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Univalence and starlikeness of a function defined by convolution of analytic function and hypergeometric function $_{3}F_{2}$

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Abstract

We consider functions defined by a condition of functions in the subclass $\mathcal{U}(\lambda)$ of analytic functions with generalized Gauss hypergeometric functions. In this paper, we give a condition of the parameter $\lambda$ for which the function to be univalent and starlike.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $S$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ that are univalent in $\mathbb{U}$.

Obradović and Ponnusamy define in [4] the class $\mathcal{U}(\lambda)$ of $f(z) \in \mathcal{A}$ satisfying the condition

\[
\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| \leq \lambda \quad (z \in \mathbb{U})
\]

for some real $\lambda > 0$, where $f'$ denotes the derivative of $f$ with respect to the variable $z$. We set $\mathcal{U}(1) = \mathcal{U}$. It is easy to see that the the condition (1.2) is equivalent to

\[
\left| z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right) \right| \leq \lambda \quad (z \in \mathbb{U}).
\]

If $f(z) \in S$ maps $\mathbb{U}$ onto a starlike domain (with respect to the origin), i.e. if $tw \in f(\mathbb{U})$ whenever $t \in [0,1]$ and $w \in f(\mathbb{U})$, then we say that $f$ is a starlike function. The class of all starlike functions is denoted by $S^*$. A necessary and sufficient condition for $f(z) \in \mathcal{A}$ to be starlike is that the inequality

\[
|z| \leq |w| \quad (z, w \in \mathbb{U}).
\]

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\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{D})
\]

holds.

For these facts, the following lemmas hold.

**Lemma 1 ([3])** If \( f(z) \in \mathcal{U}(\lambda), \ a := \frac{|f''(0)|}{2} \leq 1 \) and \( 0 \leq \lambda \leq \frac{\sqrt{2-a^2} - a}{2} \), then \( f(z) \in S^* \).

**Lemma 2 ([7])** If \( f(z) = z + a_{n+1}z^{n+1} + \cdots \ (n \geq 2) \) belongs to \( \mathcal{U}(\lambda) \) and

\[
0 \leq \lambda \leq \frac{n-1}{\sqrt{(n-1)^2 + 1}},
\]

then \( f(z) \in S^* \).

For analytic functions \( f(z) \) and \( g(z) \) on \( \mathbb{D} \) with \( f(z) = \sum_{n=0}^{\infty}a_{n}z^{n} \) and \( g(z) = \sum_{n=0}^{\infty}b_{n}z^{n} \), the power series \( \sum_{n=0}^{\infty}a_{n}b_{n}z^{n} \) is said the convolution of \( f(z) \) and \( g(z) \), denoted by \( f * g \) (cf ([5])).

For \( f(z) = z + \sum_{n=2}^{\infty}a_{n}z^{n} \) in \( \mathcal{A} \), we have a natural convolution operator defined by

\[
zF(a, b; c; z) * f(z) := \sum_{n=1}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_{n}z^{n}, \quad c \in \{-1, -2, -3, \ldots\}, z \in \mathbb{D},
\]

where \((a)_{n}\) denotes the Pochhammer symbol \((a)_{0} = 1, (a)_{n} = a(a+1)\cdots(a+n-1)\) for \(n \in \mathbb{N}\). Here \( F(a, b; c; z) \) denotes the Gauss hypergeometric function which is analytic in \( \mathbb{D} \). As a special case of the Euler integral representation for the hypergeometric function, one has

\[
F(1, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} \frac{1}{1-tz} t^{b-1}(1-t)^{c-b-1} dt, \quad z \in \mathbb{D}, \ \text{Re} \ c > \text{Re} \ b > 0.
\]

Using this representation, we have, for \( f(z) \in \mathcal{A} \),

\[
zF(1, c+1; z) * f(z) = z \left( F(1, c+1; z) * \frac{f(z)}{z} \right) = zc \int_{0}^{1} \frac{f(tz)}{tz} t^{c-1} dt, z \in \mathbb{D}, \ \text{Re} \ c > 0.
\]

Obradović and Ponnusamy have shown the following result.

