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Kyoto University
HOMOTOPY COMMUTATIVITY IN LOCALIZED GAUGE GROUPS

DAISUKE KISHIMOTO

1. INTRODUCTION AND STATEMENT OF THE RESULT

This is a survey the paper [KKTh] written with Akira Kono and Stephen Theriault.

Throughout the paper, we only consider the Lie group $G = SU(n)$ for simplicity, while most results hold for other simply connected, simple Lie groups. Let us recall $p$-local properties of $G$.

**Theorem 1.1** (Mimura, Nishida and Toda [MNT]). There exist $p$-local spaces $B_1, \ldots, B_{p-1}$ satisfying

$$G_{(p)} \simeq B_1 \times \cdots \times B_{p-1},$$

where the mod $p$ cohomology of $B_i$ is given by

$$H^*(B_i; \mathbb{Z}/p) = \Lambda(x_{2i+1+2k(p-1)} | 0 \leq k < \frac{n-i-1}{p-1}), \quad |x_j| = j.$$

This is called the mod $p$ decomposition of $G$. Observe that if $p \geq n$, each $B_i$ has the homotopy type of $S_{(p)}^{2i+1}$ or a point. Then we can say that the $p$-local homotopy type of $G$ degenerates as $p$ gets larger. So it is natural to consider degeneration of the $H$-structure of $G_{(p)}$ as $p$ gets larger. As for homotopy commutativity, the complete answer was given by McGibbon [M] as:

**Theorem 1.2** (McGibbon [M]). $G_{(p)}$ is homotopy commutative if and only if $p > 2n$.

Later, this result was generalized by Kaji and Kishimoto [KaKi] and Kishimoto [Ki] to homotopy nilpotency.

Our object to study is a gauge group which is the topological group of all automorphisms of a principal bundle, i.e. self-maps of the total space which are compatible with the action of the fiber and cover the identity map of the base space. Recall that principal $G$-bundles over $S^4$ are classified by $\pi_4(BG) \cong \mathbb{Z}$. We write the gauge group of the principal $G$-bundle over $S^4$ corresponding to the integer $k \in \mathbb{Z} \cong \pi_4(BG)$ by $G_k$. The homotopy theory of gauge groups has been studied in many directions (cf. [CS, Ko, KiKo]). In each work, we have seen that $G_k$ has a close relation with $G$ as is expected from definition. So we may expect that $G_k$ possesses $p$-local properties analogous to $G$. As for the mod $p$ decomposition, our expectation has been proved to be true.

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Theorem 1.3 (Kishimoto, Kono and Tsutaya [KKTs]). There exist $p$-local spaces $B_1, \ldots, B_{p-1}$ satisfying
\[ \mathcal{G}_{k(p)} \simeq B_1 \times \cdots \times B_{p-1} \]
and homotopy fibrations
\[ \Omega(\Omega_0^3 B_i) \to B_i \to B_{i-2}, \]
where we regard the spaces $B_i$ of Theorem 1.1 are indexed by $\mathbb{Z}/(p-1)$. Moreover, the homotopy fibrations are trivial if $p \geq n + 2$.

In particular, we can say that the $p$-local homotopy type of $\mathcal{G}_k$ degenerates as $p$ gets larger, analogously to $G$.

Now we naturally ask whether there is a gauge group version of Theorem 1.2. Let us state our main result.

Theorem 1.4. Suppose $n \geq 4$.

1. For $p < 2n + 1$, $\mathcal{G}_{k(p)}$ is not homotopy commutative.
2. For $p > 2n + 1$, $\mathcal{G}_{k(p)}$ is homotopy commutative.
3. For $p = 2n + 1$, $\mathcal{G}_{k(p)}$ is homotopy commutative if and only if $p$ divides $k$.

Remark 1.5. Note that the integer $k$ only appears in the border case $p = 2n + 1$.

