Univalence and starlikeness of a function defined by convolution of analytic function and hypergeometric function $_3F_2$

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Abstract

We consider functions defined by a condition of functions in the subclass $\mathcal{U}(\lambda)$ of analytic functions with generalized Gauss hypergeometric functions. In this paper, we give a condition of the parameter $\lambda$ for which the function to be univalent and starlike.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ that are univalent in $\mathbb{U}$.

Obradović and Ponnusamy define in [4] the class $\mathcal{U}(\lambda)$ of $f(z) \in \mathcal{A}$ satisfying the condition

$$\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| \leq \lambda \quad (z \in \mathbb{U})$$

for some real $\lambda > 0$, where $f'$ denotes the derivative of $f$ with respect to the variable $z$. We set $\mathcal{U}(1) = \mathcal{U}$. It is easy to see that the condition (1.2) is equivalent to

$$|z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right)'| \leq \lambda \quad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{S}$ maps $\mathbb{U}$ onto a starlike domain (with respect to the origin), i.e. if $tw \in f(\mathbb{U})$ whenever $t \in [0,1]$ and $w \in f(\mathbb{U})$, then we say that $f$ is a starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$. A necessary and sufficient condition for $f(z) \in \mathcal{A}$ to be starlike is that the inequality

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For these facts, the following lemmas hold.

Lemma 1 ([3]) If \( f(z) \in \mathcal{U}(\lambda), \) \( a := \frac{|f''(0)|}{2} \leq 1 \) and \( 0 \leq \lambda \leq \frac{\sqrt{2-a^{2}}-a}{2} \), then \( f(z) \in S^{*} \).

Lemma 2 ([7]) If \( f(z) = z + a_{n+1}z^{n+1} + \cdots \) \( (n \geq 2) \) belongs to \( \mathcal{U}(\lambda) \) and

\[
0 \leq \lambda \leq \frac{n-1}{\sqrt{(n-1)^{2}+1}},
\]

then \( f(z) \in S^{*} \).

For analytic functions \( f(z) \) and \( g(z) \) on \( U \) with \( f(z) = \sum_{n=0}^{\infty} a_{n}z^{n} \) and \( g(z) = \sum_{n=0}^{\infty} b_{n}z^{n} \), the power series \( \sum_{n=0}^{\infty} a_{m}b_{n}z^{n} \) is said the convolution of \( f(z) \) and \( g(z) \), denoted by \( f* g \) (cf ([5])).

For \( f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \) in \( \mathcal{A} \), we have a natural convolution operator defined by

\[
zF(a, b; c; z) * f(z) := \sum_{n=1}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_{n}z^{n}, \quad c \in \{-1, -2, -3, \cdots\}, z \in U,
\]

where \( (a)_{n} \) denotes the Pochhammer symbol \( (a)_{0} = 1, (a)_{n} = a(a+1)\cdots(a+n-1) \) for \( n \in \mathbb{N} \). Here \( F(a, b; c; z) \) denotes the Gauss hypergeometric function which is analytic in \( U \).

As a special case of the Euler integral representation for the hypergeometric function, one has

\[
F(1, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} \frac{1}{1-tz} t^{b-1}(1-t)^{c-b-1} dt, \quad z \in U, \quad \text{Re} c > \text{Re} b > 0.
\]

Using this representation, we have, for \( f(z) \in \mathcal{A} \),

\[
zF(1, c; c+1; z) * f(z) = z \left( F(1, c; c+1; z) * \frac{f(z)}{z} \right) = zc \int_{0}^{1} \frac{f(tz)}{tz} t^{c-1} dt, z \in U, \quad \text{Re} c > 0.
\]

Obradović and Ponnusamy have shown the following result.

**Theorem A ([5])**

*Let \( f \in \mathcal{U}(\lambda) \) and \( c \in \mathbb{C} \) with \( \text{Re} c > 0 \) such that

\[
\left( \frac{z}{f(z)} \right) * F(1, c; c+1; z) \neq 0 \quad \text{in} \quad z \in U,
\]

holds.*
and \(G(z) = G_f^c(z)\) be the transformed function defined by
\[
G(z) = \frac{z}{\left(\frac{z}{f(z)}\right) * F(1, c; c + 1; z)} \quad (z \in U).
\]

Then we have the following:

1. \(G \in U \left(\frac{\lambda|c|}{|c + 2|}\right)\). The result is sharp especially when \(f''(0) \leq 1 - \lambda\). In particular, \(G \in U\) whenever \(0 < \lambda \leq \frac{c + 2}{c}\).