**Theorem A ([5])**

Let \( f \in \mathcal{U}(\lambda) \) and \( c \in \mathbb{C} \) with \( \text{Re} \ c > 0 \) such that

\[
\left( \frac{z}{f(z)} \right) * F(1, c+1; z) \neq 0 \quad \text{in} \quad z \in \mathbb{D},
\]
and $G(z) = G_f^c(z)$ be the transformed function defined by

$$G(z) = \frac{z}{\left(\frac{z}{f(z)}\right) \cdot F(1, c; c+1; z)} (z \in U).$$

Then we have the following:

1. $G \in U \left(\frac{\lambda|c|}{|c+2|}\right)$. The result is sharp especially when $\left|\frac{f''(0)}{2}\right| \leq 1 - \lambda$. In particular, $G \in U$ whenever $0 < \lambda \leq \left|\frac{c+2}{c}\right|$.

2. $G \in S^*$ whenever $0 < \lambda \leq \frac{|c+2|}{2|c|} \left(\sqrt{2-A^2} - A\right)$ with $A = \left|\frac{c}{c+1} \cdot \frac{f''(0)}{2}\right| \leq 1$.

2 Main Result

For the generalized hypergeometric function $\,_{3}F_{2}(1, \alpha, \beta; \alpha+1, \beta+1; z)$, we obtain

**Theorem 1**

Let $f(z) \in U(\lambda)$. Let $\alpha, \beta \in \mathbb{C}$ satisfying

$$\text{Re} \, \alpha \geq 0, \text{Re} \, \beta \geq 0, \frac{1}{|\alpha + \beta|} \left(\frac{|\alpha|}{|\beta+2|} + \frac{|\beta|}{|\alpha+2|}\right) < 1 \text{ and } |\alpha + \beta| > |\alpha \beta|$$

and

$$\frac{z}{f(z)} \cdot \,_{3}F_{2}(1, \alpha, \beta; \alpha+1, \beta+1; z) \neq 0, \quad z \in U.$$

Denote by $G(z) = G_f^{\alpha, \beta}(z)$ the function defined by

$$G(z) = \frac{z}{\frac{z}{f(z)} \cdot \,_{3}F_{2}(1, \alpha, \beta; \alpha+1, \beta+1; z)}, \quad z \in U,$$

where $\,_{3}F_{2}(1, \alpha, \beta; \alpha+1, \beta+1; z)$ is the generalized hypergeometric function. Then we have the following:

1. $G(z) \in U \left(\frac{\lambda|\alpha + \beta|}{|\alpha + \beta + 4|}\right)$. The result is sharp especially when $\left|\frac{f''(0)}{2}\right| \leq 1 - \lambda$.

   In particular, $G(z) \in U$ whenever $0 < \lambda \leq \frac{|\alpha + \beta + 4|}{|\alpha + \beta|}$.

2. $G(z) \in S^*$

   whenever $0 < \lambda \leq \frac{|\alpha + \beta + 4|}{2|\alpha + \beta|} \left(\sqrt{2-A^2} - A\right)$ with $A = \left|\frac{\alpha \beta}{(\alpha+1)(\beta+1)} \cdot \frac{f''(0)}{2}\right| \leq 1$.

