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The action of 1-dimensional holomorphic semigroups for functions on symmetric cones and the Bessel functions

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Abstract

The unitary highest weight representations of Hermitian simple Lie groups of tube type can be realized on the square-integrable space on the space called symmetric cones. This can be regarded as a generalization of the Weil representation of $Sp(r,\mathbb{R})$ on $L^2(\mathbb{R}^n)$. On $L^2(\mathbb{R}^n)$, there exists a 1-dimensional holomorphic semigroup defined by integral operators by means of Mehler kernel, the kernel written by exponential functions. This semigroup is called Hermite semigroup. We can generalize this semigroup to the function spaces on symmetric cones, and this can be expressed by integral operators with kernel defined by generalized Bessel functions. In this paper we prove the upper estimate of generalized Bessel functions, and prove that the integral kernel decreases sufficiently rapidly.

1 Introduction: Hermite semigroup for $O(n)$-invariant functions

First we consider the following operator on $\mathcal{S}(\mathbb{R}^n)$:

$$H := \frac{1}{2}(-\Delta + |x|^2).$$

Then the exponential operator e^{-tH} is given as follows.

Theorem 1.1. *$e^{-tH} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ ($\mathrm{Re}\, t \geq 0$, $t \notin \pi\sqrt{-1}\mathbb{Z}$) is given by the following integral:*

$$e^{-tH} f(x) = \frac{1}{(2\pi \sinh t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{-\frac{1}{2}\coth t(|x|^2+|y|^2)+\frac{1}{\sinh t}x\cdot y} dy.$$

This family of operators is called the Hermite semigroup. If $t \in \sqrt{-1}\mathbb{R}$, then e^{-tH} is a unitary operator on $L^2(\mathbb{R}^n)$.

Remark 1.2. *We have the following isomorphisms and inclusion relations.*

$$\begin{aligned}
&\mathfrak{sp}(n,\mathbb{R}) \simeq \mathrm{span}\left\{ix_jx_k, i\frac{\partial^2}{\partial x_j \partial x_k}, x_j\frac{\partial}{\partial x_k} + \frac{\delta_{jk}}{2} : 1 \leq j,k \leq n\right\} \\
&\cup \\
&\mathfrak{u}(n) \simeq \mathrm{span}\left\{ix_jx_k - i\frac{\partial^2}{\partial x_j \partial x_k}, x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j} : 1 \leq j,k \leq n\right\} \\
&\cup \\
&\mathfrak{z}(\mathfrak{u}(n)) \simeq i\mathbb{R}H
\end{aligned}$$

We assume $f \in L^2(\mathbb{R}^n)$ is $O(n)$-invariant, and set

$$f(x) = \varphi\left(\frac{1}{2}|x|^2\right)$$

where $\varphi : \mathbb{R}_{>0} \to \mathbb{C}$. Then $e^{-tH}f$ is also $O(n)$-invariant. We set

$$e^{-tH} f(x) = \psi\left(\frac{1}{2}|x|^2\right)$$

and calculate $\psi(\xi)$.

$$\begin{aligned}
&\psi(\xi) = e^{-tH} f(\sqrt{2\xi}e_1) \\
&= \frac{1}{(2\pi \sinh t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi\left(\frac{|y|^2}{2}\right) e^{-\frac{1}{2}\coth t(2\xi+|y|^2)+\frac{\sqrt{2}}{\sinh t}\sqrt{\xi}e_1\cdot y} dy \\
&\text{(set } y = r((\cos\theta)e_1 + (\sin\theta)\sigma)) \\
&= \frac{1}{(2\pi \sinh t)^{\frac{n}{2}}} \int_0^\infty \int_0^\pi \int_{S^{n-2}} \varphi\left(\frac{r^2}{2}\right) e^{-\frac{1}{2}\coth t(2\xi+r^2)} e^{\frac{\sqrt{2}}{\sinh t}\sqrt{\xi}r\cos\theta} r^{n-1} \sin^{n-2}\theta d\sigma d\theta dr \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{1}{(2\pi \sinh t)^{\frac{n}{2}}} \int_0^\infty \varphi\left(\frac{r^2}{2}\right) e^{-\frac{1}{2}\coth t(2\xi+r^2)} \int_0^\pi e^{\frac{\sqrt{2}}{\sinh t}\sqrt{\xi}r\cos\theta} \sin^{n-2}\theta d\theta r^{n-1} dr \\
&\left(\text{set } \frac{r^2}{2} = \eta\right) \\
&= \frac{1}{\sqrt{\pi}\sinh^{\frac{n}{2}} t\,\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \varphi(\eta) e^{-\coth t(\xi+\eta)} \int_0^\pi e^{\frac{2}{\sinh t}\sqrt{\xi\eta}\cos\theta} \sin^{n-2}\theta d\theta \eta^{\frac{n}{2}-1} d\eta.
\end{aligned}$$

Now we recall the I-Bessel function.

$$I_\lambda(z) := \left(\frac{z}{2}\right)^\lambda \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\Gamma(\lambda+m+1)}.$$

We set

$$\tilde{I}_\lambda(z) := \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\Gamma(\lambda+m+1)} = \left(\frac{z}{2}\right)^{-\lambda} I_\lambda(z).$$

Then for $\lambda > -\frac{1}{2}$ we have the following integral expression:

$$\tilde{I}_\lambda(z) = \frac{1}{\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)} \int_0^\pi e^{z\cos\theta} \sin^{2\lambda}\theta d\theta.$$

Using this, we continue the calculation.

$$\begin{aligned}\psi(\xi) =& \frac{1}{\sqrt{\pi}\sinh^{\frac{n}{2}} t\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \varphi(\eta) e^{-\coth t(\xi+\eta)} \\ &\times \int_0^\pi e^{\frac{2}{\sinh t}\sqrt{\xi\eta}\cos\theta} \sin^{n-2}\theta d\theta \eta^{\frac{n}{2}-1} d\eta \\ =& \frac{1}{\sinh^{\frac{n}{2}} t} \int_0^\infty \varphi(\eta) e^{-\coth t(\xi+\eta)} \tilde{I}_{\frac{n}{2}-1}\left(\frac{2}{\sinh t}\sqrt{\xi\eta}\right) \eta^{\frac{n}{2}-1} d\eta.\end{aligned}$$

