THE SHARP GROWTH ESTIMATE FOR $\mathcal{U}(\lambda)$ (Some inequalities concerned with the geometric function theory)

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THE SHARP GROWTH ESTIMATE FOR $U(\lambda)$

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ABSTRACT. For $0 < \lambda \leq 1$ let $U(\lambda)$ be the class of analytic functions in the unit disk $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ satisfying $|f'(z)(z/f(z))^2 - 1| < \lambda$ in $\mathbb{D}$. Then it is known that every $f \in U(\lambda)$ is univalent in $\mathbb{D}$. In the present article we shall prove the sharp estimates $|f''(0)| \leq 2(1+\lambda)$ and $|z|/(1+|z|)(1+\lambda|z|) \leq |f(z)| \leq |z|/(1-|z|)(1-\lambda|z|)$. As an application we shall also give the sharp covering theorems.

1. INTRODUCTION

We denote the complex plane by $\mathbb{C}$ and the extended complex plane by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For $c \in \mathbb{C}$ and $r > 0$ let $\mathbb{D}(c, r) = \{z \in \mathbb{C} : |z-c| < r\}$ and $\mathbb{D} = \mathbb{D}(0,1)$. Similarly let $\Delta_r = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0,r) = \{z \in \hat{\mathbb{C}} : r < |z| \leq \infty\}$ and $\Delta = \Delta_1$.

Let $\mathcal{A}([\mathbb{D}])$ denote the space of analytic functions in $\mathbb{D}$ and $\mathcal{A}_0([\mathbb{D}]) = \{f \in \mathcal{A}([\mathbb{D}]) : f(0) = f'(0) - 1 = 0\}$. Here we regard $\mathcal{A}([\mathbb{D}])$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. A function $f$ is said to be univalent in a domain $D$ if it is one-to-one in $D$. Let $S$ denote the class of univalent functions in $\mathcal{A}_0([\mathbb{D}])$.

For $0 < \lambda \leq 1$ let $U(\lambda)$ be the class of functions $f \in \mathcal{A}_0$ satisfying

\[
(1.1) \quad |f'(z) \left( \frac{z}{f(z)} \right)^2 - 1| < \lambda
\]

in $\mathbb{D}$. The boundedness of $f'(z)(z/f(z))^2$ forces $f \in U(\lambda)$ that $f(z) \neq 0$ in $\mathbb{D}\setminus\{0\}$. Hence $f'(z) \neq 0$ holds in $\mathbb{D}$ and $f$ is locally univalent in $\mathbb{D}$. Moreover it is known that $f \in U(\lambda)$ is univalent in $\mathbb{D}$, i.e., $U(\lambda) \subset S$. 
In the present article we shall prove the sharp inequalities
\[
|a_2(f)| = 2^{-1}|f''(0)| \leq 1 + \lambda,
\]
\[
\frac{|z|}{(1 - \lambda|z|)(1 - |z|)} \leq |f(z)| \leq \frac{|z|}{(1 - \lambda|z|)(1 - |z|)}
\]
for \( f \in \mathcal{U}(\lambda) \). To this end we introduce three classes of meromorphic functions in \( \Delta \) closely related to \( \mathcal{U}(\lambda) \). For \( 0 < \lambda \leq 1 \) let \( \mathcal{M}(\lambda) \) be the class of meromorphic functions \( g \) in \( \Delta \) which has a Laurent expansion of a form
\[
(1.2) \quad g(w) = w + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots, \quad 1 < |w| < \infty
\]
and satisfying
\[
|g'(w) - 1| < \lambda.
\]
Now for \( f \in \mathcal{U}(\lambda) \) put
\[
T(f)(w) = \frac{1}{f(\frac{1}{w})}, \quad w \in \Delta.
\]
Then we have
\[
\frac{d}{dw}T(f)(w) = \frac{f'(1/w)}{w^2 f^2(\frac{1}{w})} = f'(z) \left( \frac{z}{f(z)} \right)^2, \quad z = \frac{1}{w}
\]
and hence \( Tf \in \mathcal{M}(\lambda) \). Thus \( T \) is a transformation which maps \( \mathcal{U}(\lambda) \) injectively into \( \mathcal{M}(\lambda) \). The image \( T(\mathcal{U}(\lambda)) \) is a proper subset of \( \mathcal{M}(\lambda) \) and it is easy to see that
\[
\mathcal{M}_0(\lambda) = \{ g \in \mathcal{M}(\lambda) : g(w) \neq 0 \text{ in } \Delta \} = T(\mathcal{U}(\lambda)).
\]
Notice that \( a_2(f) = -c_0(T(f)) \) hold for \( f \in \mathcal{U}(\lambda) \). Moreover let
\[
\mathcal{M}(\lambda) = \{ g \in \mathcal{M}(\lambda) : c_0(g) = 0 \}.
\]
In the next section we shall show that every \( g \in \mathcal{M}(\lambda) \) satisfies \( g(w) \neq 0 \) in \( \Delta \). Thus the relation
\[
T(\mathcal{U}(\lambda)) = \mathcal{M}(\lambda) \subset T(\mathcal{U}(\lambda)) = \mathcal{M}_0(\lambda) \subset \mathcal{M}(\lambda)
\]
holds.

