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Embedding \(\alpha\)-convex functions in the class \(\mathcal{U}\)

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Abstract

In this paper the relation between classes

\[
\mathcal{M}(\alpha, \beta) = \left\{ f \in \mathcal{A} : \Re \left( 1 - \alpha \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, z \in \mathbb{D} \right\} \quad (\alpha \in \mathbb{R}, 0 \leq \beta < 1)
\]

and

\[
\mathcal{U}(\lambda, \mu) = \left\{ f \in \mathcal{A} : \frac{f(z)}{z} \neq 0 \text{ and } \left| \left( \frac{z}{f(z)} \right)^{1+\mu} f'(z) - 1 \right| < \lambda, z \in \mathbb{D} \right\} \quad (\mu \in \mathbb{C}, \lambda > 0)
\]

is studied and sharp sufficient conditions that imply

\[
\mathcal{M}(\alpha, \beta) \subset \mathcal{U}(\lambda, \mu)
\]

are given, together with several examples.

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1 Introduction

Let \(\mathcal{H}(\mathbb{D})\) be the class of functions that are analytic in the unit disk \(\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}\) and

\[
\mathcal{H}[a, n] = \left\{ f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \right\},
\]

where \(n\) is a positive integer and \(a \in \mathbb{C}\), with \(\mathcal{H}_n \equiv \mathcal{H}[1, n]\). Also, let

\[
\mathcal{A} = \{ f \in \mathcal{H}(\mathbb{D}) : f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \}.
\]

The class of starlike functions of order \(\alpha, 0 \leq \alpha < 1\), which is a subclass of the class of univalent functions, is defined by

\[
S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{D} \right\}.
\]

The class of starlike functions \(S^*\) \(\equiv S^*(0)\) consists of functions \(f\) that map the unit disk onto a starlike region, i.e. if \(w \in f(\mathbb{D})\), then \(tw \in f(\mathbb{D})\) for all \(t \in [0, 1]\).

Another subclass of univalent functions is the class of convex functions of order \(\alpha, 0 \leq \alpha < 1\), defined by

\[
\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > \alpha, z \in \mathbb{D} \right\}.
\]

Here \(\mathcal{K} \equiv \mathcal{K}(0)\) is the class of convex functions such that \(f \in \mathcal{K}\) if and only if \(f(\mathbb{D})\) is a convex region, i.e., if for any \(w_1, w_2 \in f(\mathbb{D})\) follows \(tw_1 + (1-t)w_2 \in f(\mathbb{D})\) for all \(t \in [0, 1]\).

Further, using operators

\[
J(f, \alpha; z) \equiv (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \quad (\alpha \in \mathbb{R})
\]
and
\[ U(f, \mu; z) = \left( \frac{z}{f(z)} \right)^{1+\mu} \cdot f'(z) \quad (\mu \in \mathbb{C}), \]

let define classes
\[
\mathcal{M}(\alpha, \beta) = \{ f \in \mathcal{A} : \Re J(f, \alpha; z) > \beta, z \in \mathbb{D} \} \quad (\alpha \in \mathbb{R}, 0 \leq \beta < 1),
\]
\[
\mathcal{M}'(\alpha, \gamma) = \{ f \in \mathcal{A} : |J(f, \alpha; z) - 1| < \gamma, z \in \mathbb{D} \} \quad (\alpha \in \mathbb{R}, \gamma > 0)
\]

and
\[
\mathcal{U}(\lambda, \mu) = \{ f \in \mathcal{A} : f(z)/z \neq 0 \text{ and } |U(f, \mu; z) - 1| < \lambda, z \in \mathbb{D} \} \quad (\mu \in \mathbb{C}, \lambda > 0).
\]

Specially, \( \mathcal{M}(\alpha) \equiv \mathcal{M}(\alpha, 0) \) is the well known class of \( \alpha \)-convex functions for which ([4], p.10):

\[ \mathcal{M}(\alpha) \subset S^* \text{ for all } \alpha \in \mathbb{R} \]

and
\[ \mathcal{M}(\alpha) \subset \mathcal{K} \subset S^* \text{ for } \alpha \geq 1. \]

More details on all these classes can be found in [2] and [4].