Since $L^2(\mathbb{R}_{>0}, \xi^{\frac{n}{2}-1}d\xi) \to L^2(\mathbb{R}^n)$, $\varphi \mapsto f(x) := \varphi\left(\frac{|x|^2}{2}\right)$ is an isometry (up to const.),

$$\varphi \mapsto \frac{1}{\sinh^{\frac{n}{2}} t} \int_0^\infty \varphi(\eta) e^{-\coth t(\xi+\eta)} \tilde{I}_{\frac{n}{2}-1}\left(\frac{2}{\sinh t}\sqrt{\xi\eta}\right) \eta^{\frac{n}{2}-1} d\eta$$

is unitary on $L^2(\mathbb{R}_{>0}, \xi^{\frac{n}{2}-1}d\xi)$ if $t \in \sqrt{-1}\mathbb{R}$.

So far we have assumed that n is a positive integer, but the above map is valid for any positive real number n. In the next section we replace $\frac{n}{2}$ to λ where λ is any positive real number.

2 Holomorphic semigroup on $\mathbb{R}_{>0}$

For $\lambda > 0$, we take $\varphi \in L^2(\mathbb{R}_{>0}, \xi^{\lambda-1}d\xi)$. For $t \in \mathbb{C}$ with $\operatorname{Re} t \geq 0$, $t \notin \sqrt{-1}\pi\mathbb{Z}$, we set

$$\tau_\lambda(t)\varphi(\xi) := \frac{1}{\sinh^\lambda t} \int_0^\infty \varphi(\eta) e^{-\coth t(\xi+\eta)} \tilde{I}_{\lambda-1}\left(\frac{2}{\sinh t}\sqrt{\xi\eta}\right) \eta^{\lambda-1} d\eta.$$

In this section we seek the properties of $\tau_\lambda(t)$.

First we prepare some notations and theorems. We write

$$\mathbf{H} := \mathbb{R} + \sqrt{-1}\mathbb{R}_{>0}, \qquad \mathbf{D} := \{w \in \mathbb{C} : |w| < 1\}.$$

For $\lambda > 0$ we set

$$L^2_\lambda(\mathbb{R}_{>0}) := \left\{ \varphi : \mathbb{R}_{>0} \to \mathbb{C} : \frac{2^\lambda}{\Gamma(\lambda)} \int_0^\infty |\varphi(\xi)|^2 \xi^{\lambda-1} d\xi < \infty \right\}.$$

Also, for $\lambda > 1$ we set

$$\mathcal{H}^2_\lambda(\mathbf{H}) := \left\{ F \in \mathcal{O}(\mathbf{H}) : \frac{\lambda-1}{4\pi} \int_{\mathbf{H}} |F(z)|^2 (\mathrm{Im}\, z)^{\lambda-2} dz < \infty \right\},$$

$$\mathcal{H}^2_\lambda(\mathbf{D}) := \left\{ f \in \mathcal{O}(\mathbf{D}) : \frac{\lambda-1}{\pi} \int_{\mathbf{D}} |f(w)|^2 (1-|w|^2)^{\lambda-2} dw < \infty \right\}.$$

Then $\widetilde{SL}(2,\mathbb{R})$ (resp. $\widetilde{SU}(1,1)$) acts on $\mathcal{H}^2_\lambda(\mathbf{H})$ (resp. $\mathcal{H}^2_\lambda(\mathbf{D})$) unitarily by,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} : F(z) \mapsto (cz+d)^{-\lambda} F\left(\frac{az+b}{cz+d}\right).$$

We define Laplace transformation and Cayley transformation by

$$\mathcal{L}_\lambda : L^2_\lambda(\mathbb{R}_{>0}) \to \mathcal{O}(\mathbf{H}), \qquad \mathcal{L}_\lambda\varphi(z) := \frac{2^\lambda}{\Gamma(\lambda)} \int_0^\infty \varphi(\xi) e^{iz\xi} \xi^{\lambda-1} d\xi,$$

$$\gamma_\lambda : \mathcal{O}(\mathbf{H}) \to \mathcal{O}(\mathbf{D}), \qquad \gamma_\lambda F(w) := (1-w)^{-\lambda} F\left(i\frac{1+w}{1-w}\right).$$

Then we have the following properties.

Theorem 2.1. *(1) If $\lambda > 1$, then* $L^2_\lambda(\mathbb{R}_{>0}) \xrightarrow[\text{unitary}]{\mathcal{L}_\lambda} \mathcal{H}^2_\lambda(\mathbf{H}) \xrightarrow[\text{unitary}]{\gamma_\lambda} \mathcal{H}^2_\lambda(\mathbf{D})$.
(2) If $\lambda > 0$, then $L^2_\lambda(\mathbb{R}_{>0}) \xrightarrow[\substack{\text{inj. with} \\ \text{dense image}}]{\mathcal{L}_\lambda} \mathcal{O}(\mathbf{H}) \xrightarrow[\text{isom.}]{\gamma_\lambda} \mathcal{O}(\mathbf{D})$.

Remark 2.2. *For $\lambda > 1$ and $f(w) = \sum_{m=0}^\infty a_m w^m \in \mathcal{O}(\mathbf{D})$, the norm $\|f\|_{\lambda,\mathbf{D}}$ is given by*

$$\|f\|^2_{\lambda,\mathbf{D}} = \sum_{m=0}^\infty \frac{m!}{(\lambda)_m} |a_m|^2.$$

This equality is proved as

$$\begin{aligned}
&\frac{\lambda-1}{\pi}\int_{|w|>1} w^m\bar{w}^n(1-|w|^2)^{\lambda-2}dw\\
=&\frac{\lambda-1}{\pi}\int_0^\infty\int_0^{2\pi} r^{m+n}e^{i\theta(m-n)}(1-r^2)^{\lambda-2}rd\theta dr\\
\overset{r^2=s}{=}&(\lambda-1)\delta_{mn}\int_0^1 s^m(1-s)^{\lambda-2}ds\\
=&\delta_{mn}(\lambda-1)B(m+1,\lambda-1)=\delta_{mn}\frac{\Gamma(m+1)\Gamma(\lambda)}{\Gamma(m+\lambda)}=\delta_{mn}\frac{m!}{(\lambda)_m}
\end{aligned}$$