In Section 2 we shall derive an integral representation of \( g \in \mathcal{M}(\lambda) \) and prove existence of the boundary limit \( g(\eta) = \lim_{\Delta \ni w \to \eta} g(w) \) for each \( \eta \in \partial \Delta \). Further we shall precisely study the boundary values of \( g \in \mathcal{M}(\lambda) \) and obtain the sharp estimate \(|g(\eta)| \leq 1 + \lambda\) for \( g \in \mathcal{M}(\lambda) \) and \( \eta \in \partial \Delta \), which is equivalent to the sharp upper bound \(|a_2(f)| \leq 1 + \lambda\) for \( f \in \mathcal{U}(\lambda) \).
In Section 3 for each fixed $w_0 \in \Delta \setminus \{\infty\}$ we shall treat the sharp estimate on $|g(w_0)|$ for $g \in \mathcal{M}_0(\lambda)$. The result has an immediate counterpart in $\mathcal{U}(\lambda)$ and we shall derive the sharp growth estimate and the sharp covering theorem for $\mathcal{U}(\lambda)$.

2. INTEGRAL REPRESENTATION

For $g \in \mathcal{M}(\lambda)$ let
\begin{align}
(2.1) \quad b_g(w) &= \frac{w^2}{\lambda}(1 - g'(w)), \quad w \in \Delta, \\
(2.2) \quad \beta_g(z) &= b_g(1/z), \quad z \in \mathbb{D}.
\end{align}
Note that $g'(w) - 1 = O(w^{-2})$ as $w \to \infty$ and that $g'(1/z)$ is analytic in $\mathbb{D}$. Applying the maximum modulus principle to $g'(1/z) - 1$ we have
\[
|g'(w) - 1| \leq \frac{\lambda}{|w|^2}, \quad 1 < |w| < \infty.
\]
Hence $\beta_g \in H_1^\infty(\mathbb{D})$ and for any $w, w_0 \in \Delta \setminus \{\infty\}$ by integrating $g'(w) - 1 = -\lambda b_g(w)w^{-2}$ we obtain
\[
g(w) - w = g(w_0) - w_0 - \lambda \int_{w_0}^{w} \frac{b_g(\zeta)}{\zeta^2} d\zeta.
\]
Since $\lim_{w_0 \to \infty}(g(w_0) - w_0) = c_0$, we have
\[
g(w) = w + c_0 - \lambda \int_{\infty}^{w} \frac{b_g(\zeta)}{\zeta^2} d\zeta = w + c_0 + \lambda \int_{0}^{1/w} \beta_g(\zeta) d\zeta.
\]
Converse is also true and we have the following.

**Theorem 2.1.** For a meromorphic function $g$ in $\Delta$, $g \in \mathcal{M}(\lambda)$ if and only if there exist $\beta \in H_1^\infty(\mathbb{D})$ and $c \in \mathbb{C}$ such that
\[
g(w) = w + c + \lambda \int_{0}^{1/w} \beta(\zeta) d\zeta.
\]

**Corollary 2.2.** Each $g \in \mathcal{M}(\lambda)$ is Lipschitz continuous and satisfies
\begin{align}
(2.3) \quad \left(1 - \frac{\lambda}{|w_0w_1|}\right)|w_1 - w_0| &\leq |g(w_1) - g(w_0)| \leq \left(1 + \frac{\lambda}{|w_0w_1|}\right)|w_1 - w_0|\end{align}
for $w_0, w_1 \in \Delta$. Particularly

