general position with respect to an invariant submanifold $P$ of $N$, then it is stratumwise transversal to $P$. In other words, for every isotropy subgroup $H$ of $M$, $F|_{M^H} : M^H \to N^H$ is transversal to $P^H$.

Our Theorem 1.1 is an equivariant analogue of A. Hattori [6, Ch.6, §3, Theorem 3.6].

3. Maps between spheres

We mean by a homotopy sphere a closed manifold being homotopy equivalent to a sphere. Let $X$ be a finite $G$-CW complex such that $G$ acts freely on $X$. For a $G$-map $f : X \to X$, the Lefschetz number $L(f)$ is congruent to $0 \mod |G|$. In the case where $X$ is a homotopy sphere of dimension $n$, we have

\[ L(f) = 1 + (-1)^n \deg f \equiv 0 \mod |G|. \]  

Using this property, we can prove the next fact without difficulties.

**Lemma 3.1** ([17, 4, 9]). Let $X$ be a connected homotopy sphere with a free $G$-action. Then for any $G$-map $f : X \to X$, $\deg f$ is congruent to $1 \mod |G|$. 

In addition, by standard arguments using Steenrod's obstruction theory [18], we can prove the next fact.

**Lemma 3.2 ([17, 4]).** Let $X$ and $Y$ be connected homotopy spheres of same dimension with free $G$-actions. Then the following conclusions hold.

1. There exist a $G$-map $X \rightarrow Y$ and a $G$-map $Y \rightarrow X$.
2. For any $G$-maps $f_0, f_1 : X \rightarrow Y$, $\deg f_0 \equiv \deg f_1 \mod |G|$.
3. For any $G$-map $f_0 : X \rightarrow Y$ and any integer $m$, there exists a $G$-map $f_1 : X \rightarrow Y$ such that $\deg f_1 = \deg f_0 + m|G|$.

These lemmas provide the next proposition.

**Proposition 3.3.** Let $X$ and $Y$ be connected homotopy spheres of same dimension with free $G$-actions and let $f : X \rightarrow Y$ be a $G$-map. Then $\deg f$ is prime to $|G|$.

**Proof.** By Lemma 3.2, there is a $G$-map $g : Y \rightarrow X$. Moreover by Lemma 3.1 we have

$$\deg(g \circ f) \equiv 1 \mod |G|$$

and $\deg(g \circ f) = \deg g \cdot \deg f$. Thus $\deg f$ is prime to $|G|$.

$$\square$$

4. TANGENTIAL REPRESENTATIONS

Let $V$ be a real $G$-module such that the $G$-action on $V \setminus \{0\}$ is free. We adopt an orientation of the ambient space of $V$. Let $\Sigma$ be an oriented homotopy sphere equipped with a free smooth $G$-action such that $\dim \Sigma = \dim S(V)$. Then there exists a smooth $G$-map $f_{V,\Sigma} : S(V) \rightarrow \Sigma$ and $\deg(f_{V,\Sigma})$ is prime to $|G|$.

**Proof of Theorem 1.2.** Let us fix an arbitrary point $a \in M^G$. The tangential representation $V = T_a(M)$ has the orientation inherited from that of $M$. Since the $G$-action on $S(V)$ preserves the orientation, $\dim S(V)$ is odd. Without any loss of generality, we can assume $\Sigma = S(V)$. Set $Y = S(\mathbb{R} \oplus V)$. There is a canonical orientation preserving $G$-diffeomorphism from the $G$-disk $D(V)$ to the upper hemisphere $S_+$ of $Y$. This diffeomorphism carries the center of $D(V)$ to the north pole $p_+ = (1,0)$ of $Y$, where $1 \in \mathbb{R}$ and $0 \in V$. Take small $G$-disk neighborhoods $D_x (\cong D(T_x(M)))$ of points $x \in M^G$ in $M$, respectively, so that $D_{x_1} \cap D_{x_2} = \emptyset$.
for distinct $x_1, x_2 \in M^G$. For each point $x \in M^G$, there is a smooth $G$-map $f_x : \partial D_x \to S(V) = \partial S_+$. Let $Df_x : D_x \to D(V) = S_+$ denote the radial extension of the map $f_x$. Clearly $Df_x$ is transversal on a color neighborhood of $\partial D_x$ to $p_+$ in $Y$. In addition, it holds that

$$\deg(Df_x : (D_x, \partial D_x) \to (S_+, \partial S_+)) = \deg(f_x : \partial D_x \to \partial S_+).$$