Thus if we redefine $\mathcal{H}^2_\lambda(\mathbf{D})$ for $\lambda>0$ by

$$\mathcal{H}^2_\lambda(\mathbf{D}):=\left\{f(w)=\sum_{m=0}^\infty a_m w^m\in\mathcal{O}(\mathbf{D}):\sum_{m=0}^\infty\frac{m!}{(\lambda)_m}|a_m|^2<\infty\right\},$$

Then $\gamma_\lambda\circ\mathcal{L}_\lambda : L^2_\lambda(\mathbb{R}_{>0})\to\mathcal{H}^2_\lambda(\mathbf{D})$ is unitary for $\lambda>0$.

We now assume $t\in\mathbb{R}_{>0}$, and calculate $\mathcal{L}_\lambda\tau_\lambda(t)\varphi(z)$, assuming that Fubini's theorem is valid.

$$\begin{aligned}
&\frac{\Gamma(\lambda)}{2^\lambda}\mathcal{L}_\lambda\tau_\lambda(t)\varphi(z)\\
=&\frac{1}{\sinh^\lambda t}\int_0^\infty\int_0^\infty\varphi(\eta)e^{-\coth t(\xi+\eta)}\tilde{I}_{\lambda-1}\left(\frac{2\sqrt{\xi\eta}}{\sinh t}\right)e^{iz\xi}\eta^{\lambda-1}\xi^{\lambda-1}d\eta d\xi\\
&\left(\text{set }\frac{\xi\eta}{\sinh^2 t}=:\xi'\right)\\
=&\frac{1}{\sinh^\lambda t}\int_0^\infty\int_0^\infty\varphi(\eta)e^{(-\coth t+iz)\frac{(\sinh^2 t)\xi'}{\eta}}e^{-(\coth t)\eta}\\
&\quad\times\tilde{I}_{\lambda-1}\left(2\sqrt{\xi'}\right)\eta^{\lambda-1}\left(\frac{(\sinh^2 t)\xi'}{\eta}\right)^\lambda\xi'^{-1}d\xi'd\eta\\
=&\sinh^\lambda t\int_0^\infty\varphi(\eta)e^{-(\coth t)\eta}\eta^{-1}\int_0^\infty e^{-\frac{(\cosh t-iz\sinh t)(\sinh t)}{\eta}\xi'}\tilde{I}_{\lambda-1}\left(2\sqrt{\xi'}\right)\xi'^{\lambda-1}d\xi'd\eta.
\end{aligned}$$

Now we have the following formula:

$$\int_0^\infty e^{-z\xi}\tilde{I}_{\lambda-1}(2\sqrt{\xi})\xi^{\lambda-1}d\xi=z^{-\lambda}e^{z^{-1}}.$$

This is proved as

$$\int_0^\infty e^{-z\xi}\tilde{I}_{\lambda-1}(2\sqrt{\xi})\xi^{\lambda-1}d\xi = \int_0^\infty e^{-z\xi}\sum_{m=0}^\infty \frac{\xi^m}{m!\Gamma(m+\lambda)}\xi^{\lambda-1}d\xi$$
$$=\sum_{m=0}^\infty \frac{1}{m!\Gamma(m+\lambda)}\int_0^\infty e^{-z\xi}\xi^{m+\lambda-1}d\xi = \sum_{m=0}^\infty \frac{1}{m!}z^{-(m+\lambda)} = z^{-\lambda}e^{z^{-1}}.$$

Using this, we continue the calculation.

$$\begin{aligned}
&\frac{\Gamma(\lambda)}{2^\lambda}\mathcal{L}_\lambda\tau_\lambda(t)\varphi(z)\\
=&\sinh^\lambda t\int_0^\infty \varphi(\eta)e^{-(\coth t)\eta}\eta^{-1}\int_0^\infty e^{-\frac{(\cosh t - iz\sinh t)(\sinh t)}{\eta}\xi'}\tilde{I}_{\lambda-1}\left(2\sqrt{\xi'}\right)\xi'^{\lambda-1}d\xi' d\eta\\
=&\sinh^\lambda t\int_0^\infty \varphi(\eta)e^{-(\coth t)\eta}\left(\frac{(\cosh t - iz\sinh t)(\sinh t)}{\eta}\right)^{-\lambda} e^{\frac{\eta}{(\cosh t - iz\sinh t)(\sinh t)}}\eta^{-1}d\eta\\
=&(-iz\sinh t+\cosh t)^{-\lambda}\int_0^\infty \varphi(\eta)e^{i\frac{z\cosh t + i\sinh t}{-iz\sinh t+\cosh t}\eta}\eta^{\lambda-1}d\eta\\
=&(-iz\sinh t+\cosh t)^{-\lambda}\frac{\Gamma(\lambda)}{2^\lambda}\mathcal{L}_\lambda\varphi\left(\frac{z\cosh t + i\sinh t}{-iz\sinh t+\cosh t}\right).
\end{aligned}$$

$$\therefore \mathcal{L}_\lambda\tau_\lambda(t)\varphi(z) = (-iz\sinh t+\cosh t)^{-\lambda}\mathcal{L}_\lambda\varphi\left(\frac{z\cosh t + i\sinh t}{-iz\sinh t+\cosh t}\right).$$

We can also check easily that

$$\gamma_\lambda\mathcal{L}_\lambda\tau_\lambda(t)\varphi(w) = e^{-\lambda t}\gamma_\lambda\mathcal{L}_\lambda\varphi(e^{-2t}w).$$

Especially we have

$$\tau_\lambda(t)\tau_\lambda(s)\varphi = \tau_\lambda(t+s)\varphi$$

if φ is a sufficiently "good" function. In order to justify the convergence and the change of order of integrals, we want to know the upper estimation of the integral kernel.