2. Noncommutativity

In this section, we give a sketch of the proof of the noncommutativity result on $\mathcal{G}_{k(p)}$. We first recall basic facts of gauge groups briefly. Let $\epsilon_i$ be a generator of $\pi_{2i-1}(G) \cong \mathbb{Z}$ for $i = 2, \ldots, n$. Recall that there is a natural homotopy equivalence
\[ BG_k \simeq \text{map}(S^4, BG; k\bar{\epsilon}_2), \]
where $\text{map}(X,Y;f)$ stands for the connected component of the space of maps from $X$ to $Y$ containing a map $f : X \to Y$ and $\bar{\epsilon}_2 : S^4 \to BG$ is the adjoint of $\epsilon_2$. See [AB]. Then the evaluation map $\text{map}(S^4, BG; k\bar{\epsilon}_2) \to BG$ induces a homotopy fibration
\[ \mathcal{G}_k \xrightarrow{\pi} G \xrightarrow{\delta} \Omega_0^3 G, \]
where $\pi$ is a loop map. The map $\delta$ is identified as:

Lemma 2.1 (Whitehead [W]). The map $\delta$ is the adjoint of the Samelson product $\langle \epsilon_2, 1_G \rangle$.

Hereafter, everything will be localized at the prime $p$.

We now sketch the proof of noncommutativity of $\mathcal{G}_k$. Suppose that there are $2 \leq i, j \leq n$ such that
\[ \langle \epsilon_2, \epsilon_i \rangle = 0, \quad \langle \epsilon_2, \epsilon_j \rangle = 0, \quad \langle \epsilon_i, \epsilon_j \rangle \neq 0. \]
Since $\delta \circ \epsilon_{\ell}$ is the adjoint of $\langle \epsilon_{2}, \epsilon_{\ell} \rangle$ by Lemma 2.1, $\delta \circ \epsilon_{\ell}$ is null homotopic for $\ell = i, j$. Then for $\ell = i, j$, $\epsilon_{\ell}$ lifts to $\tilde{\epsilon}_{\ell} : S^{2\ell-1} \to G_{k}$ through $\pi : G_{k} \to G$. Consider the Samelson product $\langle \tilde{\epsilon}_{i}, \tilde{\epsilon}_{j} \rangle$.

Since $\pi$ is an H-map, we have
\[
\pi \circ \langle \tilde{\epsilon}_{i}, \tilde{\epsilon}_{j} \rangle = \langle \pi \circ \tilde{\epsilon}_{i}, \pi \circ \tilde{\epsilon}_{j} \rangle = \langle \epsilon_{i}, \epsilon_{j} \rangle
\]
which is nontrivial by assumption. Then in particular, we obtain that $G_{k}$ is not homotopy commutative. So our task is to find $2 \leq i, j \leq n$ satisfying (2.2), which is easily done by the following classical result if $n \geq 4$.

**Theorem 2.2** (Bott [B]). If $2 \leq i, j \leq n$ and $i + j > n$, the order of the Samelson product $\langle \epsilon_{i}, \epsilon_{j} \rangle$ is a nonzero multiple of
\[
\frac{(i + j - 1)!}{(i - 1)!(j - 1)!}.
\]

3. **Commutativity**

In this section, we give a brief sketch of the proof of the commutativity result on $G_{k}$. If the map $\pi$ in the homotopy fibration (2.1) has a homotopy section, we have a decomposition
\[
G_{k} \simeq G \times \Omega(\Omega_{0}^{3}G)
\]
as spaces. If this decomposition is as H-spaces and $G$ is homotopy commutative (i.e. $p > 2n$ by Theorem 1.2), we obtain that $G_{k}$ is homotopy commutative as desired. Then we give a criterion for the decomposition being as H-spaces, where we omit the proof.

**Lemma 3.1** (cf. [KiKo]). If there is an H-map $\hat{s} : G \to G_{k}$ such that $\pi \circ \hat{s}$ is a homotopy equivalence, then there is a homotopy equivalence as H-spaces
\[
G_{k} \simeq G \times \Omega(\Omega_{0}^{3}G).
\]

In particular, if moreover $p > 2n$, $G_{k}$ is homotopy commutative.

For the rest of this section, we assume $p > 2n$. Then in particular, $G \simeq S^{3} \times S^{5} \times \cdots \times S^{2n-1}$.

Since $G$ is homotopy commutative, it follows from Lemma 2.1 that $\pi$ has a homotopy section $s : G \to G_{k}$, not necessarily an H-map. We replace this homotopy section with an H-map. To this end, we employ the loop-suspension technique.

**Theorem 3.2** (James [J]). Consider a map $f : X \to Y$ where $Y$ is a homotopy associative $H$-space. There is a unique (up to homotopy) H-map $\tilde{f} : \Omega \Sigma X \to Y$ satisfying $\tilde{f} \circ E \simeq f$ for the suspension map $E : X \to \Omega \Sigma X$, where $\tilde{f}$ is called the extension of $f$.