2. \(G \in S^*\) whenever \(0 < \lambda \leq \frac{1 + \sqrt{2 - A^2}}{A} \) with \(A = \frac{c}{c + 1}\).

2 Main Result

For the generalized hypergeometric function \(\, _3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z)\), we obtain

**Theorem 1**

Let \(f(z) \in U(\lambda)\). Let \(\alpha, \beta \in \mathbb{C}\) satisfying

\[
\text{Re} \alpha \geq 0, \text{Re} \beta \geq 0, \quad \frac{1}{|\alpha + \beta|} \left(\frac{|\alpha|}{|\beta + 2|} + \frac{|\beta|}{|\alpha + 2|}\right) < 1 \quad \text{and} \quad |\alpha + \beta| > |\alpha \beta|
\]

and

\[
\frac{z}{f(z)} * \, _3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z) \neq 0, \quad z \in U.
\]

Denote by \(G(z) = G_f^{\alpha, \beta}(z)\) the function defined by

\[
G(z) = \frac{z}{\frac{z}{f(z)} * \, _3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z)}, \quad z \in U,
\]

where \(\, _3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z)\) is the generalized hypergeometric function. Then we have the following:

1. \(G(z) \in U \left(\frac{\lambda|\alpha + \beta|}{|\alpha + \beta + 4|}\right)\). The result is sharp especially when \(f''(0) \leq 1 - \lambda\). In particular, \(G(z) \in U\) whenever \(0 < \lambda \leq \frac{\alpha + \beta + 4}{|\alpha + \beta|}\).

2. \(G(z) \in S^*\) whenever \(0 < \lambda \leq \frac{\alpha + \beta + 4}{2|\alpha + \beta|} \) with \(A = \frac{\alpha \beta}{(\alpha + 1)(\beta + 1)}\) and \(f''(0) \leq 1\).

**Proof.**

Since

\[
\, _3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z) = \sum_{n=0}^{\infty} \frac{\alpha \beta}{(\alpha + n)(\beta + n)} z^n = 1 + \sum_{n=1}^{\infty} \frac{\alpha \beta}{(\alpha + n)(\beta + n)} z^n,
\]
we have
\[
\frac{z}{f(z)} \ast {}_3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z) = 1 - \frac{\alpha \beta a_2}{(\alpha + 1)(\beta + 1)} z + \frac{\alpha \beta (a_2^2 - a_3)}{(\alpha + 2)(\beta + 2)} z^2 + \ldots
\]
\[
= \left\{ 1 - \frac{\alpha a_2}{\alpha + 1} z + \frac{\alpha (a_2^2 - a_3)}{\alpha + 2} z^2 + \ldots \right\} \ast \left\{ 1 - \frac{\beta a_2}{\beta + 1} z + \frac{\beta (a_2^2 - a_3)}{\beta + 2} z^2 + \ldots \right\}
\]
\[
= \left\{ \frac{z}{f(z)} \ast F(1, \alpha; \alpha + 1; z) \right\} \ast F(1, \beta; \beta + 1; z).
\]
Thus $G(z)$ can be written as
\[
G(z) = \frac{z}{f(z)} \ast F(1, \alpha; \alpha + 1; z) \ast F(1, \beta; \beta + 1; z)
\]
In the same manner, $G(z)$ can be also written as
\[
G(z) = \frac{z}{f(z)} \ast F(1, \beta; \beta + 1; z) \ast F(1, \alpha; \alpha + 1; z)
\]
Put
\[
h_1(z) = \frac{z}{f(z)} \ast F(1, \alpha; \alpha + 1; z), \quad h_2(z) = \frac{z}{f(z)} \ast F(1, \beta; \beta + 1; z)
\]
then
\[
\frac{z}{f(z)} \ast F(1, \alpha; \alpha + 1; z) = \frac{z}{h_1(z)} \quad \frac{z}{f(z)} \ast F(1, \beta; \beta + 1; z) = \frac{z}{h_2(z)}
\]
By the Theorem A in the introduction, we have
\[
h_1(z) \in \mathcal{U} \left( \frac{\lambda |\alpha|}{|\alpha + 2|} \right) \quad \text{i.e.} \quad \left| \left( \frac{z}{h_1(z)} \right)^2 h_1'(z) - 1 \right| < \frac{\lambda |\alpha|}{|\alpha + 2|}
\]
and
\[
h_2(z) \in \mathcal{U} \left( \frac{\lambda |\beta|}{|\beta + 2|} \right) \quad \text{i.e.} \quad \left| \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) - 1 \right| < \frac{\lambda |\beta|}{|\beta + 2|}
\]
Since
\[
\frac{z}{G(z)} = \frac{z}{h_1(z)} \ast F(1, \beta; \beta + 1; z) \quad (z \in \mathbb{U}),
\]
we have
\[
(\beta + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{G(z)} + z \left( \frac{z}{G(z)} \right)'.
\]
On the other hand, $\frac{z}{G(z)}$ can be also written as
\[
\frac{z}{G(z)} = \frac{z}{h_2(z)} \ast F(1, \alpha; \alpha + 1; z) \quad (z \in \mathbb{U}),
\]
we have