(i) The limit $g(\eta) = \lim_{\Delta \ni w \to \eta} g(w)$ exists for every $\eta \in \partial\Delta$.
(ii) $g$ is univalent in $\Delta$. Furthermore if $0 < \lambda < 1$, then $g$ is univalent on $\overline{\Delta}$.
Proof. Inequality (2.3) easily follows from Theorem 2.1 and
\[ \left| \int_{1/w_0}^{1/w_1} \beta(\zeta) \, d\zeta \right| \leq \frac{1}{w_1} - \frac{1}{w_0} = \frac{|w_1 - w_0|}{|w_0w_1|}. \]
By (2.3) the function $g$ is Lipschitz continuous in $\Delta$ and from this the boundary limit $g(\eta) = \lim_{\Delta \ni \omega \to \eta} g(w)$ exists for each $\eta \in \partial \Delta$. Inequality (2.3) also shows that $g$ is univalent in $\Delta$. Since (2.3) still holds on $\overline{\Delta}$ by continuity, $g$ is univalent on $\overline{\Delta}$, when $0 < \lambda < 1$. 

Each $f \in \mathcal{U}(\lambda)$ is univalent in $\mathbb{D}$, since $Tf$ is univalent in $\Delta$ by Corollary 2.2.

Corollary 2.3. For each $0 < \lambda \leq 1$ the inclusion relation $\tilde{\mathcal{M}}(\lambda) \subset \mathcal{M}_0(\lambda)$ holds.

Proof. For $g \in \tilde{\mathcal{M}}(\lambda)(\subset \mathcal{M}(\lambda))$ we have by Theorem 2.1
\[ |g(w)| = |w + \lambda \int_{0}^{1/w} \beta(\zeta) \, d\zeta| \geq |w| - \frac{\lambda}{|w|} > 0, \quad |w| > 1. \]
Thus $g$ has no zeros in $\Delta$ and hence $g \in \mathcal{M}_0(\lambda)$. 

For $g \in \mathcal{M}(\lambda)$ let $E(g)$ be the omitted set of $g$, i.e.,
\[ E(g) = \hat{\mathbb{C}} \setminus g(\Delta). \]
For $R > 1$ the image $g(\partial \Delta_R)$ is an analytic Jordan curve and $g(\Delta_R)$ is the domain outside $g(\partial \Delta_R)$. Let $D_R$ be the domain bounded by $g(\partial \Delta_R)$. Then $\{D_R : 1 < R\}$ is a 1-parameter family of increasing domains in $\mathbb{C}$ and
\[ E(g) = \bigcap_{R > 1} D_R. \]
In particular when $0 < \lambda < 1$, by Corollary 2.2 $E(g)$ is a closed Jordan domain bounded by $g(\partial \Delta)$. Notice that for any $g \in \mathcal{M}(\lambda)$ with $0 < \lambda \leq 1$ and $a \in \partial E(g)$ there exists $\eta \in \partial \Delta$ such that $a = g(\eta)$.

Theorem 2.4. Let $g \in \mathcal{M}(\lambda)$ and $g(w) = w + c_0 + \int_0^{1/w} \beta(\zeta) \, d\zeta$ with $\beta \in H_1^\infty(\mathbb{D})$ and $c_0 \in \mathbb{C}$. Then $g \in \mathcal{M}_0(\lambda)$ if and only if
\[ -c_0 \in E(\tilde{g}), \tag{2.4} \]
where
\[ \tilde{g}(w) = w + \int_0^{1/w} \beta(\zeta) \, d\zeta \in \tilde{\mathcal{M}}(\lambda). \]
Proof. For \( g \in \mathcal{M}(\lambda) \), by definition, \( g \in \mathcal{M}_0(\lambda) \) if and only if \( g(w) = \tilde{g}(w) + c_0 \neq 0 \) for all \( w \in \Delta \). This is equivalent to (2.4).

For \( g \in \mathcal{M}_0(\lambda) \) the coefficient \( c_0 = c_0(g) \) in the expansion (1.2) is called the conformal center of the set \( E(g) \). For more details on conformal center we refer to [8].

Notice that \( g + c \in \mathcal{M}(\lambda) \) holds for any \( g \in \mathcal{M}(\lambda) \) and \( c \in \mathbb{C} \). This implies that there are no upper bound on \( |c_0(g)| \) for \( g \in \mathcal{M}(\lambda) \). However concerning with the class \( \mathcal{M}_0(\lambda) \), it is not difficult to get the sharp estimate.