Lemma 2.3. *If $\lambda \geq -\frac{1}{2}$, then there exists a positive constant C such that*

$$|\tilde{I}_\lambda(z)| \leq Ce^{|\operatorname{Re} z|}.$$

Proof. Using the integral formula

$$\tilde{I}_\lambda(z) = \frac{1}{\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)}\int_0^\pi e^{z\cos\theta}\sin^{2\lambda}\theta d\theta,$$

we have

$$
\begin{aligned}
|\tilde{I}_\lambda(z)| \leq & \frac{1}{\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)} \int_0^\pi |e^{z\cos\theta}| \sin^{2\lambda}\theta d\theta \\
\leq & \frac{1}{\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)} \int_0^\pi e^{|\operatorname{Re} z|} \sin^{2\lambda}\theta d\theta \leq C e^{|\operatorname{Re} z|}. \qquad \square
\end{aligned}
$$

Corollary 2.4. *If $\lambda \geq \frac{1}{2}$, then there exists $C > 0$ such that for $t = u + iv$ with $u \geq 0$,*

$$
\left| e^{-\coth t(\xi+\eta)} \tilde{I}_{\lambda-1}\left(\frac{2\sqrt{\xi\eta}}{\sinh t}\right)\right| \leq C \exp\left(-\frac{\sinh u}{\cosh u + |\cos v|}(\xi+\eta)\right).
$$

Therefore, if $\operatorname{Re} t = 0$, we have

$$
\tau_\lambda(t) : L^1(\mathbb{R}_{>0}, \xi^{\lambda-1} d\xi) \longrightarrow L^\infty(\mathbb{R}_{>0}),
$$

and if $\operatorname{Re} t > 0$, we have

$$
\tau_\lambda(t) : \{\text{Polynomial growth functions}\} \longrightarrow \{\text{Exponential decay functions}\}.
$$

3 Holomorphic semigroup on general symmetric cones

In this section we generalize the previous results to more general setting. Let $\mathfrak{g}$ be a simple Hermitian Lie algebra of tube type. Then there exists a 3-graded decomposition

$$
\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{l} \oplus \mathfrak{n}^-
$$

so that l is a reductive subalgebra and $\mathfrak{n}^\pm$ are Abelian subalgebras. Let ϑ be a Cartan involution such that $\vartheta|_{\mathfrak{z}(\mathfrak{l})} = -\mathrm{id}_{\mathfrak{z}(\mathfrak{l})}$ holds, where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. Then ϑ reverses the grading. We fix an $e \in \mathfrak{n}^+$ such that

$$
-[[e, \vartheta e], e] = 2e, \qquad \mathfrak{l}^\vartheta . e = 0
$$

holds, where $\mathfrak{l}^\vartheta$ is the subalgebra of $\mathfrak{l}$ which consists of fixed points of ϑ. Then $\mathfrak{n}^+$ has a Euclidean Jordan algebra structure with the product

$$
x \cdot y := -\frac{1}{2}[[x, \vartheta e], y].
$$

That is, for any $x, y \in \mathfrak{n}^+$ we have

$$
xy = yx, \qquad x^2(xy) = x(x^2 y),
$$

and there exists an inner product $(\cdot|\cdot)$ such that for any $x, y, z \in \mathfrak{n}^+$,

$$(xy|z) = (x|yz)$$

holds. e becomes the unit element of $\mathfrak{n}^+$.

Example 3.1. When $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})$, we have the following isomorphism.

$$\begin{aligned}\mathfrak{sp}(r,\mathbb{R}) &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} : \begin{array}{l} A \in \mathfrak{gl}(r,\mathbb{R}), \\ B, C \in \mathrm{Sym}(r,\mathbb{R}) \end{array} \right\} \\ &\simeq \mathrm{Sym}(r,\mathbb{R}) \oplus \mathfrak{gl}(r,\mathbb{R}) \oplus \mathrm{Sym}(r,\mathbb{R}).\end{aligned}$$

$\mathfrak{n}^+ = \mathrm{Sym}(r, \mathbb{R})$ has a Jordan algebra structure with the product

$$x \cdot y := \frac{1}{2}(xy + yx).$$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and L, K, K_L be connected subgroups with Lie algebra $\mathfrak{l}, \mathfrak{k} = \mathfrak{g}^{\vartheta}, \mathfrak{k}_{\mathfrak{l}} = \mathfrak{l}^{\vartheta}$ respectively. We set $n := \dim \mathfrak{n}^+$, $r := \mathrm{rank}_{\mathbb{R}}\mathfrak{g}$, and $d := \dim \mathfrak{g}_{\pm\varepsilon_i \pm \varepsilon_j}$ with $i \neq j$, where $\mathfrak{g}_{\pm\varepsilon_i\pm\varepsilon_j}$ is the restricted root space with respect to $\pm\varepsilon_i \pm \varepsilon_j \in \Sigma(\mathfrak{g}, \mathfrak{a})$ of type C_r, where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} = \mathfrak{g}^{-\vartheta}$. Then we have the equality

$$n = r + \frac{1}{2}r(r-1)d.$$

Let $(\cdot|\cdot)$ be the inner product on $\mathfrak{n}^+$ such that

$$(xy|z) = (x|yz), \qquad (e|e) = r,$$

and set $\mathrm{tr}(x) := (x|e)$. This is called the *Jordan trace* of $\mathfrak{n}^+$. Also let $\Delta(x)$ be the *Jordan determinant*, that is, $\Delta(x)$ is the polynomial on $\mathfrak{n}^+$ of degree r such that

$$\Delta(lx) = \Delta(le)\Delta(x) \quad (\forall l \in L), \qquad \Delta(e) = 1$$

holds, where $lx := \mathrm{Ad}(l)x$. In addition, we denote by $h(z, w)$ the holomorphic polynomial on $\mathfrak{n}^+_{\mathbb{C}} \times \overline{\mathfrak{n}^+_{\mathbb{C}}}$ such that

$$h(lz, w) = h(z, l^*w) \quad (\forall l \in L), \qquad h(x, x) = \Delta(e - x^2) \quad (\forall x \in \mathfrak{n}^+)$$

holds.