We put $A = S^{3} \vee S^{5} \vee \cdots \vee S^{2n-1}$ and let $i : A \to G$ be the inclusion of a wedge into a product. Let $F$ be the homotopy fiber of the extension $\tilde{f} : \Omega \Sigma A \to G$, and let $\lambda : F \to \Omega \Sigma$ be the fiber inclusion. By an easy diagram chasing, we can prove:
Lemma 3.3. Consider a map $f: G \to Z$ where $Z$ is a homotopy associative $H$-space. If the composite $F \xrightarrow{\lambda} \Omega \Sigma A \xrightarrow{\bar{s} \circ i} G_k$ is null homotopic, there is an $H$-map $\hat{s}: G \to G_k$ satisfying the homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega \Sigma A & \xrightarrow{i} & G \\
\downarrow{\bar{s} \circ i} & & \downarrow{\hat{s}} \\
G_k & = & G_k
\end{array}
\]

Suppose now that the composite $F \xrightarrow{\lambda} \Omega \Sigma A \xrightarrow{\bar{s} \circ i} \mathcal{G}_k$ is null homotopic. Then it follows from Lemma 3.3 that there is an $H$-map $\hat{s}: G \to G_k$ satisfying the homotopy commutative square

\[
\begin{array}{ccc}
\Omega \Sigma A & \xrightarrow{i} & G \\
\downarrow{\bar{s} \circ i} & & \downarrow{\hat{s}} \\
\mathcal{G}_k & = & \mathcal{G}_k
\end{array}
\]

In particular, there is a chain of homotopies

\[
ip \circ \hat{s} \circ i \simeq \pi \circ \hat{s} \circ \overline{i} \circ E \simeq \pi \circ (\bar{s} \circ i) \circ E \simeq \pi \circ s \circ i \simeq i.
\]

In the mod $p$ homology, the map $i: A \to G$ induces the inclusion of ring generators. Then $\pi \circ \hat{s}$ turns out to be the identity map on ring generators in the mod $p$ homology, hence since $\pi \circ \hat{s}$ is an H-map, it is an isomorphism in the mod $p$ homology. So we obtain that $\pi \circ \hat{s}$ is a $p$-local homotopy equivalence. Then all we have to do is prove that the composite $F \xrightarrow{\lambda} \Omega \Sigma A \xrightarrow{\bar{s} \circ i} \mathcal{G}_k$ is null homotopic. To this end, we analyze the fiber inclusion $\lambda$.

Let $F'$ be the homotopy fiber of the adjoint $\Sigma A \to BG$ of the inclusion $i: A \to G$. Since the extension $\bar{i}: \Omega \Sigma A \to G$ is the loop of the above adjoint, we get:

Lemma 3.4. $F \simeq \Omega F'$ and the fiber inclusion $\lambda: \Omega F' \to \Omega \Sigma A$ is a loop map.

Let $L$ be the free Lie algebra generated by $\tilde{H}_*(A; \mathbb{Z}/p)$. Then as in [CN], the induced map $\bar{i}_*: H_*(\Omega \Sigma A; \mathbb{Z}/p) \to H_*(G; \mathbb{Z}/p)$ is identified with the map between universal envelopes

\[
U(L) \to U(L/\{L, L\})
\]

induced from the abelianization $L \to L/\{L, L\}$. Moreover, there is a splitting

\[
U(L) \cong U([L, L]) \oplus U(L/\{L, L\}),
\]

hence the image of $\lambda_*: H_*(F; \mathbb{Z}/p) \to H_*(\Omega \Sigma A; \mathbb{Z}/p)$ is identified with $U([L, L]) \subset U(L)$. A little more consideration shows that the Lie algebra generators of $[L, L]$ are spherical and lift to $F$. So we obtain:

Theorem 3.5. There is a wedge of spheres $R$ such that $F' \simeq \Sigma R$, and the composite $R \xrightarrow{E} \Omega \Sigma R \xrightarrow{\lambda} \Omega \Sigma A$ is a wedge of iterated Samelson products of

\[
\mu_j: S^{2j-1} \xrightarrow{\text{incl}} A \xrightarrow{E} \Omega \Sigma A.
\]
Corollary 3.6. If $p > 2n + 1$, the composite $F \overset{\lambda}{\to} \Omega \Sigma A \overset{\cong}{\to} G_k$ is null homotopic.