**Proof.**

Since

$$\,_{3}F_{2}(1, \alpha, \beta; \alpha+1, \beta+1; z) = \sum_{n=0}^{\infty} \frac{\alpha \beta}{(\alpha+n)(\beta+n)} z^n = 1 + \sum_{n=1}^{\infty} \frac{\alpha \beta}{(\alpha+n)(\beta+n)} z^n,$$
we have

\[
\frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z) = 1 - \frac{\alpha \beta a_2}{(\alpha + 1)(\beta + 1)} z + \frac{\alpha \beta (a_2^2 - a_3)}{(\alpha + 2)(\beta + 2)} z^2 + \cdots
\]

\[
= \left\{ 1 - \frac{\alpha a_2}{\alpha + 1} z + \frac{\alpha (a_2^2 - a_3)}{\alpha + 2} z^2 + \cdots \right\} \cdot \left\{ 1 - \frac{\beta a_2}{\beta + 1} z + \frac{\beta (a_2^2 - a_3)}{\beta + 2} z^2 + \cdots \right\}
\]

\[
= \left\{ \frac{z}{f(z)} * F(1, \alpha; \alpha + 1; z) \right\} * F(1, \beta; \beta + 1; z).
\]

Thus \( G(z) \) can be written as

\[
G(z) = \frac{z}{\frac{z}{f(z)} * F(1, \alpha; \alpha + 1; z)} * F(1, \beta; \beta + 1; z).
\]

In the same manner, \( G(z) \) can be also written as

\[
G(z) = \frac{z}{\frac{z}{f(z)} * F(1, \beta; \beta + 1; z)} * F(1, \alpha; \alpha + 1; z).
\]

Put \( h_1(z) = \frac{z}{\frac{z}{f(z)} * F(1, \alpha; \alpha + 1; z)} \), \( h_2(z) = \frac{z}{\frac{z}{f(z)} * F(1, \beta; \beta + 1; z)} \).

then

\[
\frac{z}{f(z)} * F(1, \alpha; \alpha + 1; z) = \frac{z}{h_1(z)} \quad \frac{z}{f(z)} * F(1, \beta; \beta + 1; z) = \frac{z}{h_2(z)}.
\]

By the Theorem A in the introduction, we have

\[
h_1(z) \in \mathcal{U} \left( \frac{\lambda |\alpha|}{|\alpha + 2|} \right) \quad i.e. \quad \left| \frac{z}{h_1(z)} \right|^2 h_1'(z) - 1 < \frac{\lambda |\alpha|}{|\alpha + 2|}
\]

and

\[
h_2(z) \in \mathcal{U} \left( \frac{\lambda |\beta|}{|\beta + 2|} \right) \quad i.e. \quad \left| \frac{z}{h_2(z)} \right|^2 h_2'(z) - 1 < \frac{\lambda |\beta|}{|\beta + 2|}.
\]

Since

\[
\frac{z}{G(z)} = \frac{z}{h_1(z)} * F(1, \beta; \beta + 1; z) \quad (z \in \mathbb{U}),
\]

we have

\[
(\beta + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{G(z)} + z \left( \frac{z}{G(z)} \right)'.
\]

On the other hand, \( \frac{z}{G(z)} \) can be also written as

\[
\frac{z}{G(z)} = \frac{z}{h_2(z)} * F(1, \alpha; \alpha + 1; z) \quad (z \in \mathbb{U}),
\]
we have

\[(2.4) \quad (\beta + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{G(z)} + z \left( \frac{z}{G(z)} \right)' \]

Then we have

\[(2.5) \quad (\alpha + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \alpha \frac{z}{h_2(z)} \quad (z \in \mathbb{U}) \]

and

\[(2.6) \quad (\beta + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{h_1(z)} \quad (z \in \mathbb{U}). \]

Set

\[ p(z) = \left( \frac{z}{G(z)} \right)^2 G'(z). \]

Then \( p(z) \) is analytic on \( \mathbb{U} \) with \( p(0) = 1 \) and \( p'(0) = 0 \), and

\[(2.7) \quad p(z) = (\alpha + 1) \frac{z}{G(z)} - \alpha \frac{z}{h_2(z)} \]

and

\[(2.8) \quad p(z) = (\beta + 1) \frac{z}{G(z)} - \beta \frac{z}{h_1(z)}. \]