**Theorem 2.5.** Let \( \lambda \in (0, 1] \). Then,

(a) For \( g \in \tilde{\mathcal{M}}(\lambda) \) the sharp estimate \( 1 - \lambda \leq |g(\eta)| \leq 1 + \lambda \) holds on \( \partial \Delta \). Furthermore equality \( |g(\eta)| = 1 - \lambda \) at some \( \eta \in \partial \Delta \) if and only if \( g(w) \equiv w - \lambda \eta^2/w \), and \( |g(\eta)| = 1 + \lambda \) if and only if \( g(w) \equiv w + \lambda \eta^2/w \).

(b) Inequality \( |c_0(g)| \leq 1 + \lambda \) holds for \( g \in \mathcal{M}_0(\lambda) \) with equality if and only if

\[
g(w) = w \left(1 + \frac{\lambda e^{i\theta}}{w}\right) \left(1 + \frac{e^{i\theta}}{w}\right), \quad w \in \Delta.
\]

for some real \( \theta \).

(c) Inequality \( |a_2(f)| \leq 1 + \lambda \) holds for \( f \in \mathcal{U}(\lambda) \) with equality if and only if

\[
f(z) = \frac{z}{(1 + \lambda e^{i\theta}z)(1 + e^{i\theta}z)}, \quad z \in \mathbb{D}
\]

for some real \( \theta \).

Proof. Let \( g \in \tilde{\mathcal{M}}(\lambda) \) and put \( \beta \in H_1^\infty(\mathbb{D}) \) as in (2.2). Then

\[
(2.5) \quad g(w) = w + \lambda \int_0^{1/w} \beta(\zeta) \, d\zeta.
\]

For \( \eta \in \partial \Delta \) we consider \( \int_0^{1/\eta} \beta(\zeta) \, d\zeta \) as the Lebesgue integral along a \( C^1 \)-path connecting 0 to \( 1/\eta \) and contained in \( \mathbb{D} \) except for the end point \( 1/\eta \). Then the integral does not depend on choice of path. Thus (2.5) still holds for \( \eta \in \partial \Delta \) and we have

\[
|g(\eta)| \leq |\eta| + \lambda \left| \int_0^{1/\eta} \beta(\zeta) \, d\zeta \right| \leq 1 + \lambda \int_{[0,1/\eta]} |\beta(\zeta)| \, d\zeta \leq 1 + \lambda,
\]
where $[0,1/\eta]$ is the radial line segment connecting 0 and $1/\eta$. If $|g(\eta_0)| = 1 + \lambda$ at some $\eta_0 \in \partial \Delta$, then by the maximum modulus theorem $\beta = \epsilon$ for some $\epsilon \in \partial \mathbb{D}$ and

$$\frac{\lambda \int_0^{1/\eta} \beta(\zeta) \, d\zeta}{\eta} = \frac{\lambda \epsilon}{\eta^2} > 0.$$  

Therefore $\epsilon = \eta^2$ and $g(w) \equiv w + \lambda\eta^2/w$. Similarly

$$|g(\eta)| \geq |\eta| - \lambda \int_0^{1/\eta} |\beta(\zeta)| \, d\zeta \geq 1 - \lambda \int_{[0,1/\eta]} |\beta(\zeta)| \, d\zeta \geq 1 - \lambda$$

with equality if and only if $\beta = \epsilon \in \partial \mathbb{D}$ and $\lambda \epsilon/\eta^2 < 0$, i.e., $\epsilon = -\eta^2$ and therefore $g(w) \equiv w - \lambda\eta^2/w$. This completes the proof of (a).

To show (b) let $g \in \mathcal{M}_0(\lambda)$. Then $g$ can be expressed as $g = \tilde{g} + c_0$ with $\tilde{g} \in \tilde{\mathcal{M}}(\lambda)$ and $-c_0 \in E(\tilde{g})$. By (a) and $E(\tilde{g}) = \cap_{R>1} D(R)$, where $D(R)$ is a domain bounded by the Jordan curve $\tilde{g}(\partial \Delta_R)$. Hence we have $E(\tilde{g}) \subset \overline{\mathbb{D}}(0,1+\lambda)$ and $|c_0(g)| \leq 1 + \lambda$.