Set $X = M \setminus \bigsqcup_{x \in M^G} \text{Int}(D_x)$. Then the $G$-action on $X$ is free. Since $S_+$ is contractible, the $G$-map $\bigsqcup_{x \in M^G} Df_x$ extends to a continuous $G$-map $f : M \to S_+$ such that $f$ is smooth on $X$. We will regard $f$ as a map $M \to Y$ as well. For a $G$-invariant positive function $\delta : M \to \mathbb{R}$, take a $G$-equivariant $\delta$-approximation $g : M \to Y$ of $f$ such that

1. $g$ is $G$-homotopic to $f$ relatively to $\bigsqcup_{x \in M^G} D_x$, and
2. $g|_X$ is smooth and transversal on $X$ to $\{p_+\}$ in $Y$.

Since the $G$-action on $g^{-1}(p_+) \cap X$ is free, each $G$-orbit in $g^{-1}(p_+) \cap X$ consists of $|G|$ points. Thus it holds that

$$\deg(g) \equiv \sum_{x \in M^G} \deg(f_x : \partial D_x = S(T_x(M)) \to S(V)) \mod |G|.$$

On the other hand, the equality $\deg(g) = \deg(f) = 0$ follows from the fact that $f : M \to Y$ is not a surjection. Hence we can conclude

$$\sum_{x \in M^G} \deg(f_x) \equiv 0 \mod |G|.$$

Proof of Theorem 1.3. Set $V = T_a(M)$ and $W = T_b(M)$. Then $G$ freely acts on $S(V)$ and $S(W)$.

If $|G| = 2$ then $V$ and $W$ are isomorphic as real $G$-representations, and hence $S(V)$ and $S(W)$ are $G$-diffeomorphic.

Thus we consider the other case, namely one where $|G| \geq 3$. The real $G$-modules $V$ and $W$ have the inherited orientations from that of $M$, respectively. Since the $G$-action on $S(V)$ preserves the orientation, so does the $G$-action on $M$. Let $f_{V,V}$ be the identity map on $S(V)$ and take a smooth $G$-map $f_{W,V} : S(W) \to S(V)$. By
Theorem 1.2, we get
\[ \deg(f_{V,V}) + \deg(f_{W,V}) = 1 + \deg(f_{W,V}) \equiv 0 \mod |G|, \]
and hence \( \deg(f_{W,V}) \equiv -1 \mod |G| \). Thus there is a smooth \( G \)-map \( f : S(W) \to S(V) \) satisfying \( \deg(f) = -1 \). On the other hand, there exists a smooth \( G \)-map \( h : S(V) \to S(W) \). We have \( \deg(h \circ f) \equiv 1 \mod |G| \) and hence \( \deg(h) \equiv -1 \mod |G| \). There exists a smooth \( G \)-map \( g : S(V) \to S(W) \) such that \( \deg(g) = -1 \). These \( f \) and \( g \) are \( G \)-homotopy inverses to each other.

Proof of Theorem 1.5. Set \( \Sigma^G = \{a, b\} \). If a connected component \( A \) of \( \Sigma^H \) containing either \( a \) or \( b \) has positive dimension then by Proposition 1 \( A \) contains both \( a \) and \( b \). By Theorem 1.3, \( S(V^H) \) and \( S(W^H) \) are \( G/H \)-homotopy equivalent. If \( \dim V^H = 0 \) and \( \dim W^H = 0 \) both hold then \( S(V^H) \) and \( S(W^H) \) are the empty set and hence they are \( G/H \)-homotopy equivalent.

\[ \square \]

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