From (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8) one then obtain that

\[
\alpha p(z) + z p'(z) = (\alpha + 1)\alpha \frac{z}{G(z)} + (\alpha + 1)z \left( \frac{z}{G(z)} \right)' - \alpha^2 \frac{z}{h_2(z)} - \alpha z \left( \frac{z}{h_2(z)} \right)' \\
= \alpha \left[ (\alpha + 1) \frac{z}{h_2(z)} - \alpha \frac{z}{h_2(z)} - z \left( \frac{z}{h_2(z)} \right) \right] \\
= \alpha \left[ \frac{z}{h_2(z)} - z \left( \frac{z}{h_2(z)} \right) \right] \\
= \alpha \left( \frac{z}{h_2(z)} \right)^2 h_2'(z)
\]

and

\[
\beta p(z) + z p'(z) = (\beta + 1)\beta \frac{z}{G(z)} + (\beta + 1)z \left( \frac{z}{G(z)} \right)' - \beta^2 \frac{z}{h_1(z)} - \beta z \left( \frac{z}{h_1(z)} \right)' \\
= \beta \left[ (\beta + 1) \frac{z}{h_1(z)} - \beta \frac{z}{h_1(z)} - z \left( \frac{z}{h_1(z)} \right) \right] \\
= \beta \left[ \frac{z}{h_1(z)} - z \left( \frac{z}{h_1(z)} \right) \right] \\
= \beta \left( \frac{z}{h_1(z)} \right)^2 h_1'(z).\]
Since
\[(\alpha + \beta)p(z) + 2zp'(z) = \alpha \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) + \beta \left( \frac{z}{h_1(z)} \right)^2 h_1'(z),\]
we have
\[p(z) + \frac{2}{\alpha + \beta}zp'(z) = \frac{\alpha}{\alpha + \beta} \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) + \frac{\beta}{\alpha + \beta} \left( \frac{z}{h_1(z)} \right)^2 h_1'(z).\]

Now, as \(h_1(z) \in U\left( \frac{\lambda|\alpha|}{|\alpha + 2|} \right)\) and \(h_2(z) \in U\left( \frac{\lambda|\beta|}{|\beta + 2|} \right)\), it follows that
\[\left| p(z) + \frac{2}{\alpha + \beta}zp'(z) - 1 \right| = \left| \frac{\alpha}{\alpha + \beta} \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) - 1 \right| + \left| \frac{\beta}{\alpha + \beta} \left( \frac{z}{h_1(z)} \right)^2 h_1'(z) - 1 \right| \leq \left| \frac{\alpha}{\alpha + \beta} \right| \left| \frac{z}{h_2(z)} \right|^2 + \left| \frac{\beta}{\alpha + \beta} \right| \left| \frac{z}{h_1(z)} \right|^2 \leq \lambda \left( \frac{|\alpha||\beta|}{|\beta + 2|} + \frac{|\beta||\alpha|}{|\alpha + 2|} \right).
\]

By the assumption, we have
\[(2.9) \quad \left| p(z) + \frac{2}{\alpha + \beta}zp'(z) - 1 \right| < \lambda.\]

From the work of Hallenbeck and Rusheweyh ([2],[6]), we deduce that
\[(2.10) \quad |p(z) - 1| \leq \frac{\lambda(\alpha + \beta)}{|\alpha + \beta + 4|} \quad (z \in U).\]

Thus we have \(G(z) \in U\left( \frac{\lambda(\alpha + \beta)}{|\alpha + \beta + 4|} \right)\).