\[(2.4) \quad (\beta + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{G(z)} + z \left( \frac{z}{G(z)} \right)'.\]

Then we have

\[(2.5) \quad (\alpha + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \alpha \frac{z}{h_2(z)} \quad (z \in \mathbb{U})\]

and

\[(2.6) \quad (\beta + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = \beta \frac{z}{h_1(z)} \quad (z \in \mathbb{U}).\]

Set

\[p(z) = \left( \frac{z}{G(z)} \right)^2 G'(z).\]

Then \(p(z)\) is analytic on \(\mathbb{U}\) with \(p(0) = 1\) and \(p'(0) = 0\), and

\[(2.7) \quad p(z) = (\alpha + 1) \frac{z}{G(z)} - \alpha \frac{z}{h_2(z)}\]

and

\[(2.8) \quad p(z) = (\beta + 1) \frac{z}{G(z)} - \beta \frac{z}{h_1(z)} .\]

From (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8) one then obtain that

\[
\alpha p(z) + z p'(z) = (\alpha + 1) \alpha \frac{z}{G(z)} + (\alpha + 1)z \left( \frac{z}{G(z)} \right)'/2 - \alpha^2 \frac{z}{h_2(z)} - az \left( \frac{z}{h_2(z)} \right)'
\]

\[
= \alpha \left[ (\alpha + 1) \frac{z}{h_2(z)} - \alpha \frac{z}{h_2(z)} - z \left( \frac{z}{h_2(z)} \right)' \right]
\]

\[
= \alpha \left[ \frac{z}{h_2(z)} - z \left( \frac{z}{h_2(z)} \right)' \right]
\]

\[
= \alpha \left( \frac{z}{h_2(z)} \right)^2 h_2'(z).
\]

and

\[
\beta p(z) + z p'(z) = (\beta + 1) \beta \frac{z}{G(z)} + (\beta + 1)z \left( \frac{z}{G(z)} \right)' - \beta^2 \frac{z}{h_1(z)} - \beta z \left( \frac{z}{h_1(z)} \right)'
\]

\[
= \beta \left[ (\beta + 1) \frac{z}{h_1(z)} - \beta \frac{z}{h_1(z)} - z \left( \frac{z}{h_1(z)} \right)' \right]
\]

\[
= \beta \left[ \frac{z}{h_1(z)} - z \left( \frac{z}{h_1(z)} \right)' \right]
\]

\[
= \beta \left( \frac{z}{h_1(z)} \right)^2 h_1'(z).
\]
Since
\[(\alpha + \beta)p(z) + 2zp'(z) = \alpha \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) + \beta \left( \frac{z}{h_1(z)} \right)^2 h_1'(z),\]
we have
\[p(z) + \frac{2}{\alpha + \beta}zp'(z) = \frac{\alpha}{\alpha + \beta} \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) + \frac{\beta}{\alpha + \beta} \left( \frac{z}{h_1(z)} \right)^2 h_1'(z).\]

Now, as \(h_1(z) \in U\left(\frac{\lambda|\alpha|}{|\alpha+2|}\right)\) and \(h_2(z) \in U\left(\frac{\lambda|\beta|}{|\beta+2|}\right)\), it follows that
\[
\left| p(z) + \frac{2}{\alpha + \beta}zp'(z) - 1 \right| = \left| \frac{\alpha}{\alpha + \beta} \left\{ \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) - 1 \right\} + \frac{\beta}{\alpha + \beta} \left\{ \left( \frac{z}{h_1(z)} \right)^2 h_1'(z) - 1 \right\} \right|
\leq \left| \frac{\alpha}{\alpha + \beta} \left( \frac{z}{h_2(z)} \right)^2 h_2'(z) - 1 \right| + \left| \frac{\beta}{\alpha + \beta} \left( \frac{z}{h_1(z)} \right)^2 h_1'(z) - 1 \right|
\leq \frac{|\alpha|}{|\alpha + \beta|} \frac{\lambda|\beta|}{|\beta + 2|} + \frac{|\beta|}{|\alpha + \beta|} \frac{\lambda|\alpha|}{|\alpha + 2|}
= \lambda \left\{ \frac{1}{|\alpha + \beta|} \left( \frac{|\alpha||\beta|}{|\beta + 2|} + \frac{|\beta||\alpha|}{|\alpha + 2|} \right) \right\}.
\]