Suppose now that $|c_0(g)| = 1 + \lambda$. Combining $-c_0(g) \in E(\tilde{g}) \cap \partial \mathbb{D}(0,1+\lambda)$ and $E(\tilde{g}) \subset \overline{\mathbb{D}}(0,1+\lambda)$, we have $-c_0(g) \in \partial E(\tilde{g})$. By Lipschitz continuity of $\tilde{g}$ there exists $\eta \in \partial \Delta$ with $-c_0(g) = \tilde{g}(\eta)$. Since $|\tilde{g}(\eta)| = |c_0(g)| = 1 + \lambda$, it follows from (a) that $\tilde{g}(w) = w + \lambda\eta^2/w$ and hence

$$g(w) = \tilde{g}(w) + c_0 = \tilde{g}(w) - \tilde{g}(\eta) = w \left(1 + \frac{\lambda\eta}{w}\right) \left(1 + \frac{\eta}{w}\right).$$

By letting $e^{i\theta} = -\eta$ we obtain (b).

Since $a_2(f) = -c_0(T(f))$ holds for $f \in \mathcal{U}(\lambda)$, (c) follows directly from (b).

\[\square\]

3. Growth Estimates

Let $0 < \lambda \leq 1$ and $1 < |w_0| < \infty$. Combining Theorem 2.1 and Theorem 2.5 (b) it is easily seen that if $g \in \mathcal{M}_0(\lambda)$ then

$$|g(w_0)| \leq |w_0| + \lambda \int_{[0,1/w_0]} |\beta(\zeta)| \, d\zeta + |c_0|$$

$$\leq |w_0| + \frac{\lambda}{|w_0|} + 1 + \lambda = |w_0| \left(1 + \frac{\lambda}{|w_0|}\right) \left(1 + \frac{1}{|w_0|}\right)$$
with equality at $w_0 = R e^{i\theta}$ if and only if $g(w) \equiv w(1 + \lambda e^{i\theta}w^{-1})(1 + e^{i\theta}w^{-1})$. Similarly the lower estimate

$$|g(w_0)| \geq |w_0| - \frac{\lambda}{|w_0|} - (1 + \lambda)$$

holds for $g \in \mathcal{M}(\lambda)$. Clearly it is not sharp, since the right hand side is negative for all $r$ sufficiently close to 1.

In this section we deal with the region of variability $V_\lambda(w_0)$ of $g(w_0)$, when $g$ varies on $\mathcal{M}_0(\lambda)$, i.e.,

$$V_\lambda(w_0) = \{g(w_0) : g \in \mathcal{M}_0(\lambda)\}.$$

We shall show that $V_\lambda(w_0)$ is a closed Jordan domain bounded by a simple closed curve and give a parameterization of the boundary curve. Using these ideas we will obtain the sharp lower estimate on $|g(w_0)|$ when $g \in \mathcal{M}_0(\lambda)$.

First we notice that $V_\lambda(w_0)$ is a compact subset of $\mathbb{C}$. Indeed by (3.1) it is clear that $\mathcal{M}_0(\lambda)$ is a family of analytic functions in $\Delta \setminus \{\infty\}$ which is locally uniformly bounded and hence normal. Moreover if a sequence $\{g_n\}_{n=1}^\infty$ in $\mathcal{M}_0(\lambda)$ converge to $g$ locally uniformly in $\Delta \setminus \{\infty\}$, then it is not difficult to see that $g \in \mathcal{M}_0(\lambda)$. Thus $\mathcal{M}_0(\lambda)$ is a compact family with respect to the topology of locally uniform convergence. Since $V_\lambda(w_0)$ is the image of $\mathcal{M}_0(\lambda)$ with respect to the continuous mapping $\mathcal{M}_0(\lambda) \ni g \mapsto g(w_0) \in \mathbb{C}$, $V_\lambda(w_0)$ is also an compact subset of $\mathbb{C}$.

Next for $g \in \mathcal{M}_0(\lambda)$ let $g_\theta(w) = e^{-i\theta}g(e^{i\theta}w)$. Then $g_\theta \in \mathcal{M}_0(\lambda)$ for any $\theta \in \mathbb{R}$. From this it follows that

$$V_\lambda(R e^{i\theta}) = e^{i\theta}V_\lambda(R)$$

and it suffices to determine $V_\lambda(R)$ for $1 < R < \infty$. Similarly for $g \in \mathcal{M}_0(\lambda)$ let $\overline{g}(w) = \overline{g(\overline{w})}$. Then $\overline{g} \in \mathcal{M}_0(\lambda)$ and hence $V_\lambda(R)$ is symmetric with respect to $\mathbb{R}$.