To prove the sharpness, we consider functions \(f(z)\) in \(U(\lambda)\) of the form
\[f(z) = \frac{z}{1 - a_2z + \lambda z^2},\]
where \(a_2 = \frac{f''(0)}{2}\) and \(|a_2| \leq 1 - \lambda\), so that \(1 - a_2z + \lambda z^2 \neq 0\) for all \(z \in U\). Since \(\text{Re} \alpha \geq 0\) and \(\text{Re} \beta \geq 0\), it follows that \(|\alpha + 2| > |\alpha + 1| > |\alpha|\) and \(|\beta + 2| > |\beta + 1| > |\beta|\) and, therefore
\[\left| 1 - a_2 \frac{\alpha \beta}{(\alpha + 1)(\beta + 1)}z + \lambda \frac{\alpha \beta}{(\alpha + 2)(\beta + 2)} z^2 \right| \neq 0\]
for all \(z \in U\), provided \(|a_2| \leq 1 - \lambda\). By the series expansion (2.2) of \(3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z)\), we have
\[G(z) = \frac{z}{1 - a_2 \alpha \beta \left( 1 - \frac{\lambda(\alpha \beta)}{(\alpha + 1)(\beta + 1)}z + \frac{\lambda(\alpha \beta)}{(\alpha + 2)(\beta + 2)} z^2 \right)}.\]
Obviously, $G(z)$ is analytic on $U$ and $\frac{z}{G(z)} \neq 0$ on $U$. Since

$$\left(\frac{z}{G(z)}\right)^2 G'(z) - 1 = -\frac{\lambda \alpha \beta}{(\alpha + 2)(\beta + 2)} z^2,$$

we have that

$$(2.11) \quad \left|\left(\frac{z}{G(z)}\right)^2 G'(z) - 1\right| \leq \frac{\lambda |\alpha \beta|}{|(\alpha + 2)(\beta + 2)|}.$$  

Now, let us compare the right hand sides of (2.10) and (2.11). Firstly, since $|\alpha + \beta + 4| < |(\alpha + 2)(\beta + 2)|$, then $\frac{1}{|(\alpha + 2)(\beta + 2)|} < \frac{1}{|\alpha + \beta + 4|}$. From the assumption, we see

$$\frac{|\alpha \beta|}{|(\alpha + 2)(\beta + 2)|} < \frac{|\alpha + \beta|}{|(\alpha + 2)(\beta + 2)|} < \frac{|\alpha + \beta|}{|\alpha + \beta + 4|}.$$  

Then, we have that

$$\left|\left(\frac{z}{G(z)}\right)^2 G'(z) - 1\right| \leq \frac{\lambda |\alpha \beta|}{|(\alpha + 2)(\beta + 2)|} < \frac{|\alpha + \beta|}{|\alpha + \beta + 4|}.$$  

Thus, we have that the bound $\frac{|\alpha + \beta|}{|\alpha + \beta + 4|}$ is sharp. We conclude that the first assertion of Theorem 1.

The second assertion is a direct consequence of Lemma 1. In fact, obviously

$$A = \frac{G''(0)}{2} = \frac{\alpha \beta}{(\alpha + 1)(\beta + 1)} \frac{f''(0)}{2}$$

is smaller than or equal to 1.

**Theorem 2**

For a fixed $n \geq 2$, let $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belong to $\mathcal{U}(\lambda)$. Let $\alpha, \beta \geq 0$ and

$$\text{Re} \alpha \geq 0, \text{Re} \beta \geq 0, \frac{1}{|\alpha + \beta|} \left(\frac{|\alpha||\beta|}{|\beta+n|} + \frac{|\alpha||\beta|}{|\alpha+n|}\right) < 1,$$

and

$$\frac{z}{f(z)} \ast_{3} F_{2}(1, \alpha, \beta; \alpha + 1, \beta + 1; z) \neq 0, \quad z \in \mathbb{U}.$$  

and $G(z) = G_{f}^{\alpha \beta}(z)$ be the transform function defined by (2.1). Then we have the following:

1. $G(z) \in \mathcal{U} \left(\frac{\lambda |\alpha + \beta|}{|\alpha + \beta + 2n|}\right)$. In particular, $G(z) \in \mathcal{U}$ whenever $0 < \lambda \leq \frac{|\alpha + \beta + 2n|}{|\alpha + \beta|}$.