Proof. Put $\bar{\mu}_j = (s \circ \bar{i}) \circ \mu_j$. We consider the Samelson product $\langle \bar{\mu}_i, \bar{\mu}_j \rangle$. Since $\pi$ is an $H$-map and $G$ is homotopy commutative, we have

$$\pi \circ \langle \bar{\mu}_i, \bar{\mu}_j \rangle = \langle \pi \circ \bar{\mu}_i, \pi \circ \bar{\mu}_j \rangle = 0.$$

Then $\langle \bar{\mu}_i, \bar{\mu}_j \rangle$ lifts to a map $S^{2i_1+2i_2-2} \to \Omega \Omega_0^2 G$ by the homotopy fibration $\Omega \Omega_0^2 G \to G_k \overset{\pi}{\to} G$. Since $p > 2n + 1$, we have $\pi_{2m}(\Omega \Omega_0^2 G) = 0$ for $m \leq 2n - 1$ by [To], implying that the above lift is null homotopic. Then we obtain $\langle \bar{\mu}_i, \bar{\mu}_j \rangle = 0$, hence

$$0 = \langle \bar{\mu}_j, \langle \cdots \langle \bar{\mu}_{j_{m-1}}, \bar{\mu}_{j_m} \rangle \cdots \rangle \rangle = (s \circ \bar{i}) \circ (\mu_j, \langle \cdots \langle \mu_{j_{m-1}}, \mu_{j_m} \rangle \cdots \rangle)$$

since $s \circ \bar{i}$ is an $H$-map. Thus by Theorem 3.5, the composite $R \overset{\lambda}{\to} \Omega \Sigma A \overset{\cong}{\to} G_k$ is null homotopic. Therefore we obtain the desired result by the uniqueness of the extension and Lemma 3.4.

4. THE CASE $p = 2n + 1$

Throughout this section, we assume $p = 2n + 1$.

As in the previous section, it is sufficient for proving the commutativity result to show that the homotopy section $s : G \to G_k$ is an $H$-map. This is equivalent to show that the adjoint

$$\bar{s} : \Sigma G \to BG_k \simeq \text{map}(S^4, BG : k\ell_2)$$

extends to the projective plane $P^2G$. By the exponential law, this is equivalent to existence of a map $\mu : S^4 \times P^2G \to BG$ satisfying a homotopy commutative diagram

$$\begin{array}{ccc}
S^4 \vee \Sigma G \overset{\text{incl}}{\longrightarrow} & BG \\
\downarrow & & \\
S^4 \times P^2G \overset{\mu}{\longrightarrow} & BG.
\end{array}$$

Since $P^2G$ is the cofiber of the Hopf construction $\Sigma G \wedge G \to \Sigma G$ and $\Sigma G \wedge G$ has the homotopy type of a wedge of spheres of dimension $\leq 2n^2 - 1 = \frac{(p-1)^2}{2} - 1$, we see that the obstruction for existence of $\mu$ lies in $\pi_*(BG)$ for $* \leq \frac{(p-1)^2}{2} + 3$. Since the obstruction is torsion in $\pi_*(BG)$, we see from [To] that it is of order at most $p$. Moreover, we also see that the obstruction is linear in $k$. Then we get:

Proposition 4.1. If $p$ divides $k$, the homotopy section $s$ is an $H$-map, hence $G_k$ is homotopy commutative.

When $p$ does not divide $k$, we can prove that the obstruction is nontrivial by looking at the Steenrod operation on the mod $p$ cohomology of $BG$. Then we have:

Proposition 4.2. If $p$ does not divide $k$, the homotopy section $s$ cannot be an $H$-map.
**Corollary 4.3.** If $p$ does not divide $k$, $\mathcal{G}_k$ is not homotopy commutative.

*Proof.* Suppose that $\mathcal{G}_k$ is homotopy commutative. Then the argument in the previous section ensures that there is an H-map $\hat{s} : G \to \mathcal{G}_k$ such that the composite $e = \pi \circ \hat{s}$ is a homotopy equivalence. If we put $s = \hat{s} \circ e^{-1}$, $s$ is a homotopy section of $\pi$ and is an H-map, which contradicts to Proposition 4.2. 

**REFERENCES**