By the assumption, we have
\[(2.9) \quad \left| p(z) + \frac{2}{\alpha + \beta}zp'(z) - 1 \right| < \lambda.
\]

From the work of Hallenbeck and Rusheweyh ([2],[6]), we deduce that
\[(2.10) \quad |p(z) - 1| \leq \frac{\lambda|\alpha + \beta|}{|\alpha + \beta + 4|} (z \in U).
\]

Thus we have \(G(z) \in U\left(\frac{\lambda|\alpha + \beta|}{|\alpha + \beta + 4|}\right)\).

To prove the sharpness, we consider functions \(f(z)\) in \(U(\lambda)\) of the form
\[f(z) = \frac{z}{1 - a_2z + \lambda z^2},\]
where \(a_2 = \frac{f''(0)}{2}\) and \(|a_2| \leq 1 - \lambda\), so that \(1 - a_2z + \lambda z^2 \neq 0\) for all \(z \in U\). Since \(\text{Re} \, \alpha \geq 0\) and \(\text{Re} \, \beta \geq 0\), it follows that \(|\alpha + 2| > |\alpha + 1| > |\alpha|\) and \(|\beta + 2| > |\beta + 1| > |\beta|\) and, therefore
\[
\left| 1 - a_2 \frac{\alpha \beta}{(\alpha + 1)(\beta + 1)}z + \lambda \frac{\alpha \beta}{(\alpha + 2)(\beta + 2)} z^2 \right| \neq 0
\]
for all \(z \in U\), provided \(|a_2| \leq 1 - \lambda\). By the series expansion (2.2) of \(_3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z)\), we have
\[G(z) = \frac{z}{1 - \frac{a_2 \alpha \beta}{(\alpha + 1)(\beta + 1)} z + \frac{\lambda(\alpha \beta)}{(\alpha + 2)(\beta + 2)} z^2}.
\]
Obviously, $G(z)$ is analytic on $U$ and $\frac{z}{G(z)} \neq 0$ on $U$. Since

$$\left( \frac{z}{G(z)} \right)^2 G'(z) - 1 = -\frac{\lambda \alpha \beta}{(\alpha + 2)(\beta + 2)} z^2,$$

we have that

$$\left| \left( \frac{z}{G(z)} \right)^2 G'(z) - 1 \right| \leq \frac{\lambda |\alpha \beta|}{|\alpha + 2)(\beta + 2)|}.$$

Now, let us compare the right hand sides of (2.10) and (2.11). Firstly, since $|\alpha + \beta + 4| < |(\alpha + 2)(\beta + 2)|$, then $\frac{1}{|\alpha + 2)(\beta + 2)|} < \frac{1}{|\alpha + \beta + 4|}$. From the assumption, we see

$$\frac{|\alpha \beta|}{|\alpha + 2)(\beta + 2)|} < \frac{|\alpha + \beta|}{|\alpha + 2)(\beta + 2)|} < \frac{|\alpha + \beta|}{|\alpha + \beta + 4|}.$$

Then, we have that

$$\left| \left( \frac{z}{G(z)} \right)^2 G'(z) - 1 \right| \leq \frac{\lambda |\alpha \beta|}{|\alpha + 2)(\beta + 2)|} < \frac{|\alpha + \beta|}{|\alpha + \beta + 4|}.$$

Thus, we have that the bound $\frac{|\alpha + \beta|}{|\alpha + \beta + 4|}$ is sharp. We conclude that the first assertion of Theorem 1.

The second assertion is a direct consequence of Lemma 1. In fact, obviously

$$A = \frac{G''(0)}{2} = \frac{\alpha \beta}{(\alpha + 1)(\beta + 1)} \frac{f''(0)}{2}$$

is smaller than or equal to 1.