Example 3.2. When $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})$ and $\mathfrak{n}^+ = \mathrm{Sym}(r, \mathbb{R})$, then we have

$$r = r, \quad n = \frac{1}{2}r(r+1), \quad d = 1.$$

$L = GL(r,\mathbb{R})$ acts on $\mathfrak{n}^+ = \mathrm{Sym}(r,\mathbb{R})$ by

$$l.x := lx{}^t l.$$

The associated polynomials are given by

$$(x|y) = \mathrm{Tr}(xy), \qquad \mathrm{tr}(x) = \mathrm{Tr}(x),$$
$$\Delta(x) = \mathrm{Det}(x), \qquad h(z,w) = \mathrm{Det}(e - xy^*).$$

$\mathfrak{g}$	$\mathfrak{l}$	$\mathfrak{n}^+$	r	n	d
$\mathfrak{sp}(r,\mathbb{R})$	$\mathbb{R} \oplus \mathfrak{sl}(r,\mathbb{R})$	$\mathrm{Sym}(r,\mathbb{R})$	r	$\frac{1}{2}r(r+1)$	1
$\mathfrak{su}(r,r)$	$\mathbb{R} \oplus \mathfrak{sl}(r,\mathbb{C})$	$\mathrm{Herm}(r,\mathbb{C})$	r	r^2	2
$\mathfrak{so}^*(4r)$	$\mathbb{R} \oplus \mathfrak{sl}(r,\mathbb{H})$	$\mathrm{Herm}(r,\mathbb{H})$	r	$r(2r-1)$	4
$\mathfrak{so}(2,n)$	$\mathbb{R} \oplus \mathfrak{so}(1,n-1)$	$\mathbb{R}^{1,n-1}$	2	n	$n-2$
$\mathfrak{e}_{7(-25)}$	$\mathbb{R} \oplus \mathfrak{e}_{6(-26)}$	$\mathrm{Herm}(3,\mathbb{O})$	3	27	8

Table 1: Classification of tube type Lie algebras and associated data

We set

$$\Omega := \{x^2 : x \in (\mathfrak{n}^+)^\times\} \subset \mathfrak{n}^+,$$
$$T_\Omega := \mathfrak{n}^+ + \sqrt{-1}\Omega \subset \mathfrak{n}^+_{\mathbb{C}},$$
$$D := (\text{Component of } \{w \in \mathfrak{n}^+_{\mathbb{C}} : h(w,w) > 0\} \text{ which contains } 0).$$

Then we have the following diffeomorphisms

$$\Omega \simeq L/K_L,$$
$$T_\Omega \simeq D \simeq G/K.$$

Example 3.3. When $\mathfrak{n}^+ = \mathrm{Sym}(r,\mathbb{R})$, then Ω, T_Ω, D are given by

$$\Omega = \{x \in \mathrm{Sym}(r,\mathbb{R}) : \text{Positive definite}\},$$
$$T_\Omega = \{z \in \mathrm{Sym}(r,\mathbb{C}) : \mathrm{Im}\, z \text{ is positive definite}\},$$
$$D = \{w \in \mathrm{Sym}(r,\mathbb{C}) : I - ww^* \text{ is positive definite}\}.$$

$L = GL(r,\mathbb{R})$ acts on Ω by $l.x := lx{}^t l$.

We set $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $J' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, and set

$$G = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2r,\mathbb{R}) : gJ{}^t g = J \right\} = Sp(r,\mathbb{R}),$$

$$G' = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2r,\mathbb{C}) : gJ{}^t g = J,\ gJ' = J'\bar{g} \right\} \underset{\text{transf.}}{\overset{\text{Cayley}}{\simeq}} Sp(r,\mathbb{R}).$$

Then G acts on T_Ω and G' acts on D by

$$g.z := (az+b)(cz+d)^{-1}.$$

We prepare some notations. For $\mathbf{s} \in \mathbb{C}^r$, we set

$$\Gamma_\Omega(\mathbf{s}) := (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j - \frac{d}{2}(j-1)\right).$$

Also, for $\lambda \in \mathbb{C}$, we set $\Gamma_\Omega(\lambda) := \Gamma_\Omega((\lambda, \dots, \lambda))$.

Now we define some function spaces. For $\lambda > \frac{n}{r} - 1$ and $\varphi : \Omega \to \mathbb{C}$ measurable, we set

$$\|\varphi\|^2_{\lambda,\Omega} := \frac{2^{r\lambda}}{\Gamma_\Omega(\lambda)} \int_\Omega |\varphi(x)|^2 \Delta(x)^{\lambda - \frac{n}{r}} dx.$$

Also, for $\lambda > \frac{2n}{r} - 1$, $F \in \mathcal{O}(T_\Omega)$, $f \in \mathcal{O}(D)$, we set

$$\|F\|^2_{\lambda,T_\Omega} := \frac{1}{(4\pi)^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)} \int_{T_\Omega} |F(z)|^2 \Delta(\mathrm{Im}\ z)^{\lambda - \frac{2n}{r}} dz,$$
$$\|f\|^2_{\lambda,D} := \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)} \int_D |f(w)|^2 h(w)^{\lambda - \frac{2n}{r}} dw$$

where $h(w) := h(w,w)$, and let $L^2_\lambda(\Omega)$, $\mathcal{H}^2_\lambda(T_\Omega)$, $\mathcal{H}^2_\lambda(D)$ be the spaces of all functions with finite norms. Then $\tilde{G}$ (universal covering group of G) acts on $\mathcal{H}^2_\lambda(T_\Omega)$ and $\mathcal{H}^2_\lambda(D)$ unitarily by, for $g \in \tilde{G}$,

$$F(z) \mapsto \Delta(d(g^{-1})(z)e)^{\frac{\lambda}{2}} F(g^{-1}z)$$

where $d(g^{-1})(z)$ denotes the differential of $g^{-1} : T_\Omega \to T_\Omega$ (resp. $D \to D$) at z.