2. $G(z) \in S^{*}$ whenever $0 < \lambda \leq \frac{(n-1)|\alpha + \beta + 2n|}{|\alpha + \beta|\sqrt{(n-1)^2 + 1}}$.

**Proof.** Using the Gaussian hypergeometric function, $G(z)$ can be written as
\[ G(z) = \frac{z}{\left\{ \frac{z}{f(z)} \ast F(1, \alpha; \alpha + 1; z) \right\} \ast F(1, \beta; \beta + 1; z)} \]

and
\[ G(z) = \frac{z}{\left\{ \frac{z}{f(z)} \ast F(1, \beta; \beta + 1; z) \right\} \ast F(1, \alpha; \alpha + 1; z)} \]

Put
\[ h_3(z) = \frac{z}{\frac{z}{f(z)} \ast F(1, \alpha; \alpha + 1; z)}, \quad h_4(z) = \frac{z}{\frac{z}{f(z)} \ast F(1, \beta; \beta + 1; z)} \]

Then
\[ \frac{z}{f(z)} \ast F(1, \alpha; \alpha + 1; z) = \frac{z}{h_3(z)}, \quad \frac{z}{f(z)} \ast F(1, \beta; \beta + 1; z) = \frac{z}{h_4(z)} \]

We see
\[ h_3(z) \in U \left( \frac{\lambda|\alpha|}{|\alpha+n|} \right), \quad \text{i.e.} \quad \left| \left( \frac{z}{h_3(z)} \right)^2 h_3'(z) - 1 \right| < \frac{\lambda|\alpha|}{|\alpha+n|} \]

and
\[ h_4(z) \in U \left( \frac{\lambda|\beta|}{|\beta+n|} \right), \quad \text{i.e.} \quad \left| \left( \frac{z}{h_4(z)} \right)^2 h_4'(z) - 1 \right| < \frac{\lambda|\beta|}{|\beta+n|} \]

Since
\[ \frac{z}{f(z)} = \frac{1}{1 + a_{n+1}z^n + \cdots} = 1 - a_{n+1}z^n + \cdots, \]

so that
\[ \frac{z}{f(z)} \ast _3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z) = 1 - a_{n+1} \left\{ \frac{\alpha \beta}{(\alpha+n)(\beta+n)} \right\} z^n + \cdots. \]

Thus, \( G(z) \) can be written in the form
\[ G(z) = z + a_{n+1} \left\{ \frac{\alpha \beta}{(\alpha+n)(\beta+n)} \right\} z^{n+1} + \cdots. \]

Therefore, as in the proof of Theorem 1, the function \( p(z) \) defined by
\[ p(z) = \left( \frac{z}{G(z)} \right)^2 G'(z) = 1 + (n-1)a_{n+1} \left\{ \frac{\alpha \beta}{(\alpha+n)(\beta+n)} \right\} z^n + \cdots \]

is analytic in \( U \) and \( p(0) = 1, \quad p'(0) = \cdots = p^{(n-1)}(0) = 0. \) \( p(z) \) can be written as
\[ p(z) = (\alpha + 1) \frac{z}{G(z)} - \alpha \frac{z}{h_3(z)} \]

and
\[ p(z) = (\beta + 1) \frac{z}{G(z)} - \beta \frac{z}{h_4(z)} \]

By the same argument of proof of Theorem 1 using \( h_3(z) \) and \( h_4(z) \) instead of \( h_1(z) \) and \( h_2(z) \), \( p(z) \) satisfies (2.9). Consequently, we obtain that
\[ |p(z) - 1| \leq \frac{\lambda|\alpha + \beta||z|^n}{|\alpha + \beta + 2n|} \quad (z \in U), \]

and the proof of part(1) is complete. The second part is a direct consequence of Lemma 2.
References


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