**Theorem 2**

For a fixed $n \geq 2$, let $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belong to $U(\lambda)$. Let $\alpha, \beta \geq 0$ and

$$\text{Re } \alpha \geq 0, \text{ Re } \beta \geq 0, \frac{1}{|\alpha + \beta|} \left( \frac{|\alpha| |\beta|}{|\beta + n|} + \frac{|\alpha| |\beta|}{|\alpha + n|} \right) < 1,$$

and

$$\frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha + 1, \beta + 1; z) \neq 0, \quad z \in U.$$

and $G(z) = G^\alpha_{\beta}(z)$ be the transform function defined by (2.1). Then we have the following:

1. $G(z) \in U \left( \frac{\lambda |\alpha + \beta|}{|\alpha + \beta + 2n|} \right)$. In particular, $G(z) \in U$ whenever $0 < \lambda \leq \frac{|\alpha + \beta + 2n|}{|\alpha + \beta|}$.

2. $G(z) \in S^\ast$ whenever $0 < \lambda \leq \frac{(n-1)|\alpha + \beta + 2n|}{|\alpha + \beta| \sqrt{(n-1)^2 + 1}}$.

**Proof.** Using the Gaussian hypergeometric function, $G(z)$ can be written as
\[ G(z) = \frac{z}{\left\{ \frac{z}{f(z)} \ast \text{F}(1, \alpha; \alpha + 1; z) \right\} \ast \text{F}(1, \beta; \beta + 1; z)} \]

and

\[ G(z) = \frac{z}{\left\{ \frac{z}{f(z)} \ast \text{F}(1, \beta; \beta + 1; z) \right\} \ast \text{F}(1, \alpha; \alpha + 1; z)} \]

Put

\[ h_3(z) = \frac{z}{\frac{z}{f(z)} \ast \text{F}(1, \alpha; \alpha + 1; z)}, \quad h_4(z) = \frac{z}{\frac{z}{f(z)} \ast \text{F}(1, \beta; \beta + 1; z)}. \]

Then

\[ \frac{z}{f(z)} \ast \text{F}(1, \alpha; \alpha + 1; z) = \frac{z}{h_3(z)}, \quad \frac{z}{f(z)} \ast \text{F}(1, \beta; \beta + 1; z) = \frac{z}{h_4(z)}. \]

We see

\[ h_3(z) \in \mathcal{U} \left( \frac{\lambda |\alpha|}{|\alpha + n|} \right), \quad i.e. \quad \left| \left( \frac{z}{h_3(z)} \right)^2 h_3'(z) - 1 \right| < \frac{\lambda |\alpha|}{|\alpha + n|} \]

and

\[ h_4(z) \in \mathcal{U} \left( \frac{\lambda |\beta|}{|\beta + n|} \right), \quad i.e. \quad \left| \left( \frac{z}{h_4(z)} \right)^2 h_4'(z) - 1 \right| < \frac{\lambda |\beta|}{|\beta + n|}. \]

Since

\[ \frac{z}{f(z)} = \frac{1}{1 + a_{n+1}z^n + \cdots} = 1 - a_{n+1}z^n + \cdots, \]

so that

\[ \frac{z}{f(z)} \ast \text{F}_3(1, \alpha, \beta; \alpha + 1, \beta + 1; z) = 1 - a_{n+1} \left\{ \frac{\alpha \beta}{(\alpha + n)(\beta + n)} \right\} z^n + \cdots. \]

Thus, \( G(z) \) can be written in the form

\[ G(z) = z + a_{n+1} \left\{ \frac{\alpha \beta}{(\alpha + n)(\beta + n)} \right\} z^{n+1} + \cdots. \]

Therefore, as in the proof of Theorem 1, the function \( p(z) \) defined by

\[ p(z) = \left( \frac{z}{G(z)} \right)^2 G'(z) = 1 + (n-1)a_{n+1} \left\{ \frac{\alpha \beta}{(\alpha + n)(\beta + n)} \right\} z^n + \cdots \]

is analytic in \( U \) and \( p(0) = 1, \quad p'(0) = \cdots = p^{(n-1)}(0) = 0. \)

\( p(z) \) can be written as

\[ p(z) = (\alpha + 1) \frac{z}{G(z)} - a_{n+1} \frac{z}{h_3(z)} \]

and

\[ p(z) = (\beta + 1) \frac{z}{G(z)} - \beta \frac{z}{h_4(z)}. \]

By the same argument of proof of Theorem 1 using \( h_3(z) \) and \( h_4(z) \) instead of \( h_1(z) \) and \( h_2(z) \), \( p(z) \) satisfies (2.9). Consequently, we obtain that

\[ |p(z) - 1| \leq \frac{\lambda |\alpha + \beta||z|^n}{|\alpha + \beta + 2n|} \quad (z \in U), \]

and the proof of part(1) is complete. The second part is a direct consequence of Lemma 2.
References


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