**Theorem 3.1.** Let $0 < \lambda \leq 1$. Then

(i) For $g \in \mathcal{M}_0(\lambda)$

$$|w| \left(1 - \frac{\lambda}{|w|}\right) \left(1 - \frac{1}{|w|}\right) \leq |g(w)| \leq |w| \left(1 + \frac{\lambda}{|w|}\right) \left(1 + \frac{1}{|w|}\right),$$

for $1 < |w| < \infty$ with equality $w_0 = R_0 e^{i\theta_0}$ if and only if

$$g(w) = w \left(1 - \frac{\lambda e^{i\theta_0}}{w}\right) \left(1 - \frac{e^{i\theta_0}}{w}\right) \text{ or } g(w) = w \left(1 + \frac{\lambda e^{i\theta_0}}{w}\right) \left(1 + \frac{e^{i\theta_0}}{w}\right),$$

respectively.
(ii) For $f \in \mathcal{U}(\lambda)$

\[
\frac{|z|}{(1+|z|)(1+\lambda|z|)} \leq |f(z)| \leq \frac{|z|}{(1-|z|)(1-\lambda|z|)}, \quad 0 < |z| < 1
\]

with equality at $z = r_0e^{i\theta_0}$ if and only if

\[
f(z) = \frac{z}{(1+\lambda e^{i\theta_0}z)(1+e^{i\theta_0}z)} \text{ or } f(z) = \frac{z}{(1-\lambda e^{i\theta_0}z)(1-e^{i\theta_0}z)}
\]

respectively.

**Theorem 3.2.** Let $f \in \mathcal{U}(\lambda)$ with $0 < \lambda \leq 1$. Then

\[
\mathbb{D} \left(0, \frac{1}{2(1+\lambda)} \right) \subset f(\mathbb{D}).
\]

Furthermore $\frac{e^{i\theta_0}}{2(1+\lambda)} \notin f(\mathbb{D})$ holds if and only if

\[
f(z) = \frac{z}{(1+\lambda e^{-i\theta_0}z)(1+e^{-i\theta_0}z)}.
\]

Now we define auxiliary functions. For $\epsilon \in \overline{\mathbb{D}}$ let

\[
\tilde{G}_{\lambda,\epsilon}(w) = w + \frac{\lambda \epsilon}{w}
\]

and

\[
E_\lambda = \begin{cases} 
\{u+iv : (u/(1+\lambda))^2 + (v/(1-\lambda))^2 \leq 1\}, & 0 < \lambda < 1 \\
[-2, 2], & \lambda = 1.
\end{cases}
\]

Notice that $E(\tilde{G}_{\lambda,\epsilon^{i\theta}}) = e^{i\theta/2}E_\lambda$ for $\theta \in \mathbb{R}$.

**Proposition 3.3.** Let $g \in \mathcal{M}_0(\lambda)$. If $g(R) \in \partial V_\lambda(R)$, then there exists $\epsilon$, $\eta$ with $|\epsilon| = |\eta| = 1$, such that $g = \tilde{G}_{\lambda,\epsilon} - \tilde{G}_{\lambda,\epsilon}(\eta)$.

**Proof.** By Theorem 2.4 $g$ can be decomposed as $g = \tilde{g} + c_0$, where $\tilde{g}(w) = w + \lambda \int_0^1 w_0 \beta_g(\zeta) d\zeta \in \tilde{M}(\lambda)$ and $-c_0 \in E(\tilde{g})$. Again by Theorem 2.4

\[
g(R) = \tilde{g}(R) + c_0 \in \tilde{g}(R) - E(\tilde{g}) \subset V_\lambda(R).
\]

Thus $-c_0$ cannot be an interior point of $E(\tilde{g})$, otherwise $g(R)$ is an interior point of $V_\lambda(R)$. Hence $-c_0 \in \partial E(\tilde{g})$. By Lipschitz continuity of $\tilde{g}$ there exists $\eta \in \partial \Delta$ such that $-c_0 = \tilde{g}(\eta)$. Therefore

\[
g(R) = \tilde{g}(R) - \tilde{g}(\eta) = R - \eta + \lambda \int_{1/\eta}^{1/R} \beta_g(\zeta) d\zeta.
\]
Notice that $R \neq \eta$, since $g(R) \neq 0$. Then we have

$$\left| \int_{1/\eta}^{1/R} \beta_g(\zeta) \, d\zeta \right| \leq \left| \frac{1}{R} - \frac{1}{\eta} \right|$$

with equality if and only if $\beta_g = \epsilon$ for some $\epsilon$ with $|\epsilon| = 1$.