Example 3.4. $\widetilde{Sp}(r,\mathbb{R})$ acts on $\mathcal{H}^2_\lambda(T_\Omega)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} : F(z) \mapsto \mathrm{Det}(cz+d)^{-\lambda} F\left((az+b)(cz+d)^{-1}\right).$$

The Laplace and Cayley transforms are defined as follows.

$$\mathcal{L}_\lambda : L^2_\lambda(\Omega) \to \mathcal{O}(T_\Omega), \quad \mathcal{L}_\lambda\varphi(z) := \frac{2^{r\lambda}}{\Gamma(\lambda)} \int_\Omega \varphi(x) e^{i(z|x)} \Delta(x)^{\lambda - \frac{n}{r}} dx,$$
$$\gamma_\lambda : \mathcal{O}(T_\Omega) \to \mathcal{O}(D), \quad \gamma_\lambda F(w) := \Delta(e-w)^{-\lambda} F\left(i(e+w)(e-w)^{-1}\right).$$

Then we have the following theorem.

Theorem 3.5 ([2, Theorem XIII.1.1, Proposition XIII.1.3, Proposition XIII.3.2]).
(1) If $\lambda > \frac{2n}{r} - 1$, *then* $L^2_\lambda(\Omega) \xrightarrow[\text{unitary}]{\mathcal{L}_\lambda} \mathcal{H}^2_\lambda(T_\Omega) \xrightarrow[\text{unitary}]{\gamma_\lambda} \mathcal{H}^2_\lambda(D)$.
(2) If $\lambda > \frac{n}{r} - 1$, *then* $L^2_\lambda(\Omega) \xrightarrow[\substack{\text{inj. with}\\ \text{dense image}}]{\mathcal{L}_\lambda} \mathcal{O}(T_\Omega) \xrightarrow[\text{isom.}]{\gamma_\lambda} \mathcal{O}(D)$.

Now we define the 1-parameter semigroup on $\mathcal{O}(T_\Omega)$. For $t \in \mathbb{C}$ with $\operatorname{Re} t \geq 0$, $t \notin \sqrt{-1}\pi\mathbb{Z}$, we define $\tilde{\tau}_\lambda(t) : \mathcal{O}(T_\Omega) \to \mathcal{O}(T_\Omega)$ by

$$\begin{aligned}\tilde{\tau}_\lambda(t)F(z) :=& \Delta(-iz\sinh t + e\cosh t)^{-\lambda} \\ & \times F\left((z\cosh t + ie\sinh t)(-iz\sinh t + e\cosh t)^{-1}\right).\end{aligned}$$

We can easily check that

$$\gamma_\lambda\tilde{\tau}_\lambda(t)\gamma_\lambda^{-1}f(w) = e^{-r\lambda t}f(e^{-2t}w),$$

so this indeed gives the action of the semigroup. From now on we find the explicit formula of

$$\mathcal{L}_\lambda^{-1}\tilde{\tau}_\lambda(t)\mathcal{L}_\lambda : L^2_\lambda(\Omega) \longrightarrow L^2_\lambda(\Omega).$$

We recall that the key formula for 1-dimensional case was given by

$$\int_0^\infty e^{-z\xi}\tilde{I}_{\lambda-1}(2\sqrt{\xi})\xi^{\lambda-1}d\xi = z^{-\lambda}e^{z^{-1}}.$$

So we generalize this formula for multi-variable case. In order to do this, we consider the decomposition of the polynomial space.

Theorem 3.6 (Hua–Kostant–Schmid, [2, Theorem XI.2.4]). *The polynomial space* $\mathcal{P}(\mathfrak{n}^+)$ *is decomposed under* L *as*

$$\mathcal{P}(\mathfrak{n}^+) = \bigoplus_{\substack{m_1\geq\cdots\geq m_r\geq 0\\ m_j\in\mathbb{Z}}} V_{m_1\gamma_1+\cdots+m_r\gamma_r}$$

where $\{\gamma_1, \ldots, \gamma_r\}$ *is a set of some strongly orthogonal roots, and each subspace* $V_{m_1\gamma_1+\cdots+m_r\gamma_r}$ *has nonzero* K_L*-invariant vectors.*

Let $\Phi_{\mathbf{m}} \in V_{m_1\gamma_1+\cdots+m_r\gamma_r}$ be the unique K_L-invariant polynomial such that $\Phi_{\mathbf{m}}(e) = 1$, and let $d_{\mathbf{m}} := \dim V_{m_1\gamma_1+\cdots+m_r\gamma_r}$. Also we use the following notation:

$$(\lambda)_{\mathbf{m}} := \frac{\Gamma_\Omega(\lambda+\mathbf{m})}{\Gamma_\Omega(\lambda)} = \prod_{j=1}^r\left(\lambda - \frac{d}{2}(j-1)\right)_{m_j}.$$

Using this, we define the *generalized I-Bessel function* as follows.

Definition 3.7 (Generalized I-Bessel function, [2, Section XV.2]).

$$\mathcal{I}_\lambda(z) := \sum_{\substack{m_1\geq\cdots\geq m_r\geq 0\\ m_j\in\mathbb{Z}}} \frac{1}{(\lambda)_{\mathbf{m}}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z).$$

If $\mathfrak{n}^+ = \mathbb{R}$, then $\mathcal{I}_\lambda(z) = \Gamma(\lambda)\tilde{I}_{\lambda-1}(2\sqrt{z})$ holds. For this function, we have the following formula.

Proposition 3.8 ([2, Proposition XV.2.1]). *For $\lambda > \frac{n}{r} - 1$ and $z \in \mathfrak{n}^+_{\mathbb{C}}$ with* Re $z \in \Omega$,

$$\int_\Omega e^{-(z|x)}\mathcal{I}_\lambda(x)\Delta(x)^{\lambda-\frac{n}{r}}dx = \Gamma_\Omega(\lambda)\Delta(z)^{-\lambda}e^{\operatorname{tr}(z^{-1})}.$$