Further, class \( \mathcal{U}(\lambda, \mu) \) and its special cases \( \mathcal{U}(\lambda) \equiv \mathcal{U}(\lambda, 1) \) and \( \mathcal{U} \equiv \mathcal{U}(1) = \mathcal{U}(1, 1) \) are widely studied in the past decades ([11], [3], [8]-[17]). It is known [1, 17] that functions in \( \mathcal{U}(\lambda) \) are univalent if \( 0 < \lambda \leq 1 \), but not necessarily univalent if \( \lambda > 1 \). Also, functions from \( \mathcal{U}(\lambda, \mu) \), in general case are not starlike. More precisely, Obradović [7], and Ponnuasamy and Singh [18], proved that

\[ \mu < 1 \text{ and } 0 \leq \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \Rightarrow \mathcal{U}(\lambda, \mu) \subset S^*; \]

extended by Fournier and Ponnuasamy [3] as:

\[ \Re \mu < 1 \text{ and } 0 \leq \lambda \leq \frac{|1-\mu|}{\sqrt{|1-\mu|^2+|\mu|^2}} \Leftrightarrow \mathcal{U}(\lambda, \mu) \subset S^*. \]

Particularly,
\[ \mathcal{U}(1, \mu) \subset S^* \Leftrightarrow \mu = 0, \]

i.e., \( \mu \not\in S^* \), which can be also verified by the function
\[ f(z) = \frac{z}{1 + \frac{i}{2}z + \frac{1}{2}z^3} \in \mathcal{U} \setminus S^*. \]

Therefore, it is of interest to study the relation between classes \( \mathcal{M}(\alpha, \beta) \), \( \mathcal{M}'(\alpha, \gamma) \) and \( \mathcal{U}(\lambda, \mu) \), which will be done in this paper. More precisely, we will obtain sufficient conditions when

\[ \mathcal{M}(\alpha, \beta) \subset \mathcal{U}(\lambda, \mu) \text{ or } \mathcal{M}'(\alpha, \gamma) \subset \mathcal{U}(\lambda, \mu). \]

For the investigation we will use methods from the theory of first order differential subordinations and we proceed with some basic definitions. Let \( f(z) \) and \( g(z) \) be analytic in the unit disk. Then we say that \( f(z) \) is subordinate to \( g(z) \), and we write \( f(z) \prec g(z) \), if \( g(z) \) is univalent in \( \mathbb{D} \), \( f(0) = g(0) \) and \( f(D) \subseteq g(D) \). Further on, we use the method of differential subordinations introduced by Miller and Mocanu [5]. In fact, if \( \phi : \mathbb{C}^2 \rightarrow \mathbb{C} \) (C is the complex plane) is analytic in domain \( D \), if \( h(z) \) is univalent in \( D \), and if \( p(z) \) is analytic in \( D \) with \( (p(z),zp'(z)) \in D \), then \( z \in D \), then we say that \( p(z) \) satisfies a first-order differential subordination if

\[ \phi(p(z),zp'(z)) \prec h(z) \quad (1) \]

The univalent function \( q(z) \) is dominant of the differential subordination (1) if \( p(z) \prec q(z) \) for all \( p(z) \) satisfying (1). If \( \bar{q}(z) \) is a dominant of (1) and \( \bar{q}(z) \prec q(z) \) for all dominants of (1), then we say that \( \bar{q}(z) \) is the best dominant of the differential subordination (1). If \( p \in \mathcal{H}[a,n] \), then \( q(z) \) is called an \((a,n)\)-dominant and \( \bar{q}(z) \) the best \((a,n)\)-dominant.

From the theory of first-order differential subordinations we will make use of the following lemma.
Lemma 1 (Suffridge [19] or Corollary 3.1.d.1 on p.76 from [4]) Let \( h \) be starlike in \( \mathbb{D} \), with \( h(0) = 0 \) and \( a \neq 0 \). If \( p \in \mathcal{H}[a, \alpha] \) satisfies
\[
\frac{zp'(z)}{p(z)} \prec h(z),
\]
then
\[
p(z) \prec q(z) = a \cdot \exp \left[ \frac{1}{n} \cdot \int_{0}^{z} \frac{h(t)}{t} \, dt \right]
\]
and \( q \) is the best \((a, \alpha)\)-dominant.