Suppose that

$$\left| \int_{1/\eta}^{1/R} \beta_g(\zeta) \, d\zeta \right| < \left| \frac{1}{R} - \frac{1}{\eta} \right| = \left| \frac{\eta - R}{R \eta} \right|.$$  

Put

$$\epsilon_0 = \frac{R \eta}{\eta - R} \int_{1/\eta}^{1/R} \beta_g(\zeta) \, d\zeta \in \mathbb{D}.$$  

Then

$$\tilde{G}_{\lambda, \epsilon_0}(R) - \tilde{G}_{\lambda, \epsilon_0}(\eta) = R - \eta + \lambda \epsilon_0 \left( \frac{1}{R} - \frac{1}{\eta} \right) = R - \eta + \lambda \int_{1/\eta}^{1/R} \beta_g(\zeta) \, d\zeta = \tilde{g}(R) - \tilde{g}(\eta) = g(R).$$

On the other hand since $\tilde{G}_{\lambda, \epsilon} - \tilde{G}_{\lambda, \epsilon}(\eta) \in \mathcal{M}_0(\lambda)$ for $c \in \overline{\mathbb{D}}$, we have $\tilde{G}_{\lambda, \epsilon}(R) - \tilde{G}_{\lambda, \epsilon}(\eta) \in V_\lambda(R)$ for $c \in \mathbb{D}$. The mapping $\mathbb{D} \ni c \mapsto \tilde{G}_{\lambda, \epsilon}(R) - \tilde{G}_{\lambda, \epsilon}(\eta) \in V_\lambda(R)$ is an analytic function of $c \in \mathbb{D}$. Since $R \neq \eta$, the mapping is not constant and hence it is an open mapping. Thus $g(R) = \tilde{G}_{\lambda, \epsilon_0}(R) - \tilde{G}_{\lambda, \epsilon_0}(\eta)$ is an interior point of $V_\lambda(R)$, which contradict the assumption that $g(R) \in \partial V_\lambda(R)$. Therefore $\beta_g = \epsilon$ for some $\epsilon \in \partial \mathbb{D}$ and $g = \tilde{G}_{\lambda, \epsilon} - \tilde{G}_{\lambda, \epsilon}(\eta)$.

\[\square\]

Proof of Theorem 3.1. Since (ii) follows directly from (i), it suffices to show (i). From compactness of $\mathcal{M}_0(\lambda)$ it follows that there exist $g_1, g_2 \in \mathcal{M}_0(\lambda)$ such that

$$|g_1(R)| = \min_{g \in \mathcal{M}_0(\lambda)} |g(R)| \quad \text{and} \quad |g_2(R)| = \max_{g \in \mathcal{M}_0(\lambda)} |g(R)|.$$

Then clearly $g_2(R) \in \partial V_\lambda(R)$. Also $g_1(R) \in \partial V_\lambda(R)$ follows from the fact that $0 \notin V_\lambda(R)$. Thus by Proposition 3.3 there exist $\epsilon_j, \eta_j$ with $|\epsilon_j| = |\eta_j| = 1$ such that $g_j = \tilde{G}_{\lambda, \epsilon_j} - \tilde{G}_{\lambda, \epsilon_j}(\eta_j)$ for $j = 1, 2$. Since

$$g_j(R) = \tilde{G}_{\lambda, \epsilon_j}(R) - \tilde{G}_{\lambda, \epsilon_j}(\eta_j) = (R - \eta_j) \left( 1 - \frac{\lambda \epsilon_j}{R \eta_j} \right),$$
we have
$$|g_1(R)| = \min_{g \in \mathcal{M}_0(\lambda)} |g(R)| \leq \tilde{G}_{\lambda, 1}(R) - \tilde{G}_{\lambda, 1}(1)$$
$$= (R - 1) \left( 1 - \frac{\lambda}{R} \right)$$
$$\leq \left| (R - \eta_1) \left( 1 - \frac{\lambda \epsilon_1}{R \eta_1} \right) \right| = |g_1(R)|.$$  
Thus \( \eta_1 = \epsilon_1 = 1 \) and hence
$$g_1(w) \equiv \tilde{G}_{\lambda, 1}(w) - \tilde{G}_{\lambda, 1}(1) = w + \frac{\lambda}{w} - (1 + \lambda).$$