Proof. Since

$$\int_\Omega e^{-(z|x)}\Phi_{\mathbf{m}}(x)\Delta(x)^{\lambda-\frac{n}{r}}dx = \Gamma_\Omega(\lambda+\mathbf{m})\Delta(z)^{-\lambda}\Phi_{\mathbf{m}}(z^{-1}),$$

$$e^{\operatorname{tr}(z)} = \sum_{\substack{m_1\geq\cdots\geq m_r\geq 0\\ m_j\in\mathbb{Z}}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z)$$

holds, the formula is proved by termwise integral. □

For $t \in \mathbb{C}$ with Re $t \geq 0$, $t \notin \sqrt{-1}\pi\mathbb{Z}$, and $\varphi \in L^2_\lambda(\Omega)$, we define

$$\tau_\lambda(t)\varphi(x) := \frac{1}{\Gamma_\Omega(\lambda)\sinh^{r\lambda}t}\int_\Omega \varphi(y)e^{-\coth t(\operatorname{tr}(x)+\operatorname{tr}(y))} \times \mathcal{I}_\lambda\left(\frac{1}{\sinh^2 t}P(x^{\frac{1}{2}})y\right)\Delta(y)^{\lambda-\frac{n}{r}}dy$$

where $P(x)y := 2x(xy) - (x^2)y$ (for example, if $\mathfrak{n}^+ = \operatorname{Sym}(r,\mathbb{R})$, then $P(x)y = xyx$). When $\mathfrak{n}^+ = \mathbb{R}$, then this coincides with the one in the previous section:

$$\tau_\lambda(t)\varphi(x) := \frac{1}{\sinh^\lambda t}\int_0^\infty \varphi(y)e^{-\coth t(x+y)}\tilde{I}_{\lambda-1}\left(\frac{2}{\sinh t}\sqrt{xy}\right)y^{\lambda-1}dy.$$

Then we can prove similarly to the 1-dimensional case that

$$\mathcal{L}_\lambda\tau_\lambda(t) = \tilde{\tau}_\lambda(t)\mathcal{L}_\lambda,$$

and especially $\tau_\lambda(t)\tau_\lambda(s) = \tau_\lambda(t+s)$ holds if the integral converges. In order to know when the integral converges, we have to know the estimate of the kernel function.

4 Speaker's results

Let U_L be the maximal compact subgroup of $L_{\mathbb{C}}$, and let $|\cdot|_1$ be the norm on $\mathfrak{n}_{\mathbb{C}}^+$ invariant under U_L, such that $|x|_1 = \mathrm{tr}(x)$ holds for any $x \in \Omega$. Then the I-Bessel function has the following upper estimate.

Theorem 4.1 (N). *If* $\mathrm{Re}\,\lambda > \frac{2n}{r} - 1 - k$ *for some* $k \in \mathbb{Z}_{\geq 0}$*, then there exists a constant* $C > 0$ *such that*

$$|\mathcal{I}_\lambda(z^2)| \leq C(1 + |z|_1^{kr})e^{2|\mathrm{Re}\, z|_1}.$$

This follows from the following integral formula.

Theorem 4.2 (N). *If* $\mathrm{Re}\,\lambda > \frac{2n}{r} - 1 - k$ *for some* $k \in \mathbb{Z}_{\geq 0}$*, then*

$$\mathcal{I}_\lambda(z^2) = c_{\lambda+k} \int_D {}_1F_1(-k, \lambda; -z, w) e^{2(z|\mathrm{Re}\, w)} h(w)^{\lambda+k-\frac{2n}{r}} dw$$

where $c_\lambda = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)}$*, and* ${}_1F_1(-k, \lambda; -z, w)$ *is a polynomial of degree* kr *with respect to* z, w*.*

Idea of proof. For simplicity, we start with $\mathfrak{n}^+ = \mathbb{R}$ case

$$\mathcal{I}_\lambda(z^2) = \frac{\lambda + k - 1}{\pi} \int_{|w|<1} {}_1F_1(-k, \lambda; -zw) e^{2z\,\mathrm{Re}\, w}(1 - |w|^2)^{\lambda+k-2} dw$$

for $k \in \mathbb{Z}_{\geq 0}$, $\mathrm{Re}\,\lambda > 1 - k$. We recall from Remark 2.2 that the inner product

$$\langle f|g \rangle_\lambda := \frac{\lambda - 1}{\pi} \int_{|w|<1} f(w)\overline{g(w)}(1 - |w|^2)^{\lambda-2} dw$$

is computed as

$$\langle f|g \rangle_\lambda = \sum_{m=0}^{\infty} \frac{m!}{(\lambda)_m} a_m \overline{b_m}$$

where $f(w) = \sum_{m=0}^{\infty} a_m w^m$, $g(w) = \sum_{m=0}^{\infty} b_m w^m$. We now define the differential operator $D^{(k)}(\lambda)$ by

$$D^{(k)}(\lambda) := w^{1-\lambda} \frac{d^k}{dw^k} w^{\lambda-1+k}.$$

Then we can prove easily that

$$D^{(k)}(\lambda) w^m = (\lambda + m)_k w^m.$$

Therefore, we have

$$\frac{1}{(\lambda)_k}\langle D^{(k)}(\lambda)f|g\rangle_{\lambda+k} = \frac{1}{(\lambda)_k}\sum_{m=0}^{\infty}\frac{m!}{(\lambda+k)_m}(\lambda+m)_k a_m\overline{b_m}$$

$$=\sum_{m=0}^{\infty}\frac{m!(\lambda+m)_k}{(\lambda)_{m+k}}a_m\overline{b_m} = \sum_{m=0}^{\infty}\frac{m!}{(\lambda)_m}a_m\overline{b_m}. \qquad (1)$$