2 Main results and consequences

Let note that for \( p(z) = U(f, -1/\alpha; z) \) we have
\[
J(f, \alpha; z) = 1 + \alpha \cdot \frac{zU'(f, -1/\alpha; z)}{U(f, -1/\alpha; z)},
\]
and \( a = p(0) = 1 \), i.e. \( p \in \mathcal{H}[1, \alpha] \). So, directly from Lemma 1, having in mind relation (2), we receive the following result.

Theorem 1 Let \( f \in \mathcal{A} \) and \( f(z) \neq 0 \) for all \( z \in \mathbb{D} \). Also, let \( h \) be starlike in \( \mathbb{D} \), \( h(0) = 0 \) and \( \alpha \neq 0 \). If
\[
\frac{1}{\alpha} \cdot [J(f, \alpha; z) - 1] \prec h(z)
\]
or equivalently
\[
J(f, \alpha; z) \prec 1 + \alpha h(z)
\]
then
\[
U(f, -1/\alpha; z) \prec \exp \left[ \frac{1}{n} \cdot \int_{0}^{z} \frac{h(t)}{t} \, dt \right] \equiv q(z),
\]
and \( q \) is the best \((1, \alpha)\)-dominant of (3). Even more, if \( f''(0) \neq 0 \), then \( n = 1 \).

Using the definition of subordination we receive the next corollary that gives information about the relation between the class \( \mathcal{U}(\lambda, \mu) \) and the classes \( \mathcal{M}(\alpha, \beta) \) and \( \mathcal{M}'(\alpha, \gamma) \).

Corollary 1 Let \( f \in \mathcal{A} \) and \( f(z) \neq 0 \) for all \( z \in \mathbb{D} \). Also, let \( \alpha \neq 0 \), \( 0 \leq \beta < 1 \) and \( \gamma > 0 \).

(i) If \( \alpha < 0 \) and \( \frac{1-\beta}{\alpha} \geq \frac{1}{2} \), then
\[
\mathcal{M}(\alpha, \beta) \subset \mathcal{U}(\lambda_1, -1/\alpha),
\]
where \( \lambda_1 = 2^{c_1} - 1 \) and \( c_1 = \frac{2(1-\beta)}{\alpha} \).

(ii) If \( \gamma \leq n|\alpha| \), then
\[
\mathcal{M}'(\alpha, \gamma) \subset \mathcal{U}(\lambda_2, -1/\alpha),
\]
where \( \lambda_2 = e^{c_2} - 1 \) and \( c_2 = \frac{\gamma}{n|\alpha|} \).

Even more, if \( f''(0) \neq 0 \), then \( n = 1 \) in the previous results. These results are sharp, i.e. given values of \( \lambda_1 \) and \( \lambda_2 \) are the smallest ones so that the corresponding inclusion holds.

Proof.
(i) Let \( f \in \mathcal{M}(\alpha, \beta) \). Then, by the definition of subordination we conclude that (3) holds for
\[
1 + \alpha h(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \quad \text{i.e.,} \quad h(z) = \frac{2z(1 - \beta)}{\alpha(1 - z)}.
\]
Now, from Theorem 1, we receive
\[
p(z) = U(f, -1/\alpha; z) \prec \exp\left[ \frac{1}{n} \int_{0}^{z} \frac{h(t)}{t} \, dt \right] = (1 - z)^{c_{1}} = q(z)
\]
and \( q \) is the best \((1, n)\)-dominant of (3). Further, having in mind that \( 0 < c_{1} \leq 1 \) we conclude that \( q(\mathbb{D}) \) is a convex region since \( q_{1}(z) = \frac{1 - q(z)}{c_{1}} \in \mathcal{K} \) because of
\[
\text{Re}\left[ 1 + \frac{zq_{2}'(z)}{q_{2}(z)} \right] > 0 \quad (z \in \mathbb{D}) \iff \text{Re}\left[ \frac{1 - c_{1}z}{1 - z} \right] > 0 \quad (z \in \mathbb{D}) \iff |c_{1}| \leq 1.
\]
Also, \( q(\mathbb{D}) \) is symmetric with respect to the real axes \((q(\overline{z}) = \overline{q(z)})\). Therefore,
\[
|q(z) - 1| < \lambda \quad (z \in \mathbb{D}),
\]
where
\[
\lambda = \sup \{|q(z) - 1| : z \in \mathbb{D} \} = \max\{q(-1) - 1, q(1) - 1\} = 2^{c_{1}} - 1 = q(-1).
\]
for \( c_{1} > 0 \). So, by the definition of subordination and subordination (4) we have \( f \in \mathcal{U}(\lambda, -1/\alpha) \). In the case when \( c_{1} < 0 \) we receive \( \lambda \to +\infty \).