We have shown that for \( g \in \mathcal{M}(\lambda) \)
$$(R - 1) \left( 1 - \frac{\lambda}{R} \right) \leq |g(R)|$$
with equality if and only if \( g(w) = w + \lambda w^{-1} - (1 + \lambda) \). Applying this to \( g_\theta(w) = e^{-i\theta} g(e^{i\theta} w) \) we have for \( w = Re^{i\theta} \) and \( g \in \mathcal{M}_0(\lambda) \)
$$(|w| - 1) \left( 1 - \frac{\lambda}{|w|} \right) = (R - 1) \left( 1 - \frac{\lambda}{R} \right) \leq |g_\theta(R)| = |g(w)|$$
with equality \( g_\theta(w) = w + \lambda w^{-1} - (1 + \lambda) \), i.e.,
$$g(w) = w + \lambda e^{2i\theta} w^{-1} - (1 + \lambda) e^{i\theta} = w \left( 1 - \frac{\lambda e^{i\theta}}{w} \right) \left( 1 - \frac{e^{i\theta}}{w} \right).$$

In the same manner we can treat the rest of the proof of (i). \( \square \)

**Proof of Theorem 3.2.** For \( f \in \mathcal{U}(\lambda) \) the relation \( \mathbb{D}(0, (2(1 + \lambda))^{-1}) \subset f(\mathbb{D}) \) directly follows from 3.1 (ii).

Suppose that \( e^{i\theta_0} \{2(1 + \lambda)\}^{-1} \notin f(\mathbb{D}) \). Then \( 2(1 + \lambda) e^{-i\theta_0} \in E(g) = E(\tilde{g}) + c_0(g) \), where \( g = Tf = \tilde{g} + c_0(g) \) with \( \tilde{g} \in \tilde{\mathcal{M}}(\lambda) \). Since \( 2(1 + \lambda) e^{-i\theta_0} - c_0(g) \in E(\tilde{g}) \subset \mathbb{D}(0, 1 + \lambda) \) and \( |c_0(g)| \leq 1 + \lambda \) by Theorem 2.5, we have
$$1 + \lambda \leq 2(1 + \lambda) - |c_0(g)| \leq |2(1 + \lambda) e^{-i\theta_0} - c_0(g)| \leq 1 + \lambda.$$  
Thus \( c_0(g) = (1 + \lambda) e^{-i\theta_0} \). By Theorem 2.5 (b) \( g(w) = w(1 + \lambda e^{i\theta} w^{-1}) (1 + e^{i\theta} w^{-1}) = w + (1 + \lambda) e^{i\theta} + \lambda e^{2i\theta} w^{-1} \) for some \( \theta \in \mathbb{R} \). Therefore \( e^{i\theta} = e^{-i\theta_0} \) and \( g(w) = w(1 + \lambda e^{-i\theta_0} w^{-1}) (1 + e^{-i\theta_0} w^{-1}) \). This implies
$$f(z) = \frac{z}{(1 + \lambda e^{-i\theta_0} z)(1 + e^{-i\theta_0} z)}.$$
\( \square \)
Proposition 3.3 implies that $\partial \partial V_{\lambda}(R)$ is contained in

$$V_{\lambda}^{*}(R) = \left\{ (R - \eta) \left( 1 - \frac{\lambda \epsilon}{R \eta} \right) : |\epsilon| = |\eta| = 1 \right\}.$$  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{\(V_{0.5}^{*}(2)\) and \(V_{0.9}^{*}(1.1)\)}
\end{figure}

One can prove $\partial V_{\lambda}^{*}(R)$ consists of two Jordan curves $J_{e}(R)$ and $J_{i}(R)$ which are starlike with respect to $R$ and $J_{i}(R)$ is contained inside of $J_{e}(R)$, and that $V_{\lambda}(R)$ is a closed Jordan domain surrounded by $J_{e}(R)$. For details see forthcoming paper [11].

\section*{References}