Especially if $f(w) = e^{zw}$ and $g(w) = e^{\overline{z}w}$, then

$$\frac{1}{(\lambda)_k}\langle D^{(k)}(\lambda)e^{zw}|e^{\overline{z}w}\rangle_{\lambda+k} = \sum_{m=0}^{\infty}\frac{m!}{(\lambda)_m}\frac{z^m}{m!}\frac{\overline{\overline{z}^m}}{m!}$$

$$=\sum_{m=0}^{\infty}\frac{1}{(\lambda)_m m!}z^{2m} = \mathcal{I}_\lambda(z^2) = \Gamma(\lambda)\tilde{I}_{\lambda-1}(2z) = \text{(LHS)}$$

holds. On the other hand, we have

$$D^{(k)}(\lambda)e^{zw} = w^{1-\lambda}\frac{d^k}{dw^k}w^{\lambda-1+k}e^{zw} = w^{1-\lambda}\sum_{j=0}^{k}\binom{k}{j}\frac{d^{k-j}(w^{\lambda-1+k})}{dw^{k-j}}\frac{d^j e^{zw}}{dw^k}$$

$$=w^{1-\lambda}\sum_{j=0}^{k}\binom{k}{j}(\lambda+j)_{k-j}w^{\lambda-1+j}z^je^{zw} = \sum_{j=0}^{k}\frac{(-1)^j(-k)_j}{j!}\frac{(\lambda)_k}{(\lambda)_j}w^jz^je^{zw}$$

$$=(\lambda)_k\sum_{j=0}^{k}\frac{(-k)_j}{(\lambda)_j j!}(-zw)^je^{zw} = (\lambda)_k {}_1F_1(-k,\lambda;-zw)e^{zw},$$

and therefore

$$\frac{1}{(\lambda)_k}\langle D^{(k)}(\lambda)e^{zw}|e^{\overline{z}w}\rangle_{\lambda+k}$$

$$=\frac{\lambda+k-1}{\pi}\int_{|w|<1}{}_1F_1(-k,\lambda;-zw)e^{zw}e^{z\overline{w}}(1-|w|^2)^{\lambda+k-2}dw = \text{(RHS)}$$

holds, and we have proved (LHS)=(RHS).

For general case, we redefine $D^{(k)}(\lambda)$ as

$$D^{(k)}(\lambda) := \Delta(w)^{\frac{n}{r}-\lambda}\Delta\left(\frac{\partial}{\partial w}\right)^k\Delta(w)^{\lambda-\frac{n}{r}+k}.$$

Then instead of (1), if $f, g \in \mathcal{O}(D)$ is decomposed as $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$, $g = \sum_{\mathbf{m}} g_{\mathbf{m}}$, according to the decomposition in Theorem 3.6, we have

$$\frac{1}{(\lambda)_k}\langle D^{(k)}(\lambda)f|g\rangle_{\lambda+k} = \sum_{\mathbf{m}}\frac{1}{(\lambda)_{\mathbf{m}}}\langle f_{\mathbf{m}}|g_{\mathbf{m}}\rangle_F$$

where this $(\lambda)_k$ means $(\lambda, \ldots, \lambda)_{(k,\ldots,k)} = \prod_{j=1}^r \left(\lambda - \frac{d}{2}(j-1)\right)_k$, and $\langle f_{\mathbf{m}} | g_{\mathbf{m}} \rangle_F$ is the *Fischer norm* defined by

$$\langle f | g \rangle_F := \frac{1}{\pi^n} \int_{\mathfrak{n}_{\mathbb{C}}^+} f(z)\overline{g(z)} e^{-|z|^2} dz$$

(see [2, Corollary XIII.2.3, Proposition XIV.2.2]). Using this, we can prove the theorem in the similar way. □

Proof of Theorem 4.1.

$$\begin{aligned}
&|\mathcal{I}_\lambda(z^2)| \\
=&|c_{\lambda+k}| \int_D |{}_1F_1(-k, \lambda; -z, w)| e^{2(\operatorname{Re} z | \operatorname{Re} w)} h(w)^{\operatorname{Re}\lambda + k - \frac{2n}{r}} dw \\
\leq&|c_{\lambda+k}| \int_D \left(1 + (|z|_1 |w|_\infty)^{kr}\right) e^{2|\operatorname{Re} z|_1 |\operatorname{Re} w|_\infty} h(w)^{\operatorname{Re}\lambda + k - \frac{2n}{r}} dw
\end{aligned}$$

Since $w \in D$ holds if and only if $|w|_\infty < 1$, where $|\cdot|_\infty$ is the dual norm of $|\cdot|_1$,

$$\begin{aligned}
\leq&|c_{\lambda+k}|(1 + |z|_1^{kr}) e^{2|\operatorname{Re} z|_1} \int_D h(w)^{\operatorname{Re}\lambda + k - \frac{2n}{r}} dw \\
\leq&C(1 + |z|_1^{kr}) e^{2|\operatorname{Re} z|_1}.
\end{aligned}$$ □

We recall that τ_λ is given by

$$\begin{aligned}
\tau_\lambda(t)\varphi(x) :=& \frac{1}{\Gamma_\Omega(\lambda)\sinh^{r\lambda} t} \int_\Omega \varphi(y) e^{-\coth t(\operatorname{tr}(x) + \operatorname{tr}(y))} \\
&\times \mathcal{I}_\lambda\left(\frac{1}{\sinh^2 t} P(x^{\frac{1}{2}}) y\right) \Delta(y)^{\lambda - \frac{n}{r}} dy.
\end{aligned}$$

Now we give the upper estimate of the integral kernel.

Corollary 4.3 (N). *If $\lambda > \frac{2n}{r} - 1 - k$ for some $k \in \mathbb{Z}_{\geq 0}$, then there exists a constant $C > 0$ such that for $t = u + iv$ with $u \geq 0$,*

$$\begin{aligned}
&\left| e^{-\coth t(\operatorname{tr}(x)+\operatorname{tr}(y))} \mathcal{I}_\lambda\left(\frac{1}{\sinh^2 t} P(x^{\frac{1}{2}}) y\right) \right| \\
\leq&C\left(1 + (\operatorname{tr}(x)\operatorname{tr}(y))^{\frac{kr}{2}}\right) \exp\left(-\frac{\sinh u}{\cosh u + |\cos v|}(\operatorname{tr}(x) + \operatorname{tr}(y))\right).
\end{aligned}$$

Therefore, if $\operatorname{Re} t = 0$ and $\lambda > \frac{2n}{r} - 1$ we have

$$\tau_\lambda(t) : L^1(\Omega, \Delta(x)^{\lambda - \frac{n}{r}} dx) \longrightarrow L^\infty(\Omega),$$

and if $\operatorname{Re} t > 0$ and $\lambda > \frac{n}{r} - 1$, we have

$$\tau_\lambda(t) : \{\text{Polynomial growth functions}\} \longrightarrow \{\text{Exponential decay functions}\}.$$

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