The result is sharp because for \( \lambda_{*} < \lambda_{1} \) and \( \mathcal{M}(\alpha, \beta) \subset \mathcal{U}(\lambda_{*}, -1/\alpha) \) we have \( p(z) \prec 1 + \lambda_{*}z \) and \( q(z) \not\in 1 + \lambda_{*}z \) which contradicts the fact that \( q \) is the best \((1, n)\)-dominant of (3).

(ii) First, let note that \(|c_{2}| \leq 1\). In a similar way as in the proof of part (i), using \( h(z) = \frac{\alpha}{\beta}z \) in Theorem 1 we receive
\[
q(z) = e^{c_{2}z}.
\]
Again, \( q(\mathbb{D}) \) is a convex region since \( q_{2}(z) = \frac{q(z) - 1}{c_{2}} \in \mathcal{K} \) due to
\[
\text{Re}\left[ 1 + \frac{zq_{2}'(z)}{q_{2}(z)} \right] > 0 \quad (z \in \mathbb{D}) \iff \text{Re}(1 - c_{2}z) > 0 \quad (z \in \mathbb{D}) \iff |c_{2}| \leq 1.
\]
Region \( q(\mathbb{D}) \) is symmetric with respect to the real axes \((q(\overline{z}) = \overline{q(z)})\), and we realize that
\[
\sup \{|q(z) - 1| : z \in \mathbb{D} \} = \max\{q(-1) - 1, q(1) - 1\} = e^{|c_{2}|} - 1.
\]
Therefore,
\[
|U(f, -1/\alpha; z) - 1| < e^{|c_{2}|} - 1 \quad (z \in \mathbb{D}),
\]
i.e., \( f \in \mathcal{U}(\lambda_{2}, -1/\alpha) \). Proof of the sharpness goes in a similar way as in part (i). \( \square \)

## 3 Examples

By specifying some concrete values for \( \alpha, \gamma \) and/or \( \beta \) we have next examples.

**Example 1** Let \( f \in \mathcal{A}, f''(0) \neq 0 \) and \( L(z) \neq 0 \) for all \( z \in \mathbb{D} \). Also, let \( \alpha \neq 0, \gamma \geq 0 \) and \( 0 \leq \beta < 1 \). The following results are sharp.
(i) If $\alpha \leq -2$ then $\mathcal{M}(\alpha) \subset \mathcal{U}(1, -1/\alpha)$, i.e.

$$\Re \left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} \right] \right\} > 0 \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \left| \frac{zf''(z)}{f'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D}).$$

($\beta = 0$ in Corollary 1(i));

(ii) If $\alpha = -2(1-\beta)$, then $\mathcal{M}(\alpha, \beta) \subset \mathcal{U}(1, -1/\alpha)$. ($c_1 = 1$ in Corollary 1(i));

(iii) $\mathcal{M}'(1,1) \subset \mathcal{U}(e-1, -1)$, i.e.

$$\left| \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \left| \frac{f(z)f'(z)}{z} - 1 \right| < e-1 \quad (z \in \mathbb{D}).$$

($\alpha = \gamma = 1$ in Corollary 1(ii));

(iv) $\mathcal{M}'(1/2,1/2) \subset \mathcal{U}(e-1, -2)$, i.e.

$$\left| \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \left| \frac{f(z)f'(z)}{z} - 1 \right| < e-1 \quad (z \in \mathbb{D}).$$

($\alpha = \gamma = 1/2$ in Corollary 1(ii));